Trishna's

Pearson SUCCESS COMPANION

Pearson IIT Foundation Series

Mathematics

CLASS 10

> Provides student-friendly content, application based problems and hints and solutions to master the art of problem solving

> Uses a graded approach to generate, build and retain interest in concepts and their applications

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CLASS

10

Pearson IIT Foundation Series

Mathematics

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CLASS

10

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Mathematics

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Trishna Knowledge Systems

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Brief Contents

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Preface

Pearson IIT Foundation Series has evolved into a trusted resource for students who aspire to be a part of the elite undergraduate institutions of India. As a result, it has become one of the best-selling series, providing authentic and class-tested content for effective preparation—strong foundation, and better scoring.

The structure of the content is not only student-friendly but also designed in such a manner that it motivates students to go beyond the usual school curriculum, and acts as a source of higher learning to strengthen the fundamental concepts of Physics, Chemistry, and Mathematics.

The core objective of the series is to be a one-stop solution for students preparing for various competitive examinations. Irrespective of the field of study that the student may choose to take up later, it is important to understand that Mathematics and Science form the basis for most modern-day activities. Hence, utmost effort has been made to develop student interest in these basic blocks through real-life examples and application-based problems. Ultimately, the aim is to ingrain the art of problem-solving in the mind of the reader.

To ensure high level of accuracy and practicality, this series has been authored by a team of highly qualified teachers with a rich experience, and are actively involved in grooming young minds. That said, we believe that there is always scope for doing things better and hence invite you to provide us with your feedback and suggestions on how this series can be improved further.

Chapter Insights

REMEMBER

Before beginning this chapter, you should be able to:

- Solve simple equations with one/two variable
- Use graphs of linear equations and in-equations
- Solve basic word problems on linear equations and in-equations

Remember section will help them to memorize and review the previous learning on a particular topic

Key points will help the students to identify the essential points in a chapter

KEY IDEAS

After completing this chapter, you would be able to:

- Solve linear equations by different methods
- Learn nature of solutions of simultaneous linear equations
- Study solving of word problems on linear equations with two variables

EQUATION

An equation is a sentence in which there is an equality sign between two algebraic expressions.

For example, $2x + 5 = x + 3$, $3y - 4 = 20$ and $5x + 6 = x + 1$ are some examples of equations. Here x and y are unknown quantities and 5, 3, 20, etc., are known quantities.

Linear Equation

An equation, in which the highest index of the unknowns present is one, is a linear equation.

$2(x + 5) = 18$, $3x - 2 = 5$, $x + y = 20$ and $3x - 2y = 5$ are some linear equations.

Text: Concepts are explained in a well structured and lucid manner

Note boxes are some add-on information of related topics

Note Two ordered pairs are said to be equal only when their first as well as the second coordinates are equal, i.e., $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

EXAMPLE 4.3

Find the roots of the equation $x^2 + 3x - 4 = 0$.

SOLUTION

$$x^2 + 3x - 4 = 0$$

$$\Rightarrow \quad x^2 - x + 4x - 4 = 0$$

$$\Rightarrow \quad x(x - 1) + 4(x - 1) = 0$$

$$\Rightarrow \quad (x + 4)(x - 1) = 0.$$

$\therefore$ $x = -4$ or $x = 1$.

Examples are given topic-wise to apply the concepts learned in a particular chapter

Illustrative examples solved in a logical and step-wise manner

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Third term of the sequence whose nth term is $2n + 5$ is _____.
2. If a is the first term and d is the common difference of an AP, then the $(n + 1)$th term of the AP is _____.
3. If the sum of three consecutive terms of an AP is 9, then the middle term is _____.
4. General term of the sequence 5, 25, 125, 625, … is _____.
5. The arithmetic mean of 7 and 8 is _____.
6. The arrangement of numbers $\frac{1}{2}, \frac{-3}{4}, \frac{-5}{6}, \frac{-7}{8}, \ldots$ is an example of sequence. [True/False]

series also will be in geometric progression. [True/False]

16. Geometric mean of 5, 10 and 20 is _____
17. Sum of the infinite terms of the GP, −3, −6, −12, … is 3. [True/False]
18. The reciprocals of all the terms of a series in geometric progression form a _____ progression.
19. The nth term of the sequence $\frac{1}{100}, \frac{1}{10000}, \frac{1}{1000000}$, … is _____.
20. In a series, $T_n = x^{2n-2}$ $(x \neq 0)$, then write the infinite series.
21. The harmonic mean of 1, 2 and 3 is $\frac{3}{2}$. [True/False]

Different levels of questions have been included in the *Test Your Concepts* as well as on *Concept Application* which will help students to develop the problem-solving skill

'Test Your Concepts' at the end of the chapter for classroom preparation

CONCEPT APPLICATION

Level 1

1. If $\sin x° = \sin \alpha\, x$, then α is

(a) $\frac{180}{\pi}$ (b) $\frac{\pi}{270}$

(c) $\frac{270}{\pi}$ (d) $\frac{\pi}{180}$

2. If in a triangle ABC, A and B are complementary, then $\tan C$ is

(a) ∞ (b) 0

(c) 1 (d) $\sqrt{3}$

4. $\sin^2 20 + \sin^2 70$ is equal to _____.

(a) 1 (b) −1

(c) 0 (d) 2

5. $\cos 50°\, 50^1 \cos 9°\, 10^1 - \sin 50°\, 50^1 \sin 9°\, 10^1 =$

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\frac{\sqrt{3}}{2}$

'Concept Application' section with problems divided as per complexity: Level 1; Level 2; and Level 3

56. $$\lim_{x\to 3} \frac{\sqrt{5x+1}-\sqrt{7x-5}}{\sqrt{7x+4}-\sqrt{5x+10}} = \frac{0}{0}$$

That is, undetermined form.

Multiply both the numerator and the denominator with $\left(\sqrt{5x+1}+\sqrt{7x-5}\right)\left(\sqrt{7x+4}+\sqrt{5x+10}\right)$

$$\Rightarrow \lim_{x\to 3} \frac{[5x+1-(7x-5)]\left(\sqrt{7x+4}+\sqrt{5x+10}\right)}{(7x+4-5x-10)\left(\sqrt{5x+1}+\sqrt{7x-5}\right)}$$

$$\Rightarrow \lim_{x\to 3} \frac{(-2x+6)\left(\sqrt{7x+4}+\sqrt{5x+10}\right)}{(2x-6)\left(\sqrt{5x+1}+\sqrt{7x-5}\right)}$$

$$= \lim_{x\to 3} \frac{-\left(\sqrt{7x+4}+\sqrt{5x+10}\right)}{\left(\sqrt{5x+1}+\sqrt{7x-5}\right)}$$

$$= \frac{-\left(\sqrt{25}+\sqrt{25}\right)}{\left(\sqrt{16}+\sqrt{16}\right)} = \frac{-10}{8} = \frac{-5}{4}.$$

Hints and Explanation for key questions along with highlights on the common mistakes that students usually make in the examination

Series Chapter Flow

Class 7

1. Number Systems
2. Expressions and Special Products
3. Ratio and Its applications
4. Indices
5. Geometry
6. Mensuration
7. Equations and their Applications
8. Formulae
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17. Percentages, Profit and Loss, Discount and Partnership
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20. Ratio, Proportion and Variation
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23. Time and Distance

Class 8

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3. Indices
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11. Time and Distance
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17. Mensuration

Class 10

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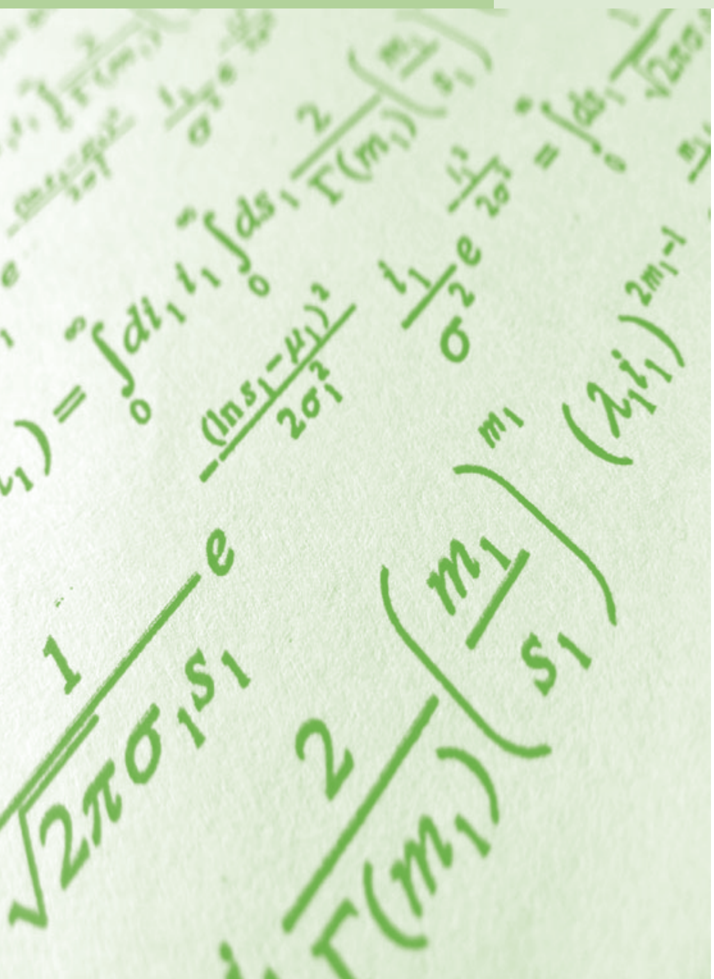
Chapter 1

Number Systems



REMEMBER

Before beginning this chapter, you should be able to:

- Review the types of numbers and understand representation of numbers on a number line
- Study properties of numbers

KEY IDEAS

After completing this chapter, you would be able to:

- Study Euclid's division lemma
- Learn fundamental arithmetic theorem
- Prove theorems related to irrational numbers and rational numbers
- Study proof of irrationality

INTRODUCTION

Earlier we have learnt about number line, irrational numbers, how to represent irrational numbers on the number line, real numbers and their decimal representation, representing real numbers on the number line and operations on the number line. We shall now present a detailed study of Euclid's division algorithm and fundamental theorem of arithmetic. Further we continue the revision of irrational numbers and the decimal expansion of rational numbers.

We are very familiar with division rule.

That is, dividend = (divisor × quotient) + remainder.

Euclid's division lemma is based on this rule. We use this result to obtain the HCF of two numbers. And also we know that, every composite number can be expressed as the product of primes in a unique way. This is the fundamental theorem of arithmetic. This result is used to prove the irrationality of a number.

In the previous class we have studied about rational numbers. In this chapter, we shall explore when exactly the decimal expansion of a rational number say $\frac{p}{q}$ $(q \neq 0)$ is terminating and when it is non-terminating and repeating.

EUCLID'S DIVISION LEMMA

For any two positive integers, say x and y, there exist unique integers say q and r satisfying $x = yq + r$, where $0 \leq r < y$.

Example: Consider the integers 9 and 19.

$$19 = 9 \times 2 + 1$$

Example: Consider the integers 6 and 24.

$$24 = 6 \times 4 + 0$$

Notes

1. Euclid's division algorithm is used for finding the greatest common divisor of two numbers.
2. An algorithm is a process of solving particular problems.

EXAMPLE 1.1

Find the HCF of 250 and 30.

SOLUTION

By using Euclid's division lemma, we get,

$$250 = 30 \times 8 + 10$$

Now consider, the divisor and remainder.

Again by using Euclid's division lemma, we get

$$30 = 10 \times 3 + 0$$

Here, we notice that the remainder is zero and we cannot proceed further.

The divisor at this stage is 10.

The HCF of 250 and 30 is 10.

It can be verified by listing out all the factors of 250 and 30.

Note Euclid's division algorithm is stated for only positive integers, it can be also extended for all negative integers.

Euclid's division algorithm has several applications. The following examples give the idea of the applications.

EXAMPLE 1.2

Show that every positive even integer is of the form $2n$ and every positive odd integer is of the form $2n + 1$.

SOLUTION

For any integer x and $y = 2$, $x = 2n + r$, where $n \geq 0$.

But $0 \leq r < 2$

$$\Rightarrow r = 0 \text{ or } 1$$

When $r = 0$, $x = 2n$

$$\Rightarrow x \text{ is a positive even integer}$$

When $r = 1$, $x = 2n + 1$

$$\Rightarrow x \text{ is a positive odd integer.}$$

EXAMPLE 1.3

A trader has 612 Dettol soaps and 342 Pears soaps. He packs them in boxes and each box contains exactly one type of soap. If every box contains the same number of soaps, then find the number of soaps in each box such that the number of boxes is the least.

SOLUTION

The required number is HCF of 612 and 342.

This number gives the maximum number of soaps in each box and the number of boxes with them be the least.

By using Euclid's division algorithm, we have

$$612 = 342 \times 1 + 270$$
$$342 = 270 \times 1 + 72$$
$$270 = 72 \times 3 + 54$$
$$72 = 54 \times 1 + 18$$
$$54 = 18 \times 3 + 0$$

Here we notice that the remainder is zero, and the divisor at this stage is 18.

$\therefore$ HCF of 612 and 342 is 18.

So, the trader can pack 18 soaps per box.

FUNDAMENTAL THEOREM OF ARITHMETIC

Every composite number can be expressed as the product of prime factors uniquely.

Note In general $a = p_1\, p_2\, p_3 \ldots p_n$, where $p_1, p_2, p_3, \ldots, p_n$ are primes in ascending order.

EXAMPLE 1.4

Write 1800 as product of prime factors.

SOLUTION

$$\begin{array}{r|l}
2 & 1800 \\ \hline
2 & 900 \\ \hline
2 & 450 \\ \hline
3 & 225 \\ \hline
3 & 75 \\ \hline
5 & 25 \\ \hline
 & 5
\end{array}$$

$\therefore$ $1800 = 2^3 \times 3^2 \times 5^2$.

Let us see the applications of fundamental theorem.

EXAMPLE 1.5

Check whether there is any value of x for which 6^x ends with 5.

SOLUTION

If 6^x ends with 5, then 6^x would contain the prime number 5.

But, $6^x = (2 \times 3)^x = 2^x \times 3^x$

$\Rightarrow$ The prime numbers in the factorization of 6^x are 2 and 3

By uniqueness of fundamental theorem, there are no prime numbers other than 2 and 3 in 6^x.

$\therefore$ 6^x never ends with 5.

EXAMPLE 1.6

Show that $5 \times 3 \times 2 + 3$ is a composite number.

SOLUTION

$$5 \times 3 \times 2 + 3 = 3(5 \times 2 + 1) = 3(11) = 3 \times 11$$

$\therefore$ The given number is a composite number.

EXAMPLE 1.7

Find the HCF and LCM of 48 and 56 by prime factorization method.

SOLUTION

$$48 = 2^4 \times 3^1$$

$$56 = 2^3 \times 7^1$$

HCF $= 2^3$ (The product of common prime factors with lesser index)

LCM $= 2^4 \times 3^1 \times 7^1$ (The product of common prime factors with greater index).

EXAMPLE 1.8

Find the HCF and LCM of 36, 48 and 60 by prime factorization method.

SOLUTION

$$36 = 2^2 \times 3^2$$
$$48 = 2^4 \times 3^1$$
$$60 = 2^2 \times 3^1 \times 5^1$$

HCF $= 2^2 \times 3^1 = 12$

LCM $= 2^4 \times 3^2 \times 5^1 = 720$.

EXAMPLE 1.9

Two bells toll at intervals of 24 minutes and 36 minutes respectively. If they toll together at 9 am, after how many minutes do they toll together again, at the earliest?

SOLUTION

The required time is LCM of 24 and 36.

$$24 = 2^3 \times 3^1$$
$$36 = 2^2 \times 3^2$$

$\therefore$ LCM of 24 and 36 is $2^3 \times 3^2 = 72$.

So, they will toll together after 72 minutes at the earliest.

EXAMPLE 1.10

There are 44 boys and 32 girls in a class. These students are arranged in rows for a prayer in such a way that each row consists of only either boys or girls, and every row contains an equal number of students. From the following 4 options, choose the minimum number of rows in which all the students can be arranged.

(a) 4 **(b)** 12 **(c)** 15 **(d)** 19

SOLUTION

For number of rows to be the least, the number of students in each row must be the highest. The number of students in each row is a common factor of 44 and 32. The highest value is the HCF of 44 and 32, which is 4.

$\therefore$ Minimum number of rows $= \dfrac{44}{4} + \dfrac{32}{4} = 19$.

EXAMPLE 1.11

Two positive numbers have their HCF as 12 and their product as 6336. Choose the correct option for the number of pairs possible for the given numbers.

(a) 2 **(b)** 3 **(c)** 4 **(d)** 5

SOLUTION

Let the numbers be $12x$ and $12y$, where x and y are co-primes.

Product of the numbers $= 144xy$

$$144xy = 6336$$

$$xy = 44$$

44 can be written as the product of two factors in three ways, i.e., 1×44, 2×22, 4×11.

As x and y are relatively prime, (x, y) can be (1, 44) or (4, 11) but not (2, 22).

Hence, two possible pairs exist.

EXAMPLE 1.12

Four persons P, Q, R and S start running around a circular track simultaneously. If they complete one round in 10, 8, 12 and 18 minutes respectively, after how much time will they next meet at the starting point? Choose the right answer from the given options.

(a) 180 minutes **(b)** 270 minutes **(c)** 360 minutes **(d)** 450 minutes

SOLUTION

The time interval for them to meet at the starting point for the next time = LCM (10, 8, 12,18) = 360 minutes.

IRRATIONAL NUMBERS

A number which cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is an irrational number.

Example: $\sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10},$ 0.123421635, …, etc.

Theorem 1

If p divides x^3, then p divides x, where x is a positive integer and p is a prime number.

Proof: Let $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, p_3, \ldots, p_n$ are primes, not necessarily distinct.

$$\Rightarrow x^3 = p_1^3 p_2^3 \cdots p_n^3$$

Given that p divides x^3.

By fundamental theorem, p is one of the primes of x^3.

By the uniqueness of fundamental theorem, the distinct primes of x^3 are same as the distinct primes of x.

$\Rightarrow p$ divides x

Hence proved.

Similarly, if p divides x^2, then p divides x, where p is a prime number and x is a positive integer.

Theorem 2

Prove that $\sqrt{2}$ is irrational.

Proof: Let us assume that, $\sqrt{2}$ is not irrational.

So, $\sqrt{2}$ is rational.

$\Rightarrow \sqrt{2} = \frac{x}{y}$, where x, y are integers and $y \neq 0$

Let x and y be co-primes.

Taking squares on both sides,

$$\Rightarrow 2 = \frac{x^2}{y^2}$$

$$\Rightarrow 2y^2 = x^2$$

$$\Rightarrow 2 \text{ divides } x^2$$

$$\Rightarrow 2 \text{ divides } x \quad (1)$$

$\therefore$ For some integer z,

$$x = 2z \quad (2)$$

From Eqs. (1) and (2),

$$2y^2 = 4z^2$$

$$\Rightarrow y^2 = 2z^2$$

$$\Rightarrow 2 \text{ divides } y^2$$

$$\Rightarrow 2 \text{ divides } y$$

$\therefore$ x and y have at least 2 as a common factor.

But it contradicts the fact that x and y are co-primes.

$\therefore$ $\sqrt{2}$ is irrational.

EXAMPLE 1.13

Prove that $\sqrt{5}$ is irrational.

SOLUTION

Let us assume that $\sqrt{5}$ is not irrational.

$\therefore$ $\sqrt{5}$ is rational.

$$\Rightarrow \sqrt{5} = \frac{p}{q}$$

Let p, q be co-primes.

Taking squares on both the sides,

$$\Rightarrow 5 = \frac{p^2}{q^2}$$

$$\Rightarrow p^2 = 5q^2 \quad (1)$$

$$\Rightarrow 5 \text{ divides } p^2$$
$$\Rightarrow 5 \text{ divides } p$$

$\therefore$ For some integer r,

$$p = 5r \quad (2)$$

From Eqs. (1) and (2),

$$25r^2 = 5q^2$$
$$\Rightarrow q^2 = 5r^2$$
$$\Rightarrow 5 \text{ divides } q^2$$
$$\Rightarrow 5 \text{ divides } q$$

$\therefore$ p and q have at least 5 as a common factor.

But it contradicts the fact that p and q are co-primes.

$\therefore$ $\sqrt{5}$ is irrational.

EXAMPLE 1.14

Show that $3+\sqrt{2}$ is irrational.

SOLUTION

Let us assume that $3+\sqrt{2}$ is rational.

$\therefore 3+\sqrt{2} = \frac{p}{q}$, where p and q are integers.

$$\Rightarrow \sqrt{2} = \frac{p}{q} - 3 \Rightarrow \sqrt{2} = \frac{p-3q}{q}$$

Since p and q are integers, $\frac{p-3q}{q}$ is rational.

But $\sqrt{2}$ is irrational.

It contradicts our assumption that $3+\sqrt{2}$ is rational.

$\therefore$ Our assumption is wrong.

Hence, $3+\sqrt{2}$ is irrational.

EXAMPLE 1.15

If $a = \sqrt{11}+\sqrt{3}$, $b = \sqrt{12}+\sqrt{2}$ and $c = \sqrt{6}+\sqrt{4}$, then which of the following holds true?

(a) $c > a > b$ **(b)** $a > b > c$ **(c)** $a > c > b$ **(d)** $b > a > c$

SOLUTION

$a = \sqrt{11}+\sqrt{3}$

$a^2 = (\sqrt{11}+\sqrt{3})^2 = 14 + 2\sqrt{33}$

$b = \sqrt{12}+\sqrt{2}$

$b^2 = (\sqrt{12} + \sqrt{2})^2 = 14 + 2\sqrt{24}$

As $\sqrt{33} > \sqrt{24}$, $a^2 > b^2$, hence $a > b$.

$c = \sqrt{6} + \sqrt{4}$

$c^2 = (\sqrt{6} + \sqrt{4})^2 = 10 + 2\sqrt{24}$

As $14 > 10$, $b^2 > c^2$, hence $b > c$.

Hence, $a > b > c$.

EXAMPLE 1.16

Choose the correct value of $\frac{1}{\sqrt{9}+\sqrt{10}} + \frac{1}{\sqrt{10}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{12}} + \cdots$ up to 91 terms from the following options:

(a) 7 **(b)** 8 **(c)** 6 **(d)** 9

SOLUTION

$$\frac{1}{\sqrt{9}+\sqrt{10}} = \frac{\sqrt{9}-\sqrt{10}}{(\sqrt{9}-\sqrt{10})(\sqrt{9}-\sqrt{10})}$$

(multiplying both its numerator and denominator by $\sqrt{9}-\sqrt{10}$)

Hence, $$\frac{1}{\sqrt{9}+\sqrt{10}} = \frac{\sqrt{9}-\sqrt{10}}{(\sqrt{9})^2-(\sqrt{10})^2}$$

$$= \frac{\sqrt{9}-\sqrt{10}}{-1}$$

By multiplying numerator and denominator of each term by the conjugate of their denominators, the given expression becomes

$$\frac{\sqrt{9}-\sqrt{10}}{-1} + \frac{\sqrt{10}-\sqrt{11}}{-1} + \frac{\sqrt{11}-\sqrt{12}}{-1} + \cdots + \frac{\sqrt{99}-\sqrt{100}}{-1}$$

$$= \frac{\sqrt{9}-\sqrt{100}}{-1} = 7.$$

RATIONAL NUMBERS

Numbers which can be written in the form of $\frac{p}{q}$ $(q \neq 0)$, where p, q are integers, are called rational numbers.

Note Every terminating decimal and non-terminating repeating decimal can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$.

Examples:

1. $0.27 = \dfrac{27}{100} = \dfrac{27}{2^2 \times 5^2}$
2. $2.356 = \dfrac{2356}{1000} = \dfrac{2^2 \times 589}{2^3 \times 5^3} = \dfrac{589}{2 \times 5^3}$
3. $2.0325 = \dfrac{20325}{10000} = \dfrac{3 \times 5^2 \times 271}{2^4 \times 5^4} = \dfrac{3 \times 271}{2^4 \times 5^2}$

From the above examples, we notice that every terminating decimal can be written in the form of $\dfrac{p}{q}$ $(q \neq 0)$, where p and q are co-primes and q is of the form $2^m \cdot 5^n$ (m and n are non-negative integers).

Let us write this result formally.

Theorem 3

Let a be a terminating decimal. Then a can be expressed as $\dfrac{p}{q}$ $(q \neq 0)$, where p and q are co-primes, and the prime factorization of q is of the form $2^m \cdot 5^n$.

Let us observe the following examples:

1. $\dfrac{1}{4} = \dfrac{1}{2^2} = \dfrac{1 \times 5^2}{2^2 \times 5^2} = \dfrac{2}{25} = 0.25$
2. $\dfrac{7}{25} = \dfrac{7}{5^2} = \dfrac{7 \times 2^2}{5^2 \times 2^2} = \dfrac{28}{100} = 0.28$
3. $\dfrac{23}{125} = \dfrac{23}{5^3} = \dfrac{23 \times 2^3}{5^3 \times 2^3} = \dfrac{184}{1000} = 0.184$
4. $\dfrac{147}{50} = \dfrac{147}{2 \times 5^2} = \dfrac{147 \times 2}{2^2 \times 5^2} = \dfrac{294}{100} = 2.94$

From the above examples, we notice that every rational number of the form $\dfrac{p}{q}$ $(q \neq 0)$, where q is of the form $2^m \cdot 5^n$ can be written as $\dfrac{x}{y}$, where y is of the form 10^k, k being a natural number.

Let us write this result formally.

Theorem 4

If $\dfrac{p}{q}$ is a rational number where q is of the form $2^m \cdot 5^m$ $(m \in W)$, then $\dfrac{p}{q}$ has a terminating decimal expansion.

Let us observe the following examples.

Examples:

1. $\dfrac{5}{3} = 1.66666... = 1.\overline{6}$
2. $\dfrac{8}{7} = 1.\overline{142857}$
3. $\dfrac{1}{11} = 0.\overline{09}$

Theorem 5

If $\frac{p}{q}$ is a rational number and q is not of the form $2^m \cdot 5^n$ (m and $n \in W$), then $\frac{p}{q}$ has a non-terminating repeating decimal expansion.

EXAMPLE 1.17

Which of the following rational numbers are terminating decimals?

(a) $\frac{17}{2^3 \times 5^2}$ **(b)** $\frac{25}{3^2 \times 2^3}$ **(c)** $\frac{68}{2^2 \times 5^2 \times 7^2}$ **(d)** $\frac{125}{3^3 \times 7^2}$

SOLUTION

Clearly $\frac{17}{2^3 \times 5^2}$ is the only terminating decimal and the remaining are non-terminating decimals.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $\frac{7^3}{5^4}$ is a non-terminating repeating decimal. (True/False)
2. $0.\overline{5} - 0.\overline{49} =$ ________.
3. If $360 = 2^x \times 3^y \times 5^z$, then $x + y + z =$ ________.
4. If $a = 2 + \sqrt{3}$ and $b = \sqrt{2} - \sqrt{3}$, then $a + b$ is ________. (rational/irrational)
5. If 19 divides a^3 (where a is a positive integer), then 19 divides a. (True/False)
6. If $0.\overline{7} = \frac{p}{q}$, then $p + q =$ ________.
7. Product of two irrational numbers is an irrational number. (True/False)
8. $2 \times 3 \times 15 + 7$ is a ________. (prime number/composite number)
9. Product of LCM and HCF of 25 and 625 is ________.
10. For what values of x, $2^x \times 5^x$ ends in 5.

Short Answer Type Questions

11. Find the HCF of 72 and 264 by using Euclid's division algorithm.
12. Show that $7 \times 5 \times 3 \times 2 + 7$ is a composite number.
13. Find the HCF and LCM of 108 and 360 by prime factorization method.
14. Prove that $\sqrt{6}$ is irrational.
15. Without performing the actual division, check whether the following rational numbers are terminating or non-terminating.

    (i) $\frac{23}{175}$ (ii) $\frac{125}{325}$ (iii) $\frac{73}{40}$ (iv) $\frac{157}{125}$

Essay Type Questions

16. A fruit vendor has 732 apples and 942 oranges. He distributes these fruits among the students of an orphanage, such that each of them gets either only apples or only oranges in equal number. Find the least possible number of students.
17. Write 75600 as the product of prime factors.
18. Aloukya and Manoghna run around a circular track and they take 180 seconds and 150 seconds respectively to complete one revolution. If they start together at 9 am from the same point, how long it would take for them to meet again for the first time at the starting point?
19. Prove that $5 - \sqrt{5}$ is irrational.
20. Express $23.3\overline{24}$ in $\frac{p}{q}$ form, where p and q are integers.

CONCEPT APPLICATION

Level 1

1. If n is a natural number, then $9^{2n} - 4^{2n}$ is always divisible by ________.

   (a) 5
   (b) 13
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)

2. N is a natural number such that when N^3 is divided by 9, it leaves remainder a. It can be concluded that

   (a) a is a perfect square.
   (b) a is a perfect cube.
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)

3. The remainder of any perfect square divided by 3 is ________.
   (a) 0
   (b) 1
   (c) Either (a) or (b)
   (d) Neither (a) nor (b)

4. Find the HCF of 432 and 504 using prime factorization method.
   (a) 36 (b) 72
   (c) 96 (d) 108

5. If n is any natural number, then $6^n - 5^n$ always ends with ________.
   (a) 1 (b) 3
   (c) 5 (d) 7

6. The LCM of two numbers is 1200. Which of the following cannot be their HCF?
   (a) 600 (b) 500
   (c) 200 (d) 400

7. Which of the following is always true?
   (a) The rationalizing factor of a number is unique.
   (b) The sum of two distinct irrational numbers is rational.
   (c) The product of two distinct irrational numbers is irrational.
   (d) None of these

8. Find the remainder when the square of any number is divided by 4.
   (a) 0 (b) 1
   (c) Either (a) or (b) (d) Neither (a) nor (b)

9. Ashok has two vessels which contain 720 ml and 405 ml of milk, respectively. Milk in each vessel is poured into glasses of equal capacity to their brim. Find the minimum number of glasses which can be filled with milk.
   (a) 45 (b) 35
   (c) 25 (d) 30

10. If n is an odd natural number, $3^{2n} + 2^{2n}$ is always divisible by
   (a) 13 (b) 5
   (c) 17 (d) 19

11. For what values of x, $2^x \times 5^x$ ends in 5?
   (a) 0 (b) 1
   (c) 2 (d) No value of x

12. Which of the following is a terminating decimal?
   (a) $\frac{4}{7}$ (b) $\frac{3}{7}$
   (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

13. LCM of two co-primes (say x and y) is ________.
   (a) $x + y$ (b) $x - y$
   (c) xy (d) $\frac{x}{y}$

14. HCF of two co-primes (say x and y) is ________.
   (a) x (b) y
   (c) xy (d) 1

15. If we apply Euclid's division lemma for two numbers 15 and 4, then we get,
   (a) $15 = 4 \times 3 + 3$. (b) $15 = 4 \times 2 + 7$.
   (c) $15 = 4 \times 1 + 11$. (d) $15 = 4 \times 4 + (-1)$.

Level 2

16. Given that the units digits of A^3 and A are the same, where A is a single digit natural number. How many possibilities can A assume?
   (a) 6 (b) 5
   (c) 4 (d) 3

17. If the product of two irrational numbers is rational, then which of the following can be concluded?
   (a) The ratio of the greater and the smaller numbers is an integer.
   (b) The sum of the numbers must be rational.
   (c) The excess of the greater irrational number over the smaller irrational number must be rational.
   (d) None of these

18. The LCM and HCF of two numbers are equal, then the numbers must be ________.

(a) prime (b) co-prime

(c) composite (d) equal

19. Which of the following is/are always true?

(a) Every irrational number is a surd.
(b) Any surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ can be rationalized by a surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$, where $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are surds.
(c) Both (a) and (b)
(d) Neither (a) nor (b)

20. The sum of LCM and HCF of two numbers is 1260. If their LCM is 900 more than their HCF, find the product of two numbers.

(a) 203400 (b) 194400

(c) 198400 (d) 205400

21. The following sentences are the steps involved in finding the HCF of 29 and 24 by using Euclid's division algorithm. Arrange them in sequential order from first to last.

(A) $5 = 1 \times 5 + 0$

(B) $29 = 24 \times 1 + 5$

(C) $24 = 5 \times 4 + 1$

(a) BAC (b) ABC

(c) BCA (d) CAB

22. The following are the steps involved in finding the LCM of 72 and 48 by prime factorization method. Arrange them in sequential order from first to last.

(A) $72 = 2^3 \times 3^2$ and $48 = 2^4 \times 3^1$

(B) LCM $= 24 \times 32$

(C) All the distinct factors with highest exponents are 24 and 32

(a) ABC (b) ACB

(c) CAB (d) BCA

Level 3

23. Find the remainder when the square of any prime number greater than 3 is divided by 6.

(a) 1 (b) 3

(c) 2 (d) 4

24. If HCF $(72, q) = 12$ then how many values can q take? (Assume q be a product of a power of 2 and a power of 3 only)

(a) 1 (b) 2

(c) 3 (d) 4

25. Find the HCF of 120 and 156 using Euclid's division algorithm.

(a) 18 (b) 12

(c) 6 (d) 24

26. The HCF of the polynomials $(x^2 - 4x + 4)(x + 3)$ and $(x^2 + 2x - 3)(x - 2)$ is ________.

(a) $x + 3$

(b) $x - 2$

(c) $(x + 3)(x - 2)$

(d) $(x + 3)(x - 2)^2$

27. $\sqrt{3+\sqrt{5}} =$ ________.

(a) $\sqrt{2}+1$ (b) $\sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}}$

(c) $\sqrt{\frac{7}{2}}-\sqrt{\frac{1}{2}}$ (d) $\sqrt{\frac{9}{2}}-\sqrt{\frac{3}{2}}$

28. The value of $(37)^{3^x} - (33)^{3^x}$ ends in ________ $(x \in N)$.

(a) 4 (b) 6

(c) 0 (d) Either (a) or (b)

29. $P = 2(4)(6)\ldots(20)$ and $Q = 1(3)(5)\ldots(19)$. What is the HCF of P and Q?

(a) $3^3 \cdot 5 \cdot 7$ (b) $3^4 \cdot 5$

(c) $3^4 \cdot 5^2 \cdot 7$ (d) $3^3 \cdot 5^2$

30. The LCM and the HCF of two numbers are 1001 and 7 respectively. How many such pairs are possible?

(a) 0 (b) 1

(c) 2 (d) 7

PRACTICE QUESTIONS

31. There are 96 apples and 112 oranges. These fruits are packed in boxes in such a way that each box contains fruits of the same variety, and every box contains an equal number of fruits. Find the minimum number of boxes in which all the fruits can be packed.

(a) 12 (b) 13
(c) 14 (d) 15

32. Two runners A and B are running on a circular track. A takes 40 seconds to complete every round and B takes 30 seconds to complete every round. If they start simultaneously at 9:00 am, then which of the following is the time at which they can meet at the starting point?

(a) 9:05 am (b) 9:10 am
(c) 9:15 am (d) 9:13 am

33. In how many ways can 1500 be resolved into two factors?

(a) 18 (b) 12
(c) 24 (d) 36

34. Four bells toll at intervals of 10 seconds, 15 seconds, 20 seconds and 30 seconds respectively. If they toll together at 10:00 am at what time will they toll together for the first time after 10 am?

(a) 10:01 am (b) 10:02 am
(d) 10:00:30 am (d) 10:00:45 am

35. If $a = \sqrt[8]{6} - \sqrt[8]{5}$, $b = \sqrt[8]{6} + \sqrt[8]{5}$, $c = \sqrt[6]{6} + \sqrt[6]{5}$, $d = \sqrt[4]{6} + \sqrt[4]{5}$, and $e = \sqrt{6} + \sqrt{5}$, then which of the following is a rational number?

(a) *abcde* (b) *abde*
(c) *ab* (d) *cd*

36. Find the units digit of $(12)^{3^x} + (18)^{3^x}$ $(x \in N)$.

(a) 2 (b) 8
(c) 0 (d) 1

37. If $X = 28 + (1 \times 2 \times 3 \times 4 \times \cdots \times 16 \times 28)$ and $Y = 17 + (1 \times 2 \times 3 \times \cdots \times 17)$, then which of the following is/are true?

(A) X is a composite number
(B) Y is a prime number
(C) $X - Y$ is a prime number
(D) $X + Y$ is a composite number

(a) Both (A) and (D)
(b) Both (B) and (C)
(c) Both (B) and (D)
(d) Both (A) and (B)

38. P is the LCM of 2, 4, 6, 8, 10, Q is the LCM of 1, 3, 5, 7, 9 and L is the LCM of P and Q. Then, which of the following is true?

(a) $L = 21P$ (b) $L = 4Q$
(c) $L = 63P$ (d) $L = 16Q$

39. The LCM and the HCF of two numbers are 144 and 12 respectively. How many such pairs of numbers are possible?

(a) 0 (b) 1
(c) 2 (d) 10

40. Two bells toll in every 45 seconds and 60 seconds. If they toll together at 8:00 am, then which of the following is the probable time at which they can toll together?

(a) 8:55 am (b) 8:50 am
(c) 8:45 am (d) 8:40 am

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. False
2. $0.\overline{06}$
3. 6
4. irrational
5. True
6. 16
7. False
8. prime number
9. 15625
10. No value of x

Short Answer Type Questions

11. 24
13. 1080
15. (i), (ii) are non-terminating and (iii), (iv) are terminating.

Essay Type Questions

16. 279
17. $2^4 \times 3^3 \times 5^2 \times 7$
18. 9:15 am
20. $\dfrac{23091}{990}$

CONCEPT APPLICATION

Level 1

1. (c) **2.** (b) **3.** (c) **4.** (b) **5.** (a) **6.** (a) **7.** (d) **8.** (b) **9.** (c) **10.** (a)
11. (d) **12.** (d) **13.** (c) **14.** (d) **15.** (a)

Level 2

16. (b) **17.** (d) **18.** (d) **19.** (d) **20.** (b) **21.** (c) **22.** (b)

Level 3

23. (a) **24.** (b) **25.** (b) **26.** (c) **27.** (c) **28.** (d) **29.** (c) **30.** (c) **31.** (b) **32.** (b)
33. (b) **34.** (a) **35.** (b) **36.** (c) **37.** (a) **38.** (a) **39.** (c) **40.** (d) **41.** (c) **42.** (a)
43. (b) **44.** (a) **45.** (c)

CONCEPT APPLICATION

Level 1

1. (i) Given expression is in the form of $a^2 - b^2$.
   (ii) $a^{2n} - b^{2n}$ is divisible by both $(a - b)$ and $(a + b)$.
2. Apply trial and error method.
3. Apply trial and error method.
4. (i) Express 432 and 504 as the product of prime factors and proceed.
   (ii) Write 432 and 504 as the product of prime factors.
   (iii) HCF is the product of common prime factors with least exponents.
5. For any natural number n, 6^n and 5^n end with 6 and 5 respectively.
6. (i) Apply the method of prime factorization for finding the HCF of the given numbers.
   (ii) Then find the possibilities for q.
7. (i) Recall the concepts of rational and irrational numbers.
   (ii) Recall the concept of RF.
8. (i) Multiply and divide $3+\sqrt{5}$ by 2.
   (ii) Convert the expression $\sqrt{x+2\sqrt{y}}$ in the form of $\sqrt{(\sqrt{m})^2+(n)^2+2(mn)}$.
9. (i) First of all find the HCF of 720 and 405.
   (ii) The required number is $\frac{720}{\text{HCF}} + \frac{405}{\text{HCF}}$.
10. (i) $3^{2n} + 2^{2n} = 9^n + 4^n$.
    (ii) $a^n + b^n$ is divisible by $(a + b)$ when n is odd.
11. For no value of x, $2^x \times 5^x$ ends in 5.
12. Only $\frac{1}{2}$ is terminating decimal.
13. LCM of two co-primes is their product.
14. HCF of two co-primes is 1.
15. $15 = 4 \times 3 + 3$.

Level 2

16. (i) Write cubes of 1 to 9.
    (ii) Then check their unit digits.
17. Recall the concept of rational numbers and irrational numbers.
18. (i) Suppose the numbers to be ka and kb where k is their HCF and a and b are co-primes.
    (ii) Check from the options.
19. (i) Recall the concepts of surds and irrational numbers.
    (ii) Recall the concepts of RF.
20. Form the equations in LCM and HCF and solve for LCM and HCF.
21. BCA is the required sequential order.
22. ACB is the required sequential order.

Level 3

23. Any prime number greater than 3 is in the form $6k \pm 1$, where k is a natural number.
24. HCF is the factor of LCM.
25. (i) Write $156 = 1 \times 120 + 36$ and proceed further.
    (ii) $156 = 120 \times 1 + 36$
    $120 = 36 \times 3 + 12$.
    (iii) Use division algorithm until we get zero as remainder.

26. Factorize the given expressions.

27. Apply trial and error method.

28. For all $x \in N$,

$(37)^{3^x}$ ends in 3 or 7 and $(33)^{3^x}$ ends in 7 or 3.

If $(37)^{3^x}$ ends in 3, then $(33)^{3^x}$ ends 7.

In this case $(37)^{3^x} - (33)^{3^x}$ ends in 6 (1)

If $(37)^{3^x}$ ends in 7, then $(33)^{3^x}$ ends in 3.

In this case, $(37)^{3^x} - (33)^{3^x}$ ends in 4 (2)

From Eqs. (1) and (2), option (d) follows.

29. In P, the prime numbers that occur are 2, 3, 5, 7.

In Q, there are no 2's. So the HCF of P, Q has only 3's, 5's, 7's.

The 3's in P come from 6, 12, 18, i.e., 3^4 is the greatest power of 3 that is a factor of P, while for Q the 3's come from 3, 9, 15, i.e., 3^4 is also a factor of Q.

Similarly, 5^2 is the greatest power of 5 that for both P and Q and 7^1 is the greatest power of 7 for both P and Q.

$\therefore$ The HCF of P, Q is $3^4(5^2)(7)$.

30. Given LCM = 1001 and HCF = 7.

Let the two numbers be x and y.

$\therefore$ $x = 7a$ and $y = 7b$, where a and b are co-primes.

We have, $x \times y$ = LCM $\times$ HCF

$7a \times 7b = 1001 \times 7$

$ab = 143 \Rightarrow (a, b) = (1, 143)$ or $(11, 13)$

$\Rightarrow$ There are two pairs of numbers.

31. For number of boxes to be the least, the number of fruits in each box should be maximum.

As per the given data, the number of fruits in each box is equal to the HCF of 96 and 112, i.e., 16

$\therefore$ Minimum number of boxes $= \dfrac{96}{16} + \dfrac{112}{16} = 13.$

32. LCM of 40 seconds and 30 seconds is 120 seconds, i.e., 2 minutes.

$\therefore$ A and B meet in every 2 minutes.

$\therefore$ Option (b) follows.

33. Number of ways in which 1500 can be resolved into two factors worked out using the concept given by

$$\frac{\text{Number of factors of 1500}}{2}$$

$1500 = 3 \times 5^3 \times 2^2$.

Number of factors of 1500 = $(1 + 1)(3 + 1)(2 + 1) = 24$.

Number of ways in which 1500 can be resolved into two factors $= \dfrac{24}{2} = 12$.

34. The time interval between simultaneous tolling of the bells = LCM (10, 15, 20, 30) seconds = 60 seconds = 1 minute.

Hence the bells will toll together again for the first time after 10:00 am at 10:01 am.

35. Given that, $a = \sqrt[8]{6} - \sqrt[8]{5}$, $b = \sqrt[8]{6} + \sqrt[8]{5}$

$c = \sqrt[6]{6} + \sqrt[6]{5}$

$d = \sqrt[4]{6} + \sqrt[4]{5}$, $e = \sqrt{6} + \sqrt{5}$

By inspection, we can reject ab, cd.

$ab = \sqrt[4]{6} + \sqrt[4]{5}$

$\therefore$ $abd = \sqrt{6} - \sqrt{5}$ and $abde = 6 - 5 = 1$.

$\therefore$ $abde$ is a rational number.

36. For all $x \in N$,

$(12)^{3^x}$ ends in either 8 or 2 and $(18)^{3^x}$ ends in either 2 or 8

If $(12)^{3^x}$ ends in 8, then $(18)^{3^x}$ ends in 2.

If $(12)^{3^x}$ ends in 2, then $(18)^{3^x}$ ends in 8.

$\therefore$ $(12)^{3^x} + (18)^{3^x}$ ends in 0 only.

37. $X = 28 + (1 \times 2 \times 3 \times \cdots \times 16 \times 28)$

$\Rightarrow X = 28\,[1 + (1 \times 2 \times 3 \times \cdots \times 16)]$

$\therefore$ X is a composite number.

$Y = 17 + (1 \times 2 \times 3 \times 4 \times \cdots \times 17) = 17\,[1 + (1 \times 2 \times \cdots \times 16)]$

$\therefore$ Y is a composite number.

Now, $X - Y = [1 + (1 \times 2 \times 3 \times \cdots \times 16)]\,(28 - 17) = [1 + (1 \times 2 \times 3 \times \cdots \times 16)]\,(11)$

$\therefore$ $X - Y$ is a composite number.

$X + Y = [1 + (1 \times 2 \times 3 \times \cdots)]$

$\therefore$ $X + Y$ is a composite number.

$\therefore$ Option (a) follows.

HINTS AND EXPLANATION

38. P is the LCM of 2, 4, 6, 8, 10.

$\therefore\ P = 3(8)(5)$

Q is the LCM of 1, 3, 5, 7, 9.

$\therefore\ Q = 5(7)(9)$

L is the LCM of P, Q

$\therefore\ L = 3(8)(5)21$ or $5(7)(9)8$, i.e., $21P$ or $8Q$.

39. Given LCM = 144 and HCF = 12

Let the two numbers be x and y.

$\therefore\ x = 12a$ and $y = 12b$

Where a and b are co-primes.

We have, $x \times y = \text{LCM} \times \text{HCF}$

$12a \times 12b = 144 \times 12$

$ab = 12 \Rightarrow (a, b) = (1, 12)$ or $(3, 4)$

There are two pairs of numbers.

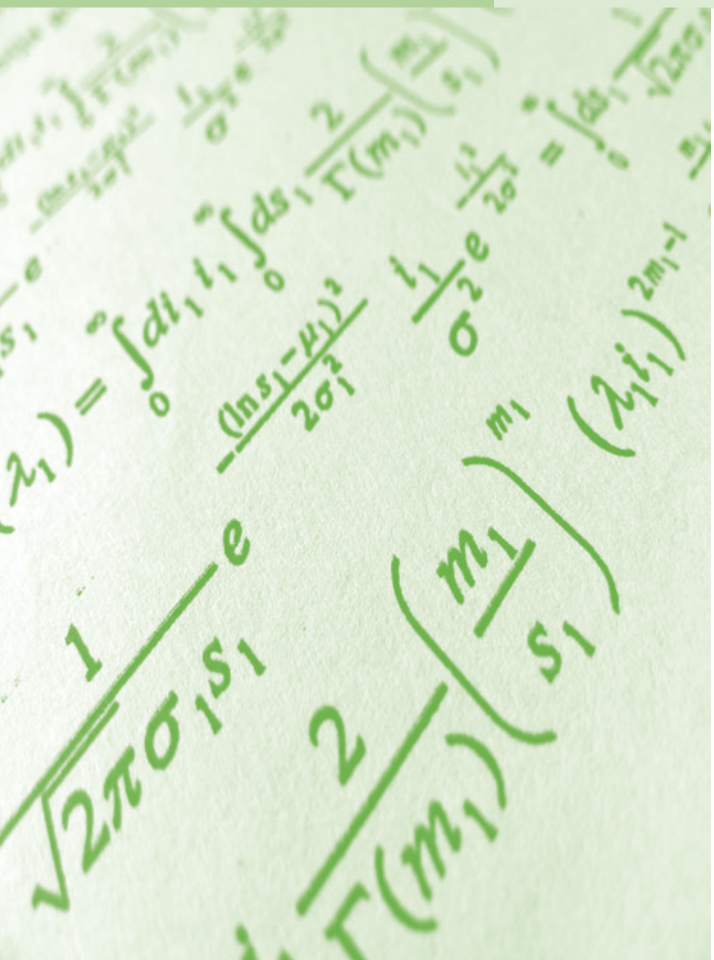
40. LCM of 45 seconds and 60 seconds is 180 seconds, i.e., 3 minutes.

$\therefore$ They toll together in every 3 minutes

$\therefore$ Option (c) follows.

Chapter 2

Polynomials and Rational Expressions



REMEMBER

Before beginning this chapter, you should be able to:

- Explain about polynomials, their types and basic operations on polynomials
- Work on factorization of polynomials

KEY IDEAS

After completing this chapter, you would be able to:

- Define terms such as HCF and LCM
- Understand the methods to find LCM and HCF of polynomials
- Study relation among the LCM, the HCF and the product of polynomials
- Learn fundamental operations on rational expressions

INTRODUCTION

In this chapter, we will learn about HCF and LCM of polynomials and the method of finding them. Further we will learn about rational expressions and also the method of expressing them in their lowest terms.

POLYNOMIAL OF *n*TH DEGREE

The expression of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is a polynomial of nth degree ($a_0 \neq 0$) in one variable. Here $a_0, a_1, \ldots, a_n$ are real numbers.

HCF of Given Polynomial

For the given two polynomials, $f(x)$ and $g(x)$, $r(x)$ can be taken as the highest common factor, if

1. $r(x)$ is a common factor of $f(x)$ and $g(x)$ and
2. every common factor of $f(x)$ and $g(x)$ is also a factor of $r(x)$.

Highest common factor is generally referred to as HCF.

Method for Finding HCF of the Given Polynomials

Step 1: Express each polynomial as a product of powers of irreducible factors which also requires the numerical factors to be expressed as the product of the powers of primes.

Step 2: If there is no common factor then HCF is 1 and if there are common irreducible factors, we find the least exponent of these irreducible factors in the factorized form of the given polynomials.

Step 3: Raise the common irreducible factors to the smallest or the least exponents found in step 2 and take their product to get the HCF.

Examples:

1. Find the HCF of $42a^2b^2$ and $48ab^3$.

   Let $f(x) = 42a^2b^2$ and $g(x) = 48ab^3$

   Writing $f(x)$ and $g(x)$ as a product of powers of irreducible factors,

   $f(x) = 2 \times 3 \times 7 \times a^2 \times b^2$

   $g(x) = 2 \times 2 \times 2 \times 2 \times 3 \times a \times b^3 = 2^4 \times 3 \times a \times b^3$

   The common factors with the least exponents are 2, 3 and ab^2.

   $\therefore$ The HCF of the given polynomials $= 2 \times 3 \times ab^2 = 6ab^2$.

2. Find the HCF of $96(x - 1)(x + 1)^2(x + 3)^3$ and $64(x^2 - 1)(x + 3)(x + 2)^2$.

   Let $f(x) = 96(x - 1)(x + 1)^2(x + 3)^3$ and $g(x) = 64(x^2 - 1)(x + 3)(x + 2)^2$.

   Writing $f(x)$ and $g(x)$ as the product of the powers of irreducible factors,

   $f(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times (x - 1) \times (x + 1)^2 \times (x + 3)^3$

   $g(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times (x - 1)(x + 1)(x + 3)(x + 2)^2$

   The common factors with the least exponents are 2^5, $(x - 1)$, $(x + 1)$, $(x + 3)$.

   $\therefore$ The HCF of the given polynomials $= 32 \times (x - 1)(x + 1)(x + 3)$

   $= 32(x^2 - 1)(x + 3)$.

LCM of the Given Polynomials

Least Common Multiple or the Lowest Common Multiple is the product of all the factors (taken once) of the polynomials given with their highest exponents respectively.

Method for Finding LCM of the Given Polynomials

Step 1: First express each polynomial as a product of powers of irreducible factors.

Step 2: Consider all the irreducible factors (only once) occurring in the given polynomials. For each of these factors, consider the greatest exponent in the factorized form of the given polynomials.

Step 3: Now raise each irreducible factor to the greatest exponent and multiply them to get the LCM.

Examples:

1. Find the LCM of $42a^2b^2$ and $48ab^3$.

Let $p(x) = 42a^2b^2$ and $q(x) = 48ab^3$

Writing $p(x)$ and $q(x)$ as the product of the powers of irreducible factors,

$p(x) = 2 \times 3 \times 7 \times a^2 \times b^2$

$q(x) = 2 \times 2 \times 2 \times 2 \times 3 \times a \times b^3$

Now all the factors (taking only once) with the highest exponents are 2^4, 3, 7, a^2, b^3

$\Rightarrow$ The LCM of the given polynomials $= 2^4 \times 3 \times 7 \times a^2 \times b^3 = 336a^2b^3$.

2. Find the LCM of $96(x-1)(x+1)^2(x+3)^3$ and $64(x^2-1)(x+3)(x+2)^2$

Let $f(x) = 96(x-1)(x+1)^2(x+3)^3$ and $g(x) = 64(x^2-1)(x+3)(x+2)^2$

$f(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 3\ (x-1)(x+1)^2\ (x+3)^3$

$g(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 2\ (x-1)(x+1)(x+3)(x+2)^2$

Now all the factors (taking only once) with the highest exponents are 2^6, 3, $(x-1)$, $(x+1)^2$, $(x+3)^3$, $(x+2)^2$.

$\Rightarrow$ The LCM of the given polynomials $= 192(x-1)(x+1)^2(x+2)^2(x+3)^3$.

Relation among the HCF, the LCM and the Product of the Polynomials

If $f(x)$ and $g(x)$ are two polynomials then we have the following relation,
(HCF of $f(x)$ and $g(x)$) $\times$ (LCM of $f(x)$ and $g(x)$) $= \pm\ (f(x) \times g(x))$.

Example:

Let $f(x) = (x+3)^2(x-1)(x+2)^3$ and $g(x) = (x+3)(x-1)^2(x+2)^2$ be two polynomials.

The common factors with the least exponents are $(x+3)$, $(x-1)$, $(x+2)^2$.

$\Rightarrow$ HCF $= (x-1)(x+3)(x+2)^2$

All the factors (taken only once) with the highest exponents are $(x+3)^2$, $(x-1)^2$, $(x+2)^3$.

$\Rightarrow$ LCM $= (x-1)^2(x+2)^3(x+3)^2$

Now, $f(x) \times g(x) = \{(x+3)^2(x-1)(x+2)^3\} \times \{(x+3)(x-1)^2(x+2)^2\} = (x-1)^3(x+2)^5\,(x+3)^3$

LCM $\times$ HCF $= \{(x-1)(x+3)(x+2)^2\} \times \{(x-1)^2(x+2)^3(x+3)^2\} = (x-1)^3(x+3)^3\,(x+2)^5$

Thus, we say, LCM $\times$ HCF $=$ Product of two polynomials.

EXAMPLE 2.1

If $f(x) = (x - 2)(x^2 - x - a)$, $g(x) = (x + 2)(x^2 + x - b)$ and their HCF is $x^2 - 4$, then find the value of $(a - b)$(a and b are constants) from the following options.

(a) 2 **(b)** 3 **(c)** -4 **(d)** 0

SOLUTION

$f(x) = (x - 2)(x^2 - x - a)$

$g(x) = (x + 2)(x^2 + x - b)$

$\text{HCF} = x^2 - 4 = (x + 2)(x - 2)$

$\therefore\ x + 2$ is a factor of $(x^2 - x - a)$

$\therefore\ (-2)^2 - (-2) - a = 0 \Rightarrow a = 6$

$x - 2$ is a factor of $x^2 + x - b$

$\therefore\ 2^2 + 2 - 3 = 0$

$b = 6\ \therefore\ a - b = 0.$

EXAMPLE 2.2

If the LCM of the polynomials $f(x) = (x + 1)^5\,(x + 2)^a$ and $g(x) = (x + 1)^b\,(x + 2)^a$ is $(x + 1)^a$ $(x + 2)^b$, then find the minimum value of $a + b$ from the following options.

(a) is 10 **(b)** is 14 **(c)** is 15 **(d)** Cannot say

SOLUTION

$f(x) = (x + 1)^5\,(x + 2)^a$

$g(x) = (x + 1)^b\,(x + 2)^5$

$\text{LCM} = (x + 1)^a\,(x + 2)^b$

As $(x + 1)^a$ is a multiple of both $(x + 1)^5$ and $(x + 1)^b$, $a \geq 5$ and $a \geq b$ (1)

As $(x + 2)^b$ is a multiple of both $(x + 2)^a$ and $(x + 2)^5$, $b \geq 5$ and $b \geq a$ (2)

From (1) and (2),

$a = b$, $a \geq 5$ and $b \geq 5$

$\therefore$ Minimum value of $a + b$ is $5 + 5$, i.e., 10.

Rational Expressions

We know that any number of the form $\dfrac{p}{q}$, where $p, q \in Z$ and $q \neq 0$, is called a rational number.

As integers and polynomials behave in the same manner, we observe most of the properties satisfied by rational numbers are also satisfied by the algebraic expressions of the form $\dfrac{f(x)}{g(x)}$ which are called rational expressions. Rational expression is an algebraic expression which is of the form $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and $g(x)$ is not a zero polynomial.

For any rational number of the form $\frac{p}{q}$, where $p, q \in Z$ and $q \neq 0$, p and q are called numerator and denominator respectively. Even though p and q are integers $\frac{p}{q}$ need not be an integer. Similarly for any rational expression $\frac{f(x)}{g(x)}$, $f(x)$ is called numerator and $g(x)$ is called denominator.

Even though $f(x)$ and $g(x)$ are polynomials, $\frac{f(x)}{g(x)}$ need not be a polynomial.

Examples:

1. $\frac{2x-1}{x^2-3x+1}$ is a rational expression.

2. $\frac{x^3+5x^2-\sqrt{3}x+\sqrt{5}}{2x^2-\sqrt{5}x+8}$ is a rational expression.

3. $\frac{x^2-5\sqrt{x}-1}{3x-5}$ is not a rational expression as the numerator is not a polynomial.

Notes

1. Every polynomial is a rational expression as $f(x)$ can be written as $\frac{f(x)}{1}$.

2. $\frac{f(x)}{g(x)}$ is not a rational expression if either numerator $f(x)$ or denominator $g(x)$ or both $f(x)$ and $g(x)$ are not polynomials.

EXAMPLE 2.3

Which of the following algebraic expressions is/are not polynomials?

(A) $x^3+2x^3+\sqrt{7}x+4$

(B) $5x^2+4\sqrt{x}-11$

(C) $\frac{x^3+3x^2-8x+11}{4x\sqrt{x}-3x+3}$

(D) $\frac{x^3+3x^2-6x+13}{x^2+1}$

(a) A, B and C **(b)** A and C **(c)** B and C **(d)** A and D

HINT

Recall definition of polynomials.

Rational Expressions in Lowest Terms

Let $f(x)$ and $g(x)$ have integer coefficients and HCF of $f(x)$ and $g(x)$ is 1, then the rational expression $\frac{f(x)}{g(x)}$ is said to be in its lowest terms.

If HCF of $f(x)$ and $g(x)$ is not equal to 1, then by cancelling HCF of $f(x)$ and $g(x)$ from both numerator and denominator, we can reduce the given rational expression in its lowest terms.

EXAMPLE 2.4

Verify whether the rational expression $\dfrac{x^2-1}{(2x+1)(x+2)}$ is in its lowest terms or not.

SOLUTION

The given rational expression can be written as $\dfrac{(x+1)(x-1)}{(2x+1)(x+2)}$ and clearly HCF of numerator and denominator of the given expression is 1.

$\therefore$ It is in its lowest terms.

EXAMPLE 2.5

Express the rational expression $\dfrac{x^2-2x-3}{2x^2-3x-5}$ in its lowest terms.

SOLUTION

Factorizing both the numerator and the denominator of the given expression, we have $\dfrac{x^2-2x-3}{2x^2-3x-5}=\dfrac{(x+1)(x-3)}{(2x-5)(x+1)}$

HCF of numerator and denominator is $x+1$.

Dividing both numerator and denominator of the given expression by $(x+1)$, we have $\dfrac{x-3}{2x-5}$ which is in its lowest terms.

Addition/Subtraction of Rational Expressions

The sum of any two rational expressions $\dfrac{f(x)}{g(x)}$ and $\dfrac{h(x)}{p(x)}$ is written as

$$\frac{f(x)}{g(x)}+\frac{h(x)}{p(x)}=\frac{f(x)\,p(x)+h(x)\,g(x)}{g(x)\,p(x)}$$

If the denominators $g(x)$ and $p(x)$ are equal then $\dfrac{f(x)}{g(x)}+\dfrac{h(x)}{p(x)}=\dfrac{f(x)+h(x)}{g(x)}$.

The difference of the above rational expressions can be written as

$$\frac{f(x)}{g(x)}-\frac{h(x)}{p(x)}=\frac{f(x)\,p(x)-h(x)\,g(x)}{g(x)\,p(x)}.$$

Notes

1. Sum or difference of two rational expressions is also a rational expression.
2. For any rational expression $\dfrac{f(x)}{g(x)}+\dfrac{-f(x)}{g(x)}$ is called the additive inverse of $\dfrac{f(x)}{g(x)}$.

   That is, $\dfrac{f(x)}{g(x)}+\left(\dfrac{-f(x)}{g(x)}\right)=0$.

EXAMPLE 2.6

Simplify $\dfrac{x+1}{2x-1}+\dfrac{3x-2}{x-1}$.

SOLUTION

$$\frac{x+1}{2x-1}+\frac{3x-2}{x-1}=\frac{(x+1)(x-1)+(3x-2)(2x-1)}{(2x-1)(x-1)}=\frac{x^2-1+6x^2-7x+2}{2x^2-3x+1}=\frac{7x^2-7x+1}{2x^2-3x+1}.$$

EXAMPLE 2.7

Find the sum of the rational expressions $\dfrac{2x-3}{x^2+x-2}$ and $\dfrac{3x-1}{2x^2+5x+2}$.

SOLUTION

$$\begin{aligned}\frac{2x-3}{x^2+x-2}+\frac{3x-1}{2x^2+5x+2}&=\frac{2x-3}{(x+2)(x-1)}+\frac{3x-1}{(2x+1)(x+2)}\\&=\frac{(2x-3)(2x+1)+(3x-1)(x-1)}{(x-1)(x+2)(2x+1)}\\&=\frac{4x^2-4x-3+3x^2-4x+1}{(x-1)(x+2)(2x+1)}=\frac{7x^2-8x-2}{2x^2+3x^2-4x-2}.\end{aligned}$$

EXAMPLE 2.8

If $A=\dfrac{x+1}{2x-1}$, $B=\dfrac{2x+1}{3x+2}$ and $C=\dfrac{4x-5}{2x^2+5x-3}$, then find $4A-3B+C$.

SOLUTION

$$4A-3B+C=\frac{4(x+1)}{2x-1}-\frac{3(2x-1)}{3x+2}+\frac{4x-5}{(2x-1)(x+3)}\qquad (\because 2x^2+5x-3=(2x-1)(x+3))$$

$$\begin{aligned}&=\frac{4(x+1)(3x+2)(x+3)-3(2x-1)^2(x+3)+(4x-5)(3x+2)}{(2x-1)(3x+2)(x+3)}\\&=\frac{4(3x^3+14x^2+17x+6)-3(4x^3+8x^2-11x+3)+12x^2-7x-10}{(2x-1)(3x+2)(x+3)}\\&=\frac{44x^2+94x+5}{6x^2+19x^2+x-6}.\end{aligned}$$

EXAMPLE 2.9

Which one should be added among the following options to $\dfrac{1}{x-2}+\dfrac{1}{x+2}$ to get $\dfrac{4x^3}{x^4-16}$?

(a) $\dfrac{2x^2}{x^2+4}$

(b) $\dfrac{2x}{x^2+4}$

(c) $\dfrac{2x^2}{x^2-4}$

(d) $\dfrac{2}{x^2+4}$

SOLUTION

Let A be the required expression

$$\frac{1}{x-2}+\frac{1}{x+2}+A=\frac{4x^3}{x^4-16}$$

$$\therefore\ A=\frac{4x^3}{x^4-16}-\left(\frac{1}{x-2}+\frac{1}{x+2}\right)$$

$$=\frac{4x^3}{x^4-16}-\frac{2x}{x^2-4}$$

$$=\frac{2x}{x^2-4}\left[\frac{2x^2}{x^2+4}-1\right]$$

$$=\frac{2x}{x^2-4}\left[\frac{2x^2-x^2-4}{x^2+4}\right]=\frac{2x}{x^2+4}.$$

Multiplication of Rational Expressions

The product of two rational expressions $\frac{f(x)}{g(x)}$ and $\frac{h(x)}{p(x)}$ is given by

$$\frac{f(x)}{g(x)}\times\frac{h(x)}{p(x)}=\frac{f(x)\cdot h(x)}{g(x)\cdot p(x)}.$$

Notes

1. The process of finding the
   (i) product of two rational expressions is similar to the process of finding the product of two rational numbers.
   (ii) product of two rational expressions is also a rational expression.
2. After finding the product of two rational expressions the resultant rational expression must be put in its lowest terms.

EXAMPLE 2.10

Find the product of the rational expressions $\frac{3x^2+8x-3}{2x^2-x-6}$ and $\frac{x^2-4}{x+3}$.

SOLUTION

Product of the given expressions is

$$\frac{3x^2+8x-3}{2x^2-x-6}\times\frac{x^2-4}{x+3}$$

$$=\frac{(x+3)(3x-1)}{(2x+3)(x-2)}\times\frac{(x-2)(x+2)}{(x+3)}$$

$$=\frac{(3x-1)(x+2)}{2x+3}=\frac{3x^2+5x-2}{2x+3}.$$

EXAMPLE 2.11

Simplify $\left[\dfrac{2x-1}{x+3}-\dfrac{x^2-4}{2x+1}\right]\times\dfrac{2x^2+7x+3}{x+2}$.

SOLUTION

$$\left[\frac{2x-1}{x+3}-\frac{x^2-4}{2x+1}\right]\times\frac{2x^2+7x+3}{x+2}$$

$$=\frac{(2x-1)(2x+1)-(x^2-4)(x+3)}{(x+3)(2x+1)}\times\frac{(x+3)(2x+1)}{x+2}$$

$$=\frac{4x^2-1-x^2-3x^2+4x+12}{x+2}=\frac{-(x^3-x^2-4x-11)}{x+2}.$$

Note For every rational expression of the form $\dfrac{f(x)}{g(x)}$, $(g(x)\neq 0)$ there exists a rational expression of the form $\dfrac{g(x)}{f(x)}$ such that $\dfrac{f(x)}{g(x)}\times\dfrac{g(x)}{f(x)}=1$, then $\dfrac{g(x)}{f(x)}$ is called the multiplicative inverse of $\dfrac{f(x)}{g(x)}$ and vice versa.

Division of Rational Expressions

Let $\dfrac{f(x)}{g(x)}$ and $\dfrac{h(x)}{p(x)}$ be two non-zero rational expressions, then $\dfrac{f(x)}{g(x)}\div\dfrac{h(x)}{p(x)}=\dfrac{f(x)}{g(x)}\times\dfrac{p(x)}{h(x)}$. That is, $\dfrac{f(x)p(x)}{g(x)h(x)}$ which is also a rational expression.

Note The process of dividing two rational expressions is similar to the process of dividing two rational numbers.

EXAMPLE 2.12

Express $\dfrac{2x^2+6x}{3x^2+7x+2}\div\dfrac{x^2+x-6}{3x^2+7x+2}$ as a rational expression in its lowest terms.

SOLUTION

Given, $\dfrac{2x^2+6x}{3x^2+7x+2}\div\dfrac{x^2+x-6}{3x^2+7x+2}$

$$\frac{2x(x+3)}{(x+2)(3x+1)}\div\frac{(x+3)(x-2)}{(3x+1)(x+2)}$$

$$\frac{2x(x+3)}{(x+2)(3x+1)}\times\frac{(3x+1)(x+2)}{(x+3)(x-2)}=\frac{2x}{x-2}.$$

EXAMPLE 2.13

If $A = 2x^3 + 5x^2 + 4x + 1$ and $B = 2x^2 + 3x + 1$, then find the quotient from the following four option, when A is divided by B.

(a) $x - 1$ **(b)** $x + 1$ **(c)** $2x + 1$ **(d)** $2x - 1$

SOLUTION

$$\frac{A}{B} = \frac{2x^3 + 5x^2 + 4x + 1}{2x^2 + 3x + 1}$$

$$= \frac{2x^3 + (3x^2 + 2x^2) + (x + 3x) + 1}{2x^2 + 3x + 1}$$

$$= \frac{(2x^3 + 3x^2 + x) + (2x^2 + 3x + 1)}{2x^2 + 3x + 1}$$

$$= \frac{(2x^2 + 3x + 1)(x + 1)}{2x^2 + 3x + 1} = x + 1.$$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. The LCM of $18x^2y3$ and $8x3y^2$ is ______.
2. The HCF of $24a^2b$ and $32ab^2$ is ______.
3. The HCF of $9a^2 - 16b^2$ and $12a^2 - 16ab$ is ______.
4. The LCM of $2(x - 3)^2$ and $3(x - 2)^2$ is ______.
5. The HCF of $24x^5$ and $36x^6y^k$ is $12x^5$, then the value of k is ______.
6. The HCF of $a^m + b^m$ and $a^n - b^n$, where m is odd positive integer and n is even positive integer is ______.
7. The LCM of $12x^3y^2$ and $18x^py^3$ is $36x^4y^3$. Then the number of integer values of p is ______.
8. The HCF of $(x + 2)(x^2 - 7x + k)$ and $(x - 3)(x^2 + 3x + l)$ is $(x + 2)(x - 3)$. Then the values of k and l are ______ respectively.
9. The LCM of $x^2 + 2x - 8$ and $x^2 + 3x - 4$ is ______.
10. The rational expression $\frac{x^2+1}{x^2+4x+3}$ is not in its lowest terms. (True/False)
11. The additive inverse of $\frac{x^2+1}{x^2-1}$ is ______.
12. $\frac{x^2}{x^2-1} \div \frac{x^3}{x^2+1} =$ ______.
13. The product of two rational expressions is not always a rational expression. (True/False)
14. The rational expression $\frac{(x+1)^2}{x^2-1}$ in its lowest terms is ______.
15. The reciprocal of $x - \frac{1}{x}$ is $\frac{1}{x} - x$. (True/False)
16. If the rational expression $\frac{x-a}{x^2-b}$ is in its lowest terms, then ______. $[a \neq b/a^2 \neq b]$
17. Every polynomial is a rational expression. (True/False)
18. $\frac{x^3+\sqrt{6}x^2-7}{2x^3+\sqrt{x}+1}$ is a rational expression. (True/False)
19. The HCF of the polynomials $8a^3b^2c$, $16a^2bc^3$ and $20ab^3c^2$ is ______.
20. The LCM of the polynomials $8(x^3 + 8)$ and $12(x^2 - 4)$ is ______.
21. The HCF of the polynomials $12xy^2z^3$, $18x^2y^2z$ and $28x^3yz^2$ is ______.
22. The LCM of the polynomials $15a^2b(a^2 - b^2)$ and $40ab^2(a - b)$ is ______.
23. The HCF of the polynomials $15(a + 1)^2 (a - 2)$, $65(a - 1)^2 (a + 1)$ and $90 (a - 2)^2 (a - 1)$ is ______.
24. The HCF of the polynomials $(x + 3)^2 (x - 2) (x + 1)^2$ and $(x + 1)^3 (x + 3)(x + 4)$ is ______.
25. The LCM of the polynomials $x^2 - 1$, $x^2 + 1$ and $x^4 - 1$ is ______.
26. The multiplicative inverse of $x - \frac{x-1}{1-x}$ is ______.
27. Which of the following algebraic expressions are polynomials?
    (a) $x^3 - \sqrt{7}x + 13$
    (b) $3x^4 + 2x^3 - \sqrt{5x} + 8$
    (c) $11 - x^2 + x\sqrt{x} + 7x$
    (d) $-\sqrt{3}x^3 + 8x^2 - 9$
28. The additive inverse of $\frac{x+1}{x^2-1}$ is ______.
29. If $A = \frac{p(x)}{q(x)}$ and $B = \frac{f(x)}{g(x)}$ are two rational expressions where $q(x)$, $g(x) \neq 0$, then $A + B$ and $A - B$ are also rational expressions. (True/False)
30. The rational expression whose numerator is a linear polynomial with -3 as zero and whose denominator is a quadratic polynomial with zeroes $\frac{1}{2}$ and -1 is ______.

Short Answer Type Questions

31. Find the HCF and the LCM of the following monomials.

(i) x^3y^6 and x^2y^8
(ii) $3a^2b^3c^4$ and $9a^4b^3c^2$
(iii) $p^4q^2r^3$ and $q^3p^6r^5$

32. Find the LCM and HCF of the polynomials $(x^2 - 4)(x^2 - x - 2)$ and $(x^2 + 4x + 4)(x^2 - 3x + 2)$. Verify that the product of the LCM and HCF is equal to the product of the polynomials.

33. If HCF of the two polynomials $(x - 1)(x - 6)(x - 2)$ and $(x - 6)^3(x - 4)(x - 2)^2$ is $(x - 6)(x - 2)$, then find their LCM.

34. If LCM of the two polynomials $(x^2 + 3x)(x^2 + 3x + 2)$ and $(x^2 + 6x + 8)(x^2 + kx + 6)$ is $x(x + 1)(x + 2)^2 (x + 3)(x + 4)$, then find k.

35. If $(x + 2)(x + 5)$ is the HCF of the polynomials $(x + 2)(x^2 + 6x + a)$ and $(x + 5)(x^2 + 8x + b)$, then find the values of a and b.

36. If the LCM and HCF of the two polynomials are $(x - 1)^2(x - 2)^2(x - 3)^3$ and $(x - 1)(x - 2)(x - 3)$ respectively and one of the polynomials is $(x - 1)(x - 2)^2 (x - 3)$, then find the other polynomial.

37. Find the LCM and HCF of the polynomials $(8 - x^3)$ and $(x^2 - 4)(x + 3)$. Verify that the product of the LCM and HCF is equal to the product of the polynomials.

38. The HCF and LCM of the two polynomials are $(x - 3)(x + 1)$ and $(x^2 - 9)(x^2 - 1)$ respectively. If one of the polynomial is $(x - 3)(x^2 - 1)$, then find the second polynomial.

39. Find the HCF and LCM of $(x - 1)(x + 2)(x - 3)$, $(x + 2)^2 (x + 1)$ and $(x + 3)(x^2 - 4)$.

40. Find the sum of the following rational expressions.

$\dfrac{x+1}{(x-1)^2}$ and $\dfrac{x-2}{x^2-1}$.

41. If $P = \dfrac{3x+1}{3x-1}$ and $Q = \dfrac{3x-1}{3x+1}$, then find $P - Q$ and $P + Q$.

42. Reduce the following rational expressions to their lowest terms.

(i) $\dfrac{16x^2 - (x^2 - 9)^2}{4x + 9 - x^2}$

(ii) $\dfrac{8x^5 - 8x}{(3x^2 + 3)(4x + 4)}$

43. If $X = \dfrac{1+a}{1-a}$ and $Y = \dfrac{1-a}{1+a}$, then find $X^2 + Y^2 + XY$.

44. What should be added to $\dfrac{a}{a-b} + \dfrac{b}{a+b}$ to get 1?

45. Simplify:

$$\frac{x+1}{x-1} - \frac{x-1}{x+1} + \frac{8x}{1+x^2} - \frac{12x^3}{x^4-1}.$$

Essay Type Questions

46. Simplify: $\left[\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2} - \dfrac{8x}{x^2-16}\right] \div \dfrac{8x}{x^4-16}$

47. Simplify the rational expression:

$$\frac{2x}{1+x^2+x^4} + \frac{1}{1+x+x^2} - \frac{1}{1-x+x^2}.$$

48. Find the product of the following rational expressions and express the result in lowest terms:

$\dfrac{8x^2 + 10x - 3}{12x^2 + x - 3}$ and $\dfrac{6x^2 - 7x - 3}{8x^2 - 10x - 3}$

49. Simplify the rational expression:

$$\frac{1}{x+p} + \frac{1}{x+q} + \frac{1}{x+r} + \frac{px}{x^3 + px^2} + \frac{qx}{x^3 + qx^2} + \frac{rx}{x^3 + rx^2}.$$

50. Simplify:

$$\frac{8x^3 - y^3 + z^3 + 6xyz}{a^3 - 8b^3 + 27c^3 + 18abc} \div \frac{4x^2 + y^2 + z^2 + 2xy + yz - 2xz}{a^2 + 4b^2 + 9c^2 + 2ab + 6bc - 3ac}.$$

PRACTICE QUESTIONS

CONCEPT APPLICATION

Level 1

1. The LCM of the polynomials $18(x^4 - x^3 + x^2)$ and $24(x^6 + x^3)$ is ________.
   (a) $72x^2(x + 1)(x^2 - x + 1)^2$
   (b) $72x^3(x^2 - 1)(x^3 - 1)$
   (c) $72x^3(x^3 - 1)$
   (d) $72x^3(x^3 + 1)$

2. The LCM of the polynomials $f(x) = 9(x^3 + x^2 + x)$ and $g(x) = 3(x^3 + 1)$ is ________.
   (a) $27x(x + 1)(x^2 + x + 1)$
   (b) $9x(x + 1)(x^2 + x + 1)(x^2 - x + 1)$
   (c) $9(x + 1)(x^2 - x + 1)(x^2 + x + 1)$
   (d) $9x(x + 1)(x^2 + x + 1)$

3. The LCM of polynomials $14(x^2 - 1)(x^2 + 1)$ and $18(x^4 - 1)(x + 1)$ is ________.
   (a) $126(x + 1)(x^2 + 1)(x - 1)$
   (b) $126(x + 1)(x^2 + 1)(x^2 - 1)$
   (c) $126(x + 1)^2 (x^2 + 1)(x - 1)^2$
   (d) $126(x + 1)(x^2 + 1)(x - 1)^2$

4. If HCF and LCM of two polynomials $P(x)$ and $Q(x)$ are $x(x + p)$ and $12x^2 (x - p)(x^2 - p^2)$ respectively. If $P(x) = 4x^2 (x + p)$, then $Q(x) =$ ________.
   (a) $3x(x - p)^2 (x + p)$
   (b) $3x(x - p)(x + p)$
   (c) $3x(x + p)(x^2 - p^2)$
   (d) $3x(x + p)(x^2 + p^2)$

5. The product of HCF and LCM of two polynomials is $(x^2 - 1)(x^4 - 1)$, then the product of the polynomials is ________.
   (a) $(x^2 - 1)(x^2 + 1)$
   (b) $(x^2 - 1)(x^2 + 1)^2$
   (c) $(x^2 - 1)^2 (x^2 + 1)$
   (d) None of these

6. If $(x + 4)(x - 2)(x + 1)$ is the HCF of the polynomials $f(x) = (x^2 + 2x - 8)(x^2 + 4x + a)$ and $g(x) = (x^2 - x - 2)(x^2 + 3x - b)$, then $(a, b) =$ ________.
   (a) $(3, -4)$ (b) $(-3, -4)$
   (c) $(-3, 4)$ (d) $(3, 4)$

7. Find the LCM of $x^3 - x^2 + x - 1$ and $x^3 - 2x^2 + x - 2$.
   (a) $(x + 1)(x - 1)$
   (b) $x - 1$
   (c) $(x^2 + 1)(x - 1)(x - 2)$
   (d) $(x^2 + 1)(x + 1)(x - 2)$

8. The HCF of the polynomials $p(x)$ and $q(x)$ is $6x - 9$, then $p(x)$ and $q(x)$ could be ________.
   (a) $3, 2x - 3$
   (b) $12x - 18, 2$
   (c) $3(2x - 3)^2, 6(2x - 3)$
   (d) $3(2x - 3), 6(2x + 3)$

9. If the HCF of the polynomials $f(x)$ and $g(x)$ is $4x - 6$, then $f(x)$ and $g(x)$ could be ________.
   (a) $2, 2x - 3$
   (b) $8x - 12, 2$
   (c) $2(2x - 3)^2, 4(2x - 3)$
   (d) $2(2x + 3), 4 (2x + 3)$

10. If the HCF of the polynomials $f(x) = (x + 3)(3x^2 - 7x - a)$ and $g(x) = (x - 3)(2x^2 + 3x + b)$ is $(x + 3)(x - 3)$, then $a + b =$ ________.
    (a) 3 (b) -15
    (c) -3 (d) 15

11. Find the HCF of the polynomials $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3 - x^2 + x - 1$.
    (a) $x(x + 1)$ (b) $x - 1$
    (c) $x^2 + 1$ (d) $x + 1$

12. If the HCF of $x^3 + 2x^2 - ax$ and $2x^3 + 5x^2 - 3x$ is $x(x + 3)$, then $a =$ ________.
    (a) 3 (b) -3
    (c) 6 (d) -4

13. The HCF of the polynomials $70(x^3 - 1)$ and $105(x^2 - 1)$ is ________.
    (a) $15(x - 1)$
    (b) $35(x - 1)$
    (c) $35(x^2 - 1)(x^2 + x + 1)$
    (d) $15(x^2 - 1)$

14. What should be subtracted from $\frac{7x}{(x^2-x-12)}$ to get $\frac{3}{x+3}$?

(a) $\frac{5}{x+4}$ (b) $\frac{4}{x-4}$

(c) $\frac{2}{x-4}$ (d) $\frac{1}{x-4}$

15. Which of the following is/are true?

(A) The sum of two rational expressions is always a rational expression.

(B) The difference of two rational expressions is always a rational expression.

(C) $\frac{p(x)}{q(x)}$ is in its lowest terms if LCM $[p(x), q(x)] = 1$.

(D) Reciprocal of $\frac{-2x}{x^2-1}$ is $\frac{x^2-1}{2x}$.

(a) A, B (b) A, B, D

(c) A, C (d) A, B, C

16. The product of additive inverses of $\frac{x^2-1}{2x}$ and $\frac{x^2-4}{3-x}$ is ______.

(a) x^2+5x+6 (b) x^2+x-6

(c) x^2-x-6 (d) x^2-5x+6

17. What should be added to $\frac{1}{x^2-7x+12}$ to get $\frac{2}{x^2-6x+18}$?

(a) $\frac{2}{(x+3)(x-2)}$ (b) $\frac{4}{(x+3)(x+2)}$

(c) $\frac{1}{x^2-5x+6}$ (d) $\frac{-1}{x^2+5x-6}$

18. The rational expression $\frac{x^3-3x^2+2x}{x^2y-2xy}$ in lowest terms is ______.

(a) $\frac{x-2}{y}$ (b) $\frac{x+1}{xy}$

(c) $\frac{x-2}{y}$ (d) $\frac{x+1}{y}$

19. If $(x-4)$ is the HCF of $p(x) = x^2 - nx - 12$ and $q(x) = x^2 - mx - 8$, then the simplest form of $\frac{p(x)}{q(x)}$ is ______.

(a) $\frac{x-3}{x+2}$ (b) $\frac{x+3}{x-2}$

(c) $\frac{x+2}{x+3}$ (d) $\frac{x+3}{x+2}$

20. What should be added to $\frac{2}{(x^2+x-6)}$ to get $\frac{-4x}{(x^2-4)(x^2-9)}$?

(a) $\frac{4}{x^2-x-6}$ (b) $\frac{-2}{x^2-x-6}$

(c) $\frac{4x}{x^2+x+6}$ (d) $\frac{-3}{x^2+x+6}$

21. The HCF of the polynomials $4(x+8)^2(x^2-5x+6)$ and $6(x^2+12x+32)(x^2-7x+12)$ is ______.

(a) $2(x+8)(x-3)$ (b) $(x-3)(x+8)$

(c) $(x-8)(x+3)$ (d) $2(x-8)(x+3)$

22. The LCM of the polynomials $8(x^2-2x)(x-6)^2$ and $2x(x^2-4)(x-6)^2$ is ______.

(a) $8(x^2-4)(x^2-6x)^2$

(b) $(x^3-4x)(x-6)^2$

(c) $8(x^3-4x)(x-6)^2$

(d) $8x(x^2+4)(x-6)^2$

23. The HCF of polynomials $(x^2-2x+1)(x+4)$ and $(x^2+3x-4)(x+1)$ is ______.

(a) $(x+4)(x-1)$

(b) $(x+1)(x+4)$

(c) $(x+1)(x-4)$

(d) $(x^2-1)(x+4)$

24. If the zeroes of the rational expression $(3x+2a)(2x+1)$ are $\frac{-1}{2}$ and $\frac{b}{3}$, then the value of a is ______.

(a) $-2b$ (b) $\frac{-b}{2}$

(c) $\frac{-b}{3}$ (d) None of these

25. The LCM of the polynomials $(x^2-8x+16)(x^2-25)$ and $(x^2-10x+25)(x^2-2x-24)$ is ______.

(a) $(x^4-41x+400)(x-6)$

(b) $(x^4+41x+400)(x^2-9x+20)$

(c) $(x^4-41x+400)(x^2-9x+20)(x-6)$

(d) $(x^4-41x+400)(x^2-9x+20)(x+6)$

PRACTICE QUESTIONS

26. If the HCF of the polynomials $(x^2 + 8x + 16)(x^2 - 9)$ and $(x^2 + 7x + 12)$ is $x^2 + 7x + 12$, then their LCM is ______.
   (a) $(x + 4)(x^2 - 9)$
   (b) $(x + 4)^2 (x^2 - 9)$
   (c) $(x^2 - 4)(x^2 + 9)$
   (d) $(x + 4)^2 (x + 3)$

27. If $h(x) = x^2 + x$ and $g(y) = y^3 - y$, then the HCF of $h(b) - h(a)$ and $g(b) - g(a)$ is ______.
   (a) $a + b$ (b) $b - a$
   (c) $b^2 + a^2$ (d) $b^2 + ab + a^2$

28. If $h(y) = y^3$ and $g(z) = z^4$, then HCF of $h(b) - h(a)$ and $g(b) - g(a)$ is ______.
   (a) $b - a$
   (b) $b^2 - a^2$
   (c) $b^3 - a^3$
   (d) $b^2 + ab + a^2$

29. If the HCF of the polynomials $(x + 4)(2x^2 + 5x + a)$ and $(x + 3)(x^2 + 7x + b)$ is $(x^2 + 7x + 12)$ then $6a + b$ is ______.
   (a) -6 (b) 5
   (c) 6 (d) -5

Level 2

30. The HCF and LCM of the polynomials $p(x)$ and $q(x)$ are $5(x - 2)(x + 9)$ and $10(x^2 + 16x + 63)(x - 2)^2$. If $p(x)$ is $10(x + 9)(x^2 + 5x - 14)$, then $q(x)$ is ______.
   (a) $5(x + 9)(x - 2)$
   (b) $10(x - 2)^2 (x + 7)$
   (c) $10(x + 9)(x - 2)$
   (d) $5(x - 2)^2 (x + 9)$

31. If the zeroes of the rational expression $(ax + b)(3x + 2)$ are $\frac{-2}{3}$ and $\frac{1}{2}$, then $a + b =$ ______.
   (a) -1 (b) 0
   (c) $-b$ (d) $-a$

32. Simplify:
$$\frac{x^2-(y-2z)^2}{x-y+2z}+\frac{y^2-(2x-z)^2}{y+2x-z}+\frac{z^2-(x-2y)^2}{z-x+2y}.$$
   (a) 0 (b) 1
   (c) $x + y + z$ (d) None of these

33. If the HCF of the polynomials $x^2 + px + q$ and $x^2 + ax + b$ is $x + l$, then their LCM is ______.
   (a) $(x + a - l)(x + l - p)$
   (b) $(x - (l + a))(x + l - p)(x + l)$
   (c) $(x + a - l)(x + p - l)(x + l)$
   (d) $(x - l + a)(x - p + l)(x + l)$

34. The expression $\frac{1}{1-x}-\frac{1}{1+x}-\frac{x^3}{1-x}+\frac{x^2}{1+x}$ in lowest terms is ______.
   (a) $2x^3 + 1$ (b) $x^2 + 2$
   (c) $x^2 + 2x$ (d) $x^2 - 2x$

35. Simplify:
$$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}.$$
   (a) 0 (b) 1
   (c) $a + b + c$ (d) $\frac{1}{a+b+c}$

36. Simplify: $\frac{x+1}{x-1}+\frac{x-1}{x+1}-\frac{2x^2-2}{x^2+1}$.
   (a) $\frac{4x^4+2}{x^4-1}$ (b) $\frac{4x^2}{x^4-1}$
   (c) $\frac{8x^2}{x^4-1}$ (d) 1

37. If the LCM of the polynomials $x^9 + x^6 + x^3 + 1$ and $x^6 - 1$ is $x^{12} - 1$, then their HCF is ______.
   (a) $x^3 + 1$ (b) $x^6 + 1$
   (c) $x^3 - 1$ (d) $x^6 - 1$

38. If $x^2 + x - 1$ is a factor of $x^4 + px^3 + qx^2 - 1$, then the values of p and q can be
   (a) 2, 1 (b) 1, -2
   (c) $-1, -2$ (d) $-2, -1$

PRACTICE QUESTIONS

39. The HCF of two polynomials $p(x)$ and $q(x)$ using long division method was found to be $x + 5$, If their first three quotients obtained are x, $2x + 5$, and $x + 3$ respectively. Find $p(x)$ and $q(x)$. (The degree of $p(x)$ > the degree of $q(x)$)
   (a) $p(x) = 2x^4 + 21x^3 + 72x^2 + 88x + 15$
   $q(x) = 2x^3 + 21x^2 + 71x + 80$
   (b) $p(x) = 2x^4 - 21x^3 - 72x^2 - 88x + 15$
   $q(x) = 2x^3 + 21x^2 - 71x + 80$
   (c) $p(x) = 2x^4 + 21x^3 + 88x + 15$
   $q(x) = 2x^3 + 71x + 80$
   (d) $p(x) = 2x^4 - 21x^2 - 72x^2 + 80x + 15$
   $q(x) = 2x^3 - 21x^2 + 71x + 80$

40. If the HCF of the polynomials $x^3 + px + q$ and $x^3 + rx^2 + lx + x$ is $x^2 + ax + b$, then their LCM is ______. $(r \neq 0)$
   (a) $(x^2 + ax + b)(x + a)(x + a - r)$
   (b) $(x^2 + ax + b)(x - a)(x - a + r)$
   (c) $(x^2 + ax + b)(x - a)(x - a - r)$
   (d) $(x^2 - ax + b)(x - a)(x - a + r)$

41. If the HCF of the polynomials $(x - 3)(3x^2 + 10x + b)$ and $(3x - 2)(x^2 - 2x + a)$ is $(x - 3)(3x - 2)$, then the relation between a and b is ______.
   (a) $3a + 8b = 0$ (b) $8a - 3b = 0$
   (c) $8a + 3b = 0$ (d) $a - 2b = 0$

42. The HCF of the polynomials $12(x + 2)^3 (x^2 - 7x + 10)$ and $18(x^2 - 4)(x^2 - 6x + 5)$ is ______.
   (a) $(x^2 + 3x + 10)(x - 2)$
   (b) $6(x^2 + 3x + 10)(x + 2)$
   (c) $(x^2 - 3x - 10)(x - 2)$
   (d) $6(x^2 - 3x - 10)(x - 2)$

43. The HCF of the polynomials $(x^2 - 4x + 4)(x + 3)$ and $(x^2 + 2x - 3)(x - 2)$ is ________.
   (a) $x + 3$ (b) $x - 2$
   (c) $(x + 3)(x - 2)$ (d) $(x + 3)(x - 2)^2$

44. The HCF of the polynomials $5(x^2 - 16)(x + 8)$ and $10(x^2 - 64)(x + 4)$ is ______.
   (a) $x^2 + 12x + 32$ (b) $5(x^2 + 12x + 32)$
   (c) $x^2 - 12x + 32$ (d) $5(x^2 - 12x + 32)$

45. Find the HCF of $6x^4y$ and $12xy$.
   (a) $6x^2y$ (b) $6x$
   (c) $6y$ (d) $6xy$

46. Find the LCM of $p^4q^2r^3$ and $q^3p^6r^5$.
   (a) $p^4q^3r^3$ (b) $p^4q^2r^5$
   (c) $p^6q^3r^5$ (d) $p^6q^2r^5$

47. The LCM of the polynomials $12(x^3 + 27)$ and $18(x^2 - 9)$ is ______.
   (a) $6(x + 3)$
   (b) $36(x^2 - 9)(x^2 + 3x + 9)$
   (c) $36(x + 3)^2 (x^2 + 3x + 9)$
   (d) $36(x^2 - 9)(x^2 - 3x + 9)$

48. The rational expression $\dfrac{x^2 + 2x + 3}{x^4 + 4x^3 + 4x^2 - 9}$ in its lowest terms is _______.
   (a) $\dfrac{1}{x^2 + 3x + 3}$ (b) $\dfrac{1}{x^2 + 2x - 3}$
   (c) $\dfrac{1}{x^2 + 4x - 3}$ (d) $\dfrac{1}{x^2 + 2x + 3}$

49. The LCM of the polynomials $(x + 3)^2 (x - 2)(x + 1)^2$ and $(x + 1)^3 (x + 3)(x^2 - 4)$ is ______.
   (a) $(x + 1)^3 (x + 3)(x^2 - 4)$
   (b) $(x + 3)^2 (x + 1)^3 (x^2 - 4)$
   (c) $(x + 3)^2 (x + 1)^3 (x + 2)$
   (d) $(x + 3)^2 (x + 1)^2 (x - 2)$

Level 3

50. The HCF of two polynomials $p(x)$ and $q(x)$ using long division method was found in two steps to be $3x - 2$, and the first two quotients obtained are $x + 2$ and $2x + 1$. Find $p(x)$ and $q(x)$. (The degree of $p(x)$ > the degree of $q(x)$).
   (a) $p(x) = 6x^3 + 11x^2 + x + 6$, $q(x) = 6x^2 + x + 2$
   (b) $p(x) = 6x^3 + 11x^2 - x + 6$, $q(x) = 6x^2 - x + 2$
   (c) $p(x) = 6x^3 - 11x^2 + x - 6$, $q(x) = 6x^2 - x - 2$
   (d) $p(x) = 6x^3 + 11x^2 - x - 6$, $q(x) = 6x^2 - x - 2$

51. Simplify: $\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2} - \dfrac{3x^2 - 3}{x^2 + 4}$.
   (a) $\dfrac{-x^4 + 31x^2 + 20}{x^4 - 16}$ (b) $\dfrac{x^4 + 31x^2 + 20}{x^4 - 16}$
   (c) $-\dfrac{x^4 + 31x^2 - 20}{x^4 - 16}$ (d) $-\dfrac{x^4 - 21x^2 + 20}{x^4 - 16}$

PRACTICE QUESTIONS

52. If $P = \frac{x+1}{x-1}$ and $Q = \frac{x-1}{x+1}$, then $P^2 + Q^2 - 2PQ$ = ______.

(a) $\frac{16x^2}{x^4 - 2x + 1}$ (b) $\frac{4x^4 + 8x^2 + 4}{x^4 - 2x + 1}$

(c) $\frac{4x^2}{x^4 + 2x^2 + 1}$ (d) $\frac{8x^2}{x^4 - 2x^2 + 1}$

53. Simplify: $\frac{81x^4 - 16x^2 + 32x - 16}{9x^2 - 4x + 4}$.

(a) $9x^2 + 4x - 4$ (b) $9x^2 - 4x - 4$

(c) $9x^2 - 2x - 8$ (d) $9x^2 + 2x - 8$

54. The rational expression

$A = \left(\frac{x+1}{x-1} - \frac{x-1}{x+1} - \frac{4x}{x^2+1}\right)$ is multiplied with

the additive inverse of $B = \frac{1-x^4}{4x}$ to get C. Then, C = ______.

(a) $\frac{32x^2}{x^4 - 1}$ (b) $\frac{2x}{x^4 - 1}$

(c) 2 (d) 1

55. If the HCF of $x^3 + qx^2 + px$ and $qx^3 + 11x^2 - 4x$ is $x(x + 4)$, then p = ______.

(a) -4 (b) 3

(c) -2 (d) 5

56. If degree of both $f(x)$ and $[f(x) + g(x)]$ is 18, then degree of $g(x)$ can be ______.

(a) 18 (b) 9

(c) 6 (d) Any one of these

57. The product of additive inverse and multiplicative inverse of $\frac{x-2}{x^2-4}$ is ______.

(a) $x^2 + 4x + 4$ (b) $x^2 - 4x + 4$

(c) $x^2 - 6x + 9$ (d) None of these

58. What should be multiplied to $(2x^2 + 3x - 4)$ to get $4x^4 - 9x^2 + 24x - 16$?

(a) $2x^2 - 3x - 4$ (b) $2x^2 + 24x - 16$

(c) $2x^2 + 3x + 4$ (d) $2x^2 - 3x + 4$

59. If $f(x) = (x + 2)^7 (x + 4)^p$, $g(x) = (x + 2)^q (x + 4)^9$ and the LCM of $f(x)$ and $g(x)$ is $(x + 2)^q (x + 4)^9$, then find the maximum value of $p - q$.

(a) 4

(b) 3

(c) 2

(d) 1

60. If $f(x) = x^2 - 7x + 12$ and $g(x) = x^2 - 8x + 15$, then find the HCF of $f(x)$ and $g(x)$.

(a) $x - 4$ (b) $x - 3$

(c) $x - 5$ (d) $x - 6$

61. If $f(x) = (x + 2)(x^2 + 8x + 15)$ and $g(x) = (x + 3)(x^2 + 9x + 20)$, then find the LCM of $f(x)$ and $g(x)$.

(a) $(x + 2)(x + 3)(x + 4)(x + 5)$

(b) $(x + 2)^2 (x + 3)(x + 5)$

(c) $(x + 2)(x + 3)(x + 5)(x + 1)$

(d) $(x + 2)^2 (x + 3)(x + 4)$

62. If $f(x) = x^2 + 6x + a$, $g(x) = x^2 + 4x + b$, $h(x) = x^2 + 14x + c$ and the LCM of $f(x)$, $g(x)$ and $h(x)$ is $(x + 8)(x - 2)(x + 6)$, then find $a + b + c$. (a, b and c are constants).

(a) 20 (b) 16

(c) 32 (d) 10

63. Simplify: $\frac{(x^2 + 11x + 28)}{(x^2 + 13x + 40)} \div \frac{(x^2 + 6x + 8)}{(x^2 + 11x + 24)}$.

(a) $\frac{(x+2)(x+7)}{(x+3)(x+3)}$ (b) $\frac{x+4}{x+8}$

(c) $\frac{x+8}{x+4}$ (d) $\frac{(x+3)(x+7)}{(x+2)(x+5)}$

64. If LCM of $f(x)$ and $g(x)$ is $6x^2 + 13x + 6$, then which of the following cannot be the HCF of $f(x)$ and $g(x)$?

(a) $2x + 3$ (b) $3x + 1$

(c) $(2x + 3)(3x + 2)$ (d) $3x + 2$

65. If the LCM of $f(x)$ and $g(x)$ is $a^6 - b^6$, then their HCF can be ______.

(a) $a - b$ (b) $a^2 + ab + b^2$

(c) $a^2 - ab + b^2$ (d) All of these

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $72x^3y^3$
2. $8ab$
3. $3a - 4b$
4. $6(x - 2)^2 (x - 3)^2$
5. any whole number
6. $a + b$
7. 4
8. 12, 2
9. $(x - 1)(x - 2)(x + 4)$
10. False
11. $\dfrac{x^2+1}{1-x^2}$
12. $\dfrac{1}{x(x-1)}$
13. False
14. $\dfrac{x+1}{x-1}$
15. False
16. $a^2 \neq b$
17. True
18. False
19. $4abc$
20. $24(x + 2)^2 (x - 2)(x^2 - 2x + 4)$
21. $2xyz$
22. $120\, a^2b^2 (a^2 - b^2)$
23. 5
24. $(x + 3)(x + 1)^2$
25. $(x^2 + 1)(x^2 - 1)$
26. $\dfrac{1}{1+x}$
27. a, d
28. $\dfrac{1}{1-x}$
29. True
30. $\dfrac{x+3}{\left(x-\frac{1}{2}\right)(x+1)}$

Short Answer Type Questions

31. (i) HCF = x^2y^6
    LCM = x^3y^8
    (ii) HCF = $3a^2b^3c^2$, LCM = $9a^4b^3c^4$
    (iii) HCF = $p^4q^2r^3$ and LCM = $p^6q^3r^5$

33. LCM = $(x - 1)(x - 2)^2 (x - 4)(x - 6)^3$

34. $k = 5$

35. $a = 5$, $b = 12$

36. $(x - 1)^2 (x - 2)(x - 3)^3$

38. $(x + 1)(x^2 - 9)$

39. HCF = $(x + 2)$
    LCM = $(x^2 - 9)(x^2 - 4)(x^2 - 1)(x + 2)$

40. $\dfrac{2x^2-x+3}{(x-1)^2(x+1)}$

41. $P - Q = \dfrac{12x}{(3x-1)(3x+1)}$

    $P + Q = \dfrac{2(9x^2+1)}{(9x^2-1)}$

42. (i) $x^2 + 4x - 9$
    (ii) $\dfrac{2x(x-1)}{3}$

43. $\dfrac{(a^2+3)(3a^2+1)}{(1-a^2)^2}$

44. $\dfrac{2ab}{b^2-a^2}$

45. $\dfrac{4x}{1-x^4}$

ANSWER KEYS

Essay Type Questions

46. 8

47. 0

48. $\dfrac{2x+3}{4x+1}$

49. $\dfrac{3}{x}$

50. $\dfrac{(2x-y+z)}{a-2b+3c}$

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (c)	**9.** (c)	**10.** (c)
11. (c)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (a)	**16.** (b)	**17.** (c)	**18.** (c)	**19.** (d)	**20.** (b)
21. (a)	**22.** (c)	**23.** (a)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (a)	

Level 2

30. (d)	**31.** (c)	**32.** (a)	**33.** (c)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (a)	**38.** (a)	**39.** (a)
40. (b)	**41.** (b)	**42.** (d)	**43.** (c)	**44.** (b)	**45.** (d)	**46.** (c)	**47.** (d)	**48.** (b)	**49.** (b)

Level 3

50. (d)	**51.** (a)	**52.** (a)	**53.** (a)	**54.** (c)	**55.** (a)	**56.** (d)	**57.** (d)	**58.** (d)	**59.** (c)
60. (b)	**61.** (a)	**62.** (a)	**63.** (d)	**64.** (b)	**65.** (d)				

CONCEPT APPLICATION

Level 1

1. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
2. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
3. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
4. We gave, $P(x)\ Q(x) = \text{LCM} \times \text{HCF}$.
5. We have, $P(x)\ Q(x) = \text{LCM} \times \text{HCF}$.
6. If $(x - k)$ is a factor if $f(x)$, then $f(k) = 0$.
7. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
10. If $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$.
11. Factorize the given polynomials.
12. If $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$.
14. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
15. Sum or difference of two rational expressions is always a rational expression.
17. Subtract the first expression from the second expression.
18. Factorize numerator and denominator and eliminate the common factors.
20. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.
21. (i) Factorize the given polynomials.
 (ii) $x^2 - 5x + 6 = (x - 2)(x - 3)$
 $x^2 + 12x + 32 = (x + 8)(x + 4)$
 $x^2 - 7x + 12 = (x - 4)(x - 3)$.
22. (i) Factorize the given polynomials.
 (ii) $x^2 - 2x = x(x - 2)$
 $x^2 - 4 = (x + 2)(x - 2)$.
23. (i) Factorize the given polynomials.
 (ii) $x^2 - 2x + 1 = (x - 1)^2$
 $x^2 + 3x - 4 = (x + 4)(x - 1)$.
24. (i) Equate the given expression to zero and compare with the given values.
 (ii) As zero of $2x + 1$ is $-\dfrac{1}{2}$, zero of the expression $3x + 2a$ is $\dfrac{b}{3}$.
25. (i) Factorize the given polynomials.
 (ii) $(x^2 - 8x + 16)(x^2 - 25) = (x - 4)^2\ (x + 5)(x - 5)$
 $(x^2 - 10x + 25)(x^2 - 2x - 24)$
 $= (x - 5)^2\ (x - 6)(x + 4)$.
26. (i) $\text{LCM} \times \text{HCF} = \pm f(x) \times g(x)$
 (ii) $x^2 + 8x + 16 = (x + 4)^2$.
27. (i) $h(b) - h(a) = (b^2 + b) - (a^2 + a)$.
 (ii) Now factorize the above expression.
 (iii) Similarly factorize $g(b) - g(a)$.
28. (i) Divide the given polynomials with $(x + l)$.
 (ii) $h(b) - h(a) = b^3 - a^3$, $g(b) - g(a) = b^4 - a^4$.
 (iii) Now, factorize the above expressions.
29. (i) Find the values of a and b using the concept of HCF then obtain the required relation.
 (ii) $x^2 + 7x + 12 = (x + 4)(x + 3)$.
 (iii) $x + 3$ is a factor of $2x^2 + 5x + a$ and $x + 4$ is a factor of $x^2 + 7x + b$.

Level 2

30. (i) $p(x)\ q(x) = \text{LCM} \times \text{HCF}$.
 (ii) $\text{LCM} \times \text{HCF} = \pm f(x) \times g(x)$.
 (iii) $x^2 + 16x + 63 = (x + 7)(x + 9)$
 $x^2 + 5x - 14 = (x + 7)(x - 2)$.
31. (i) Equate the given expression to zero to get the values of a and b then find $a + b$.
 (ii) As the zero of $3x + 2$ is $\dfrac{-2}{3}$, zero of $ax + b$ is $\dfrac{1}{2}$.
32. (i) Factorize and then simplify.
 (ii) $x^2 - (y - 2z)^2 = (x + y - 2z)(x - y + 2z)$ and so on.
33. (i) If $(x - k)$ is a of factor $f(x)$, then $f(k) = 0$.
 (ii) Find the quotients by dividing each of the expressions by $x + l$.
34. (i) Add the terms which have the same denominators.

(ii) LCM of denominators is $(1 - x)(1 + x)$.
(iii) $a^2 - 1 = (a + 1)(a - 1)$;
$a^3 - 1 = (a - 1)(a^2 + a + 1)$.

35. (i) Factorize numerator and denominator then simplify.
(ii) $a^2 - (b - c)^2 = (a + b - c)(a - b + c)$; $(a + c)^2 - b^2 = (a + c + b)(a + c - b)$.

36. (i) Simplify the first two terms.
(ii) LCM of denominators is $(x^2 - 1)(x^2 + 1)$.

37. (i) We have, $P(x)\ Q(x) = \text{LCM} \times \text{HCF}$.
(i) $P^2 + Q^2 - 2PQ = (P - Q)^2$.
(ii) Find $P - Q$ and then find $(P - Q)^2$.

39. (i) Use division rule.
(ii) Apply the concept of finding HCF by long division method.

40. Find the quotients by dividing each of the given expressions by $x^2 + ax + b$.

41. (i) Find the values of a, b using the concept of HCF then obtain the relation between a and b.
(ii) $3x - 2$ is a factor of $3x^2 + 10x + b$.
(iii) $x - 3$ is a factor of $x^2 - 2x + a$.
(iv) Apply remainder theorem to find the values of a and b.

42. (i) Factorize the given polynomials.
(ii) $x^2 - 7x + 10 = (x - 5)(x - 2)$
$(x^2 - 6x + 5) = (x - 5)(x - 1)$.

44. (i) Factorize the given polynomials.
(ii) $x^2 - 16 = (x + 4)(x - 4)$
$x^2 - 64 = (x + 8)(x - 8)$.

45. The factors of $6x^4y$ and $12xy$ are $(6xy)x^3$ and $(6xy)\cdot 2$ respectively.
$\therefore$ HCF is $6xy$.

46. The factors of $p^4q^2r^3$ and $q^3p^6r^5$ are $(p^4q^2r^3)$ and $(p^4q^2r^3)\ p^2qr^2$ respectively.
$\therefore$ LCM is $p^6q^3r^5$.

47. Let $f(x) = 12(x^3 + 27)$ and $g(x)$
$= 18(x^2 - 9)$
$f(x) = 12(x^3 + 3^3) = 12(x + 3)(x^2 - 3x + 9)$
$g(x) = 18(x^2 - 3^2) = 18(x + 3)(x - 3)$.
LCM of 12 and 18 is 36.
$\therefore$ LCM of $f(x)$ and $g(x)$ is $36(x + 3)(x - 3)(x^2 - 3x + 9) = 36(x^2 - 9)(x^2 - 3x + 9)$.

48. $$\frac{x^2+2x+3}{x^4+4x^3+4x^2-9}$$
$$= \frac{x^2+2x+3}{(x^2+2x)^2-3^2}$$
$$= \frac{x^2+2x+3}{(x^2+2x+3)(x^2+2x-3)} = \frac{1}{x^2+2x-3}.$$

49. Let $f(x) = (x + 3)^2\ (x - 2)(x + 1)^2$ and
$g(x) = (x + 1)^3\ (x + 3)(x^2 - 4)$
$g(x) = (x + 1)^3\ (x + 3)(x + 2)(x - 2)$
$\therefore$ LCM of $f(x)$ and $g(x)$ is $(x + 3)^2\ (x + 1)^3\ (x + 2)(x - 2)$
$= (x + 3)^2\ (x + 1)^3\ (x^2 - 4)$.

Level 3

50. (i) Use division rule.
(ii) Apply the concept of finding HCF by division method.

52. (i) $P^2 + Q^2 - 2PQ = (P - Q)^2$.
(ii) Find $P - Q$ and then find $(P - Q)^2$.

54. (i) Simplify the expression A
(ii) Find additive inverses of the expression B and then multiply with A.

55. Let $f(x) = x^3 + qx^2 + px = x(x^2 + qx + p)$ and $g(x) = qx^3 + 11x^2 - 4x = x(qx^2 + 11x - 4)$.
Given, HCF $= x(x + 4)$
$\Rightarrow x + 4$ is the factor of $g(x)$
$\Rightarrow g(-4) = 0$
$\Rightarrow q(-4)^2 + 11(-4) - 4 = 0$
$\Rightarrow q = 3$
$x + 4$ is also factor of $f(x)$
$\Rightarrow f(-4) = 0$
$\Rightarrow (-4)^2 + 3\ (-4) + p = 0 \Rightarrow p = -4$.

56. Degree of $f(x) = 18$
Degree of $[f(x) + g(x)] = 18$
$\therefore$ The degree of $g(x)$ can be less than or equal to 18.

57. $$\frac{x-2}{x^2-4} = \frac{1}{x+2}$$
Additive inverse of $\frac{1}{x+2}$ is $\frac{-1}{x+2}$
Multiplicative inverse of $\frac{1}{x+2}$ is $x + 2$

HINTS AND EXPLANATION

$\therefore$ The required product is $\left(\dfrac{-1}{x+2}\right)(x+2)$, i.e., -1.

58. Let A be the required expression

$\therefore A(2x^2 + 3x - 4) = 4x^4 - 9x^2 + 24x - 16$

$$\Rightarrow A = \frac{4x^4 - (9x^2 - 24x + 16)}{2x^2 + 3x - 4}$$

$$= \frac{(2x^2)^2 - (3x-4)}{2x^2 + 3x - 4}$$

$$= \frac{(2x^2 + 3x - 4)(2x^2 - 3x + 4)}{(2x^2 + 3x - 4)}$$

$$= 2x^2 - 3x + 4.$$

59. Given

$f(x) = (x+2)^7 (x+4)^p$

$g(x) = (x+2)^q (x+4)^9$

LCM $= (x+2)^q (x+4)^9$

$\therefore q \geq 7$ and $p \leq 9$

For maximum value of $p - q$, p is maximum and q is minimum.

$\therefore$ maximum value of $p - q$ is $9 - 7 = 2$.

60. Given

$f(x) = x^2 - 7x + 12 = (x-3)(x-4)$

$g(x) = x^2 - 8x + 15 = (x-3)(x-5)$

$\therefore$ HCF $= x - 3$.

61. Given $f(x) = (x+2)(x^2 + 8x + 15)$

$= (x+2)(x+3)(x+5)$

$g(x) = (x+3)(x^2 + 9x + 20)$

$= (x+3)(x+4)(x+5)$

LCM $= (x+2)(x+3)(x+4)(x+5)$.

62. $f(x) = x^2 + 6x + a$

$g(x) = x^2 + 4x + b$

$h(x) = x^2 + 14 + c$

LCM $= (x+8)(x-2)(x+6)$

As three given polynomials are quadratic and their LCM is cubic, each polynomial is the product of two of the factors of LCM. By inspection method,

$f(x) = x^2 + 6x + a = (x+8)(x-2)$

$\Rightarrow a = -16$

$g(x) = x^2 + 4x + b = (x+2)(x+6)$

$\Rightarrow b = -12$

$h(x) = x^2 + 14x + c = (x+8)(x+6)$

$\Rightarrow c = 48$

$a + b + c = -16 - 12 + 48 = 20.$

63. $$\frac{(x^2 + 11x + 28)}{(x^2 + 13x + 40)} \div \frac{(x^2 + 6x + 8)}{(x^2 + 11x + 24)}$$

$$= \frac{(x+4)(x+7)}{(x+5)(x+8)} \div \frac{(x+2)(x+4)}{(x+3)(x+8)}$$

$$= \frac{(x+4)(x+7)(x+3)(x+8)}{(x+5)(x+8)(x+2)(x+4)}$$

$$= \frac{(x+7)(x+3)}{(x+5)(x+2)}.$$

64. LCM of $f(x)$ and $g(x) = 6x^2 + 13x + 6$

$= (2x+3)(3x+2)$.

We know that, HCF is a factor of LCM.

$\therefore 3x + 1$ cannot be the HCF.

65. LCM of $f(x)$ and $g(x)$ is $a^6 - b^6$

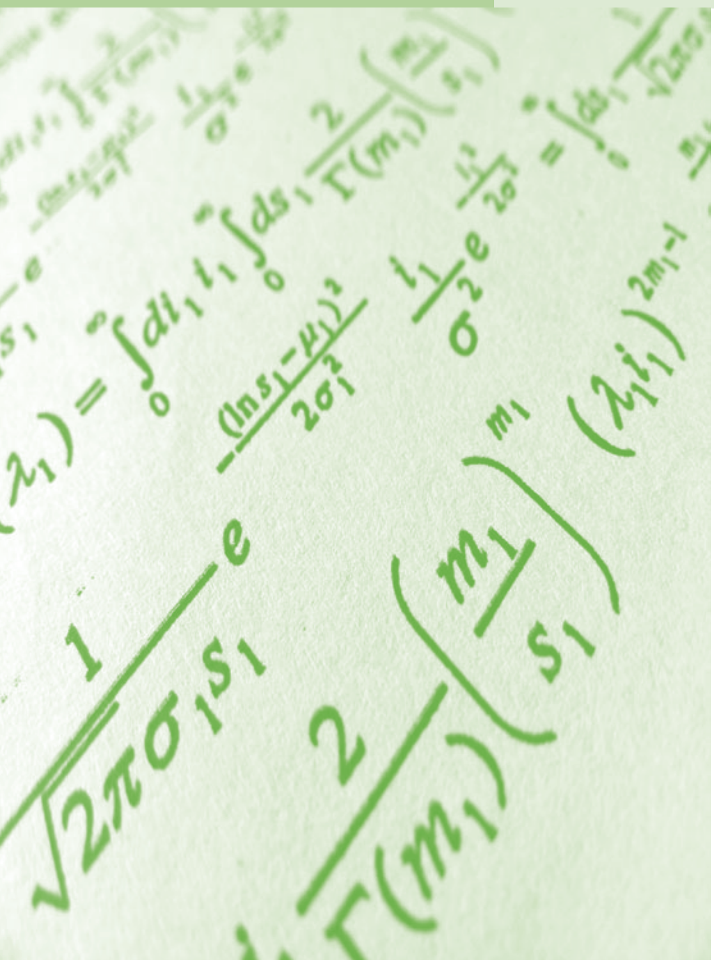
HCF of $f(x)$ and $g(x)$ is a factors $a^6 - b^6$

All $(a - b)$, $(a^2 + ab + b^2)$

and $(a^2 - ab + b^2)$ are factors of $a^6 - b^6$.

Chapter 3

Linear Equations in Two Variables



REMEMBER

Before beginning this chapter, you should be able to:

- Solve simple equations with one or two variable
- Use graphs of linear equations and inequations
- Solve basic word problems on linear equations and inequations

KEY IDEAS

After completing this chapter, you would be able to:

- Solve linear equations by different methods
- Learn nature of solutions of simultaneous linear equations
- Study solving of word problems on linear equations with two variables
- Find solution of linear equations by graphical methods

INTRODUCTION

While solving the problems, in most cases, first we need to frame an equation. In this chapter, we will learn how to frame and solve equations. Framing an equation is more difficult than solving an equation. Now, let us review the basic concepts related to this chapter.

ALGEBRAIC EXPRESSIONS

Expressions of the form $2x$, $(3x + 5)$, $(4x - 2y)$, $2x^2 + 3\sqrt{y}$, $\frac{3x^2}{2}\sqrt{y}$ are algebraic expressions. $3x$ and 5 are the terms of $(3x + 5)$ and $4x$ and $2y$ are the terms of $4x - 2y$. Algebraic expressions are made of numbers, symbols and the basic arithmetical operations. In the term $2x$, 2 is the numerical coefficient of x and x is the variable coefficient of 2.

EQUATION

An equation is a sentence in which there is an equality sign between two algebraic expressions.

For example, $2x + 5 = x + 3$, $3y - 4 = 20$ and $5x + 6 = x + 1$ are some examples of equations. Here x and y are unknown quantities and 5, 3, 20, etc., are known quantities.

Linear Equation

An equation, in which the highest index of the unknown present is one, is a linear equation.

$2(x + 5) = 18$, $3x - 2 = 5$, $x + y = 20$ and $3x - 2y = 5$ are some linear equations.

Simple Equation

A linear equation which has only one unknown is a simple equation. $3x + 4 = 16$ and $2x - 5 = x + 3$ are examples of simple linear equations.

The part of an equation which is to the left side of the equality sign is known as the left hand side, abbreviated as LHS. The part of an equation which is to the right side of the equality sign is known as the right hand side, abbreviated as RHS. The process of finding the value of an unknown in an equation is called solving the equation. The value/values of the unknown found after solving an equation is/are called the solutions or the roots of the equation.

Before we learn how to solve an equation, let us review the basic properties of equality.

1. When a term is added to both sides of the equality, the equality does not change.

   Example: If $a + b = c + d$, then $a + b + x = c + d + x$.

   This property holds good for subtraction also.

2. When the expressions on the LHS and RHS of the equation are multiplied by a non-zero term, the equation does not change.

   Example: If $a + b = c + d$, then $x(a + b) = x(c + d)$.

   This property holds good for division also.

Solving an Equation in One Variable

The following steps are involved in solving an equation.

Step 1: Always ensure that the unknown quantities are on the LHS and the known quantities or constants on the RHS.

Step 2: Add all the terms containing the unknowns on the LHS and all the knowns on the RHS, so that each side of the equation contains only one term.

Step 3: Divide both sides of the equation by the coefficient of the unknown.

EXAMPLE 3.1

If $4x + 15 = 35$, then find the value of x.

SOLUTION

Step 1: Group the known quantities as the RHS of the equation, i.e., $4x = 35 - 15$.

Step 2: Simplify the numbers on the RHS $\Rightarrow 4x = 20$.

Step 3: Since 4 is the coefficient of x, divide both the sides of the equation by 4.

$$\frac{4x}{4} = \frac{20}{4} \Rightarrow x = 5.$$

EXAMPLE 3.2

Solve: $3x + 11 = 6x - 13$.

SOLUTION

Step 1: $6x - 3x = 11 + 13$

Step 2: $3x = 24$

Step 3: $\frac{3x}{3} = \frac{24}{3} \Rightarrow x = 8.$

EXAMPLE 3.3

A swarm of 62 bees flies in a garden. If 3 bees land on each flower, 8 bees are left with no flowers. Find the number of flowers in the garden.

SOLUTION

Let the number of flowers be x.

Number of bees on the flowers $= 3x$

Total number of bees $= 62$

$\therefore 3x + 8 = 62$

$\Rightarrow 3x = 62 - 8 = 54 \Rightarrow x = 18$

$\therefore$ There are 18 flowers in the garden.

Transposition

In the above problem, $3x + 8 = 62$ can be written as $3x = 62 - 8$. When a term is moved (transposed) from one side of the equation to the other side, the sign is changed. The positive sign is changed to the negative sign and multiplication is changed to division. Moving a term from one side of the equation to the other side is called transposition. Thus solving a linear equation, in general, comprises two kinds of transposition.

Simultaneous Linear Equations

We have learnt to solve an equation with one unknown. Very often we come across equations involving more than one unknown. In such cases we require more than one condition or equation. Generally, when there are two unknowns, we require two equations to solve the problem. When there are three unknowns, we require three equations and so on.

We need to find the values of the unknowns that satisfy all the given equations. Since the values satisfy all the given equations we call them simultaneous equations. In this chapter, we deal with simultaneous (linear) equations in two unknowns.

Let us consider the equation, $2x + 5y = 19$, which contains two unknown quantities x and y.

Here, $5y = 19 - 2x$

$$\Rightarrow y = \frac{19 - 2x}{5} \tag{1}$$

In the above equation for every value of x, there exists a corresponding value for y.

When $x = 1$, $y = \frac{17}{5}$

When $x = 2$, $y = 3$ and so on.

If there is another equation, of the same kind, say, $5x - 2y = 4$.

From this, we get,

$$y = \frac{5x - 4}{2} \tag{2}$$

If we need the values of x and y such that both the equations are satisfied, then $\frac{19 - 2x}{5} = \frac{5x - 4}{2}$

$$\Rightarrow 38 - 4x = 25x - 20$$

$$\Rightarrow 29x = 58$$

$$\Rightarrow x = 2$$

On substituting the value of $x = 2$ in Eq. (1), we get

$$y = \frac{19 - 2(2)}{5} = \frac{15}{5} = 3$$

$$\Rightarrow y = 3$$

Both the equations are satisfied by the same values of x and y. Thus we can say that when two or more equations are satisfied by the same values of unknown quantities, then those equations are called simultaneous equations.

When two equations, each in two variables, are given, they can be solved in five ways.

1. Elimination by cancellation
2. Elimination by substitution
3. Adding the two equations and subtracting one equation from the other
4. Cross-multiplication method
5. Graphical method

Elimination by Cancellation

EXAMPLE 3.4

If $2x + 3y = 19$ and $5x + 4y = 37$, then find the values of x and y.

SOLUTION

In this method, the two equations are reduced to a single variable equation by eliminating one of the variables.

Step 1: Here, let us eliminate the y term, and in order to eliminate the y term, we have to multiply the first equation with the coefficient of y in the second equation and the second equation with the coefficient of y in the first equation so that the coefficients of y terms in both the equations become equal.

$$(2x + 3y = 19)4 \Rightarrow 8x + 12y = 76 \quad (1)$$

$$(5x + 4y = 37)3 \Rightarrow 15x + 12y = 111 \quad (2)$$

Step 2: Subtract Eq. (2) from Eq. (1),

$$(15x + 12y) - (8x + 12y) = 111 - 76$$

$$\Rightarrow 7x = 35$$

$$\Rightarrow x = 5.$$

Step 3: Substitute the value of x in Eq. (1) or Eq. (2) to find the value of y. Substituting the value of x in the first equation, we get,

$$2(5) + 3y = 19$$

$$\Rightarrow 3y = 19 - 10 \Rightarrow 3y = 9$$

$$\Rightarrow y = 3.$$

$\therefore$ The solution of the given pair of equation is $x = 5$; $y = 3$.

Elimination by Substitution

EXAMPLE 3.5

If $4x - 3y = 32$ and $x + y = 1$, then find the values of x and y.

SOLUTION

In this method, the two equations are reduced to a single variable equation by substituting the value of one variable, obtained from one equation, in the other equation.

Step 1: Using the second equation, find x in terms of y, i.e.,

$$x + y = 1$$
$$\Rightarrow y = 1 - x \quad (1)$$

Step 2: Substitute the value of y in the first equation to find the value of x.

$\therefore 4x - 3(1 - x) = 32$

Step 3: Simplify the equation in terms of x and find the value of x.

$$4x - 3 + 3x = 32$$
$$\Rightarrow 7x = 32 + 3 = 35$$
$$\Rightarrow x = 5$$

Step 4: Substituting the value of x in Eq. (1), we have,

$$y = 1 - x = 1 - 5$$
$$\Rightarrow y = -4.$$

$\therefore$ The solution for the given pair of equations is $x = 5$; $y = -4$.

Adding Two Equations and Subtracting One Equation from the Other

EXAMPLE 3.6

Solve $3x + 7y = 32$ and $7x + 3y = 48$.

SOLUTION

Given,

$$3x + 7y = 32 \quad (1)$$
$$7x + 3y = 48 \quad (2)$$

Step 1: Adding both the equations, we get

$$10x + 10y = 80$$
$$\Rightarrow 10(x + y) = 10 \times 8$$
$$\Rightarrow x + y = 8 \quad (3)$$

Step 2: Subtracting Eq. (1) from Eq. (2), we have

$$(7x + 3y) - (3x + 7y) = 48 - 32$$
$$\Rightarrow 4x - 4y = 16$$
$$\Rightarrow 4(x - y) = 4 \times 4$$
$$\Rightarrow x - y = 4 \quad (4)$$

Step 3: Adding the Eqs. (3) and (4), we have

$$x + y + x - y = 12$$
$$\Rightarrow 2x = 12$$
$$\Rightarrow x = 6$$

Substituting $x = 6$ in any of the Eqs. (1), (2), (3) or (4), we get, $y = 2$.

$\therefore$ The solution of the pair of equations is $x = 6$; $y = 2$.

Cross-multiplication Method

EXAMPLE 3.7

Solve $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

SOLUTION

Given,

$$a_1x + b_1y + c_1 = 0 \quad (1)$$
$$a_2x + b_2y + c_2 = 0 \quad (2)$$

Solving the above equations using elimination by cancellation method, we get,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

Applying alternendo on the above ratios, we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \text{ and } \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

The above result can be better remembered using the Fig. 3.1.

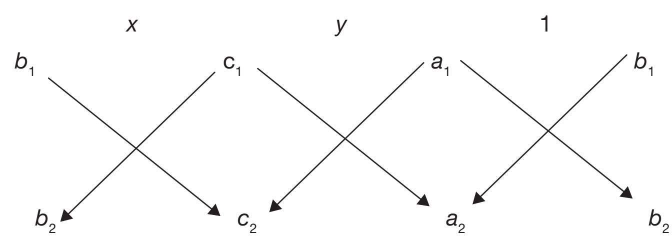


Figure 3.1

The arrows between the two numbers indicate that they are to be multiplied and second product is to be subtracted from the first.

While using this method, the following steps are to be followed.

Step 1: Write the given equations in the form $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Step 2: Write the coefficients of the equations as mentioned above.

Step 3: Find the values of x and y.

EXAMPLE 3.8

Solve $4x + 5y = 71$ and $5x + 3y = 66$.

SOLUTION

Step 1: Rewriting the equations, we get,

$$4x + 5y - 71 = 0 \quad (1)$$
$$5x + 3y - 66 = 0 \quad (2)$$

Step 2: Write the coefficients of x and y in the specified manner.

x	y	1	
5	−71	4	5
3	−66	5	3

Step 3: Find the values of x and y.

$$\frac{x}{-330+213} = \frac{y}{-355+264} = \frac{1}{12-25}$$
$$\Rightarrow \frac{x}{-117} = \frac{y}{-91} = \frac{1}{-13}$$
$$\Rightarrow x = \frac{-117}{-13}; y = \frac{-91}{-13}$$
$$\Rightarrow x = 9 \text{ and } y = 7.$$

Note Choosing a particular method to solve a pair of equations makes the simplification easier. One can learn as to which method is the easiest to solve a pair of equations by becoming familiar with the different methods of solving the equation.

Graphical Method

Plotting the Points If we consider any point in a plane, then we can determine the location of the given point, i.e., we can determine the distance of the given point from X-axis and Y-axis. Therefore, each point in the plane represents the distance from both the axes. So, each point is represented by an ordered pair and it consists of x-coordinate and y-coordinate. The first element of an ordered pair is called x-coordinate and the second element of an ordered pair is called y-coordinate. In the first quadrant Q_1, both the x-coordinate and y-coordinate are positive real numbers. In the second quadrant Q_2, x-coordinates are negative real numbers and y-coordinates are positive real numbers. In the third quadrant Q_3 both the x-coordinate and y-coordinate are negative real numbers. In the fourth quadrant Q_4, x-coordinates are positive real numbers and y-coordinates are negative real numbers. And the origin is represented by (0, 0).

Consider the point (2, 3). Here 2 is the x-coordinate and 3 is the y-coordinate. The point (2, 3) is 2 units away from the Y-axis and 3 units away from the X-axis. The point (2, 3) belongs to the first quadrant. If we consider the point (–3, –5), –3 is x-coordinate and –5 is y-coordinate. The point (–3, –5) belongs to Q_3 and is 3 units away from the Y-axis and 5 units away from the X-axis.

To plot a point say P(–3, 4), we start from the origin and proceed 3 units towards the left hand side along the X-axis (i.e., negative direction as x-coordinate is negative), and from there we move 4 units upwards along the Y-axis (i.e., positive direction as y-coordinate is positive). The method of plotting a point in a coordinate plane was explained by Rene Des Cartes, a French mathematician.

EXAMPLE 3.9

Plot the following points on the coordinate plane.
$P(2, 3)$, $Q(-4, -5)$, $R(2, -3)$, $S(-4, 4)$, $T(3, 0)$, $U(0, 5)$.

SOLUTION

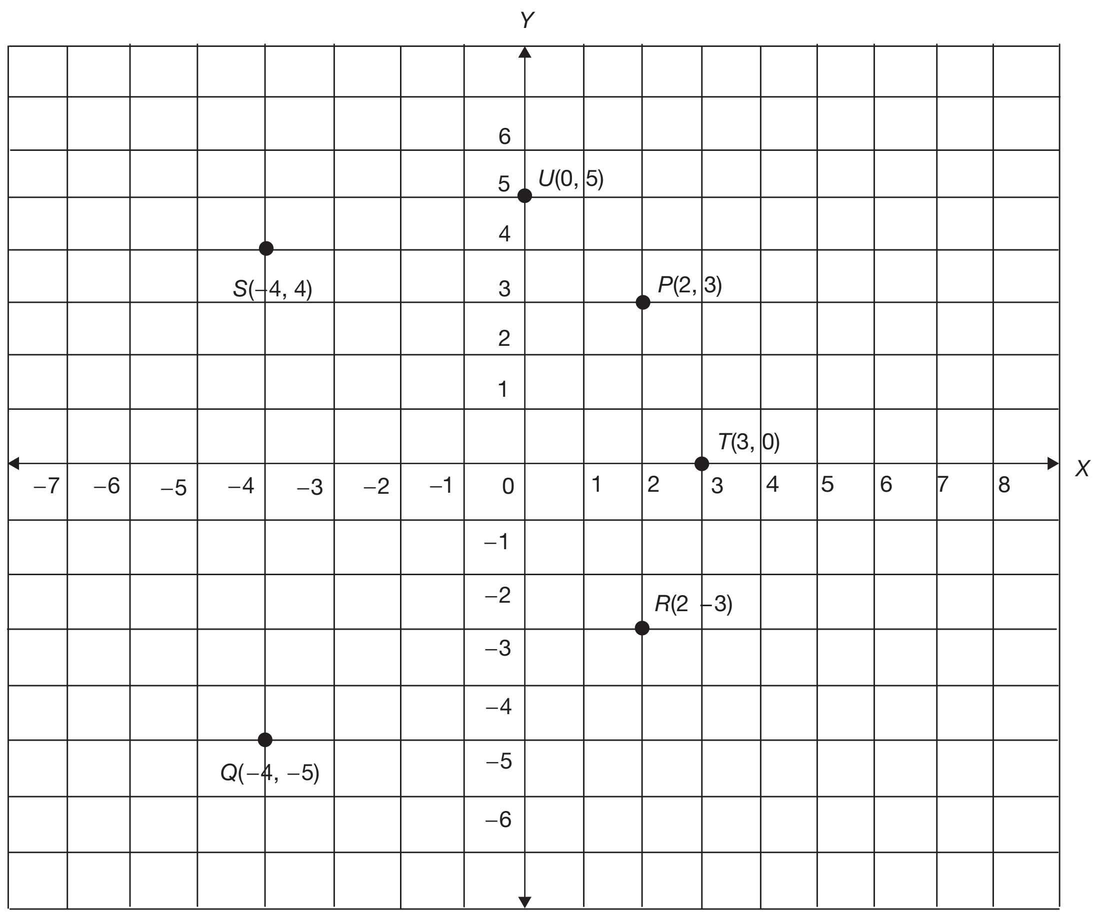


Figure 3.2

EXAMPLE 3.10

Plot the following points on the coordinate plane. What do you observe? (see Fig. 3.3)

(a) $(-2, 3)$, $(-1, 3)$, $(0,3)$, $(1, 3)$, $(2, 3)$

(b) $(4, 2)$, $(4, 1)$, $(4, 0)$, $(4, -1)$, $(4, -2)$

SOLUTION

(a) $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$, $(2, 3)$

(i) The above points lie on the same straight line which is perpendicular to the Y-axis.

(ii) The y-coordinates of all the given points are the same, i.e., $y = 3$.

(iii) Hence, the straight line passing through the given points is represented by $y = 3$.

(iv) Therefore, the line $y = 3$ is parallel to X-axis which intersects Y-axis at (0, 3).

(b) (4, 2), (4, 1), (4, 0), (4, –1), (4, –2)

(i) The above points lie on the same straight line which is perpendicular to the X-axis.

(ii) The x-coordinates of all the given points is the same, i.e., $x = 4$.

(iii) Hence, the straight line passing through the given points is represented by $x = 4$.

(iv) Therefore, the line $x = 4$ is parallel to the Y-axis which intersects the X-axis at (4, 0).

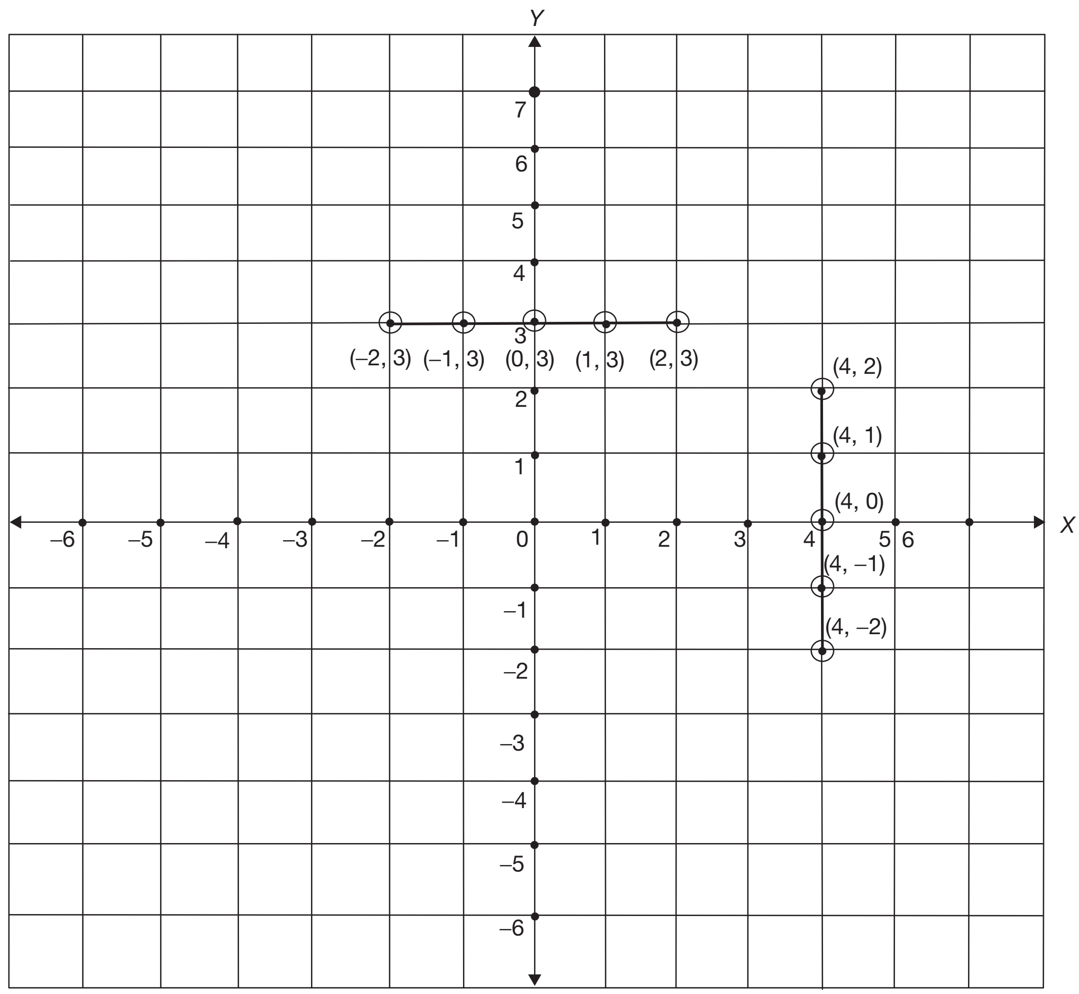


Figure 3.3

Notes

1. The y-coordinate of every point on the X-axis is zero, i.e., $y = 0$. Therefore the X-axis is denoted by $y = 0$.
2. The x-coordinate of every point on the Y-axis is zero, i.e., $x = 0$. Therefore the Y-axis is denoted by $x = 0$.

EXAMPLE 3.11

Plot the following points on the coordinate plane and what do you observe?
(–3, 3), (−2, 2), (−1, 1), (0, 0), (1, −1), (2, −2), (3, −3)

SOLUTION

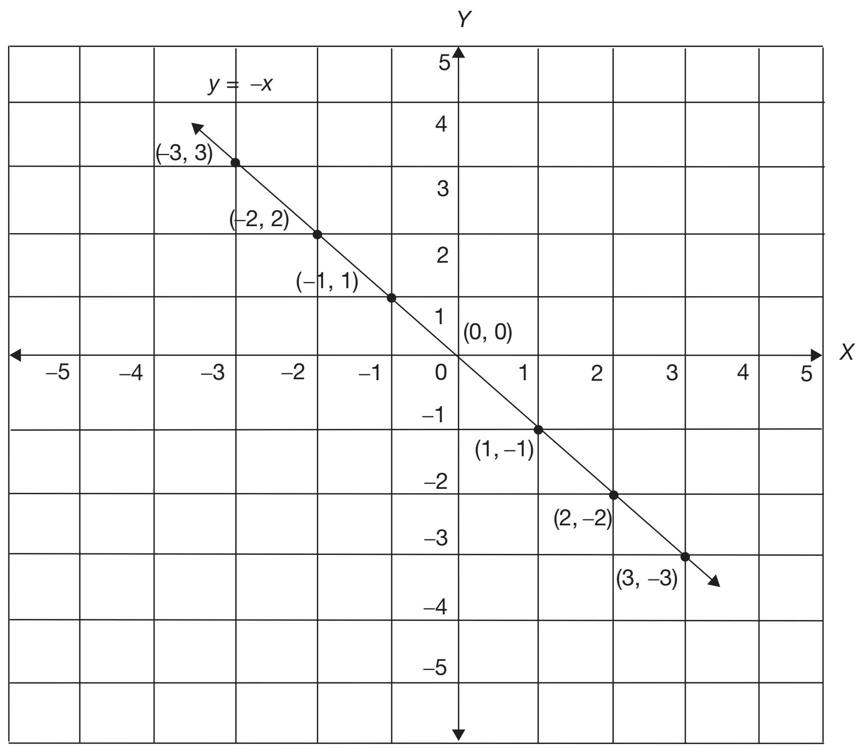


Figure 3.4

(a) All the given points lie on the same straight line.

(b) Every point on the straight line represents $y = -x$.

(c) The above line with the given ordered pairs is represented by the equation $y = -x$.

EXAMPLE 3.12

Draw the graph of the equation $y = 3x$ where R is the replacement set for both x and y.

SOLUTION

x	−2	−1	0	1	2
$y = 3x$	−6	−3	0	3	6

Some of the ordered pairs which satisfy the equation $y = 3x$ are (–1, –3), (–2, –6), (0, 0), (1, 3), (2, 6).

By plotting the above points on the graph sheet, we get the following:

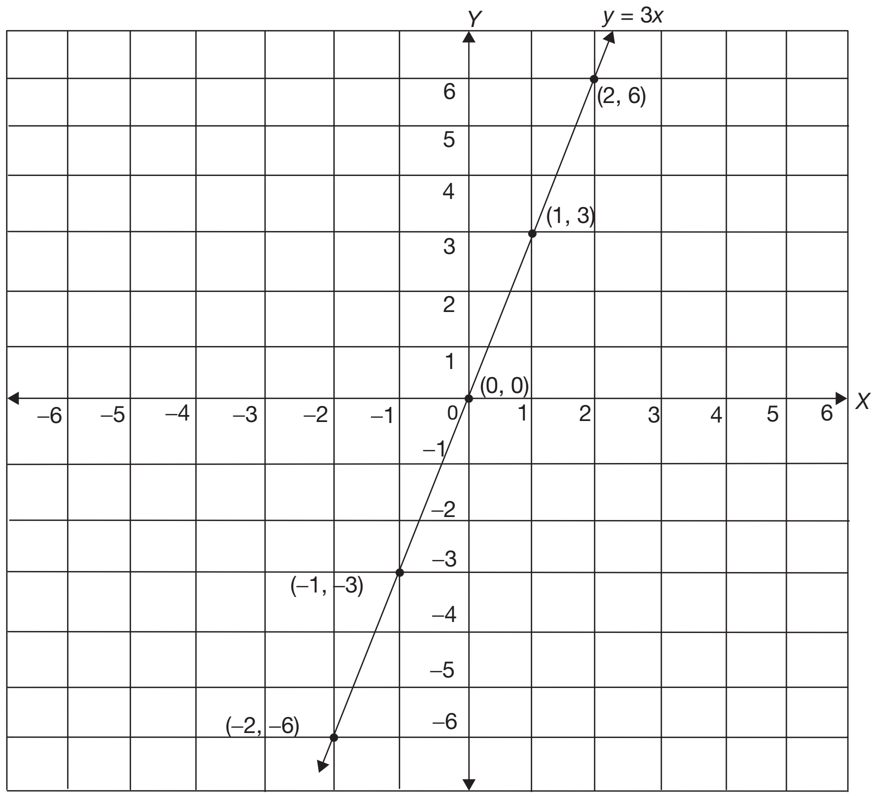


Figure 3.5

EXAMPLE 3.13

Draw the graph of the equations $x + y = -1$ and $x - y = 5$.

SOLUTION

(a) $x + y = -1$

x	-3	-2	-1	0	1	2	3
$y = -1 - x$	2	1	0	-1	-2	-3	-4

Some of the ordered pairs which satisfy the equation $x + y = -1$ are $(-3, 2)$, $(-2, 1)$, $(-1, 0)$, $(0, -1)$, $(1, -2)$, $(2, -3)$, $(3, -4)$.

(b) $x - y = 5$

x	-2	-1	0	1	2	3	4
$y = x - 5$	-7	-6	-5	-4	-3	-2	-1

$\therefore$ Some of the ordered pairs which satisfy the equation $x - y = 5$ are $(-2, -7)$, $(-1, -6)$, $(0, -5)$, $(1, -4)$, $(2, -3)$, $(3, -2)$, $(4, -1)$.

The ordered pairs which satisfy the equations $x + y = -1$ and $x - y = 5$ are plotted on a graph paper. We find that each equation represents a line.

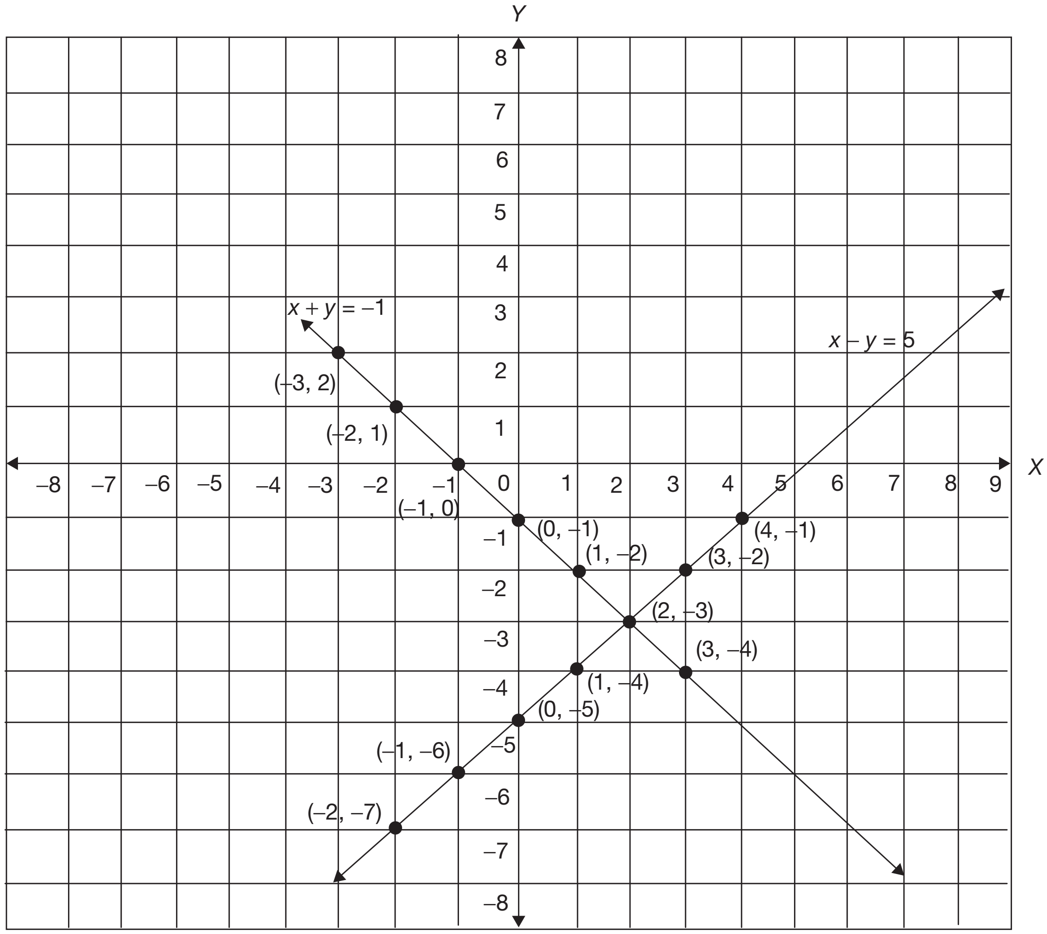


Figure 3.6

From the graph, we notice that the two given lines intersect at the point (2, –3).

That is, lines $x + y = -1$ and $x - y = 5$ have a common point (2, –3). Therefore, (2, –3) is the solution of the equations $x + y = -1$ and $x - y = 5$.

Verification:

$$x + y = -1 \qquad (1)$$

$$x - y = 5 \qquad (2)$$

Solving Eqs. (1) and (2), we get,

$x = 2$ and $y = -3$.

∴ (2, –3) is the solution of $x + y = -1$ and $x - y = 5$.

Note From the above example, we notice that we can find the solution for simultaneous equations by representing them in graphs, i.e., by using the graphical method.

Nature of Solutions

When we try to solve a pair of equations we could arrive at three possible results. They are, having

1. a unique solution.
2. an infinite number of solutions.
3. no solution.

Let the pair of equations be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where a_1 and a_2 are the coefficients of x; b_1 and b_2 are the coefficients of y; while c_1 and c_2 are the known constant quantities.

1. **A pair of equations having a unique solution:**

   If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the pair of equations will have a unique solution.

   We have solved such equations in the previous examples of this chapter.

2. **A pair of equations having infinite solutions:**

   If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will have infinite number of solutions.

   Note In fact this means that there are no two equations as such and one of the two equations is simply obtained by multiplying the other with a constant. These equations are known as dependent equations.

Example:

$$3x + 4y = 8$$
$$9x + 12y = 24$$

For these two equations, $a_1 = 3$, $a_2 = 9$, $b_1 = 4$, $b_2 = 12$, $c_1 = -8$, $c_2 = -24$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Since, $\frac{3}{9} = \frac{4}{12} = \frac{-8}{-24}$

The above pair of equations will have infinite solutions.

3. **A pair of equations having no solution at all:**

   If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will have no solution.

   Notes

   1. In other words, the two equations will contradict each other or be inconsistent with each other.
   2. A pair of equations is said to be consistent if it has a solution (finite or infinite).

Example:

$$5x + 6y = 30$$
$$10x + 12y = 40$$

For these two equations, $a_1 = 5$, $a_2 = 10$, $b_1 = 6$, $b_2 = 12$, $c_1 = -30$, $c_2 = -40$.

Here,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

$$\frac{5}{10} = \frac{6}{12} \neq \frac{-30}{-40}.$$

Hence, the pair of equations have no solution at all.

Word Problems and Applications of Simultaneous Equations

In this chapter, we have discussed earlier that it is essential to have as many equations as there are unknown quantities to be determined. In word problems also, it is necessary to have as many independent conditions, as there are unknown quantities to be determined.

Let us understand with the help of the following examples as to how word problems can be solved using simultaneous equations.

EXAMPLE 3.14

The sum of the successors of two numbers is 42 and the difference of their predecessors is 12. Find the numbers.

SOLUTION

Let the two numbers be x and y.

Given that,

$$(x + 1) + (y + 1) = 42$$

$$\Rightarrow x + y = 40 \tag{1}$$

Also,

$$(x - 1) - (y - 1) = 12$$

$$\Rightarrow x - y = 12 \tag{2}$$

Adding Eqs. (1) and (2), we get,

$$2x = 52$$

$$\Rightarrow x = 26$$

Substituting $x = 26$ in any one of Eqs. (1) and (2), we get $y = 14$.

$\therefore$ The two numbers are 26 and 14.

EXAMPLE 3.15

In a fraction, if numerator is increased by 2 and denominator is decreased by 3, then the fraction becomes 1. Instead, if numerator is decreased by 2 and denominator is increased by 3, then the fraction becomes $\frac{3}{8}$. Find the fraction.

SOLUTION

Let the fraction be $\frac{a}{b}$.

Applying the first condition, we get $\frac{a+2}{b-3} = 1$

$$\Rightarrow a + 2 = b - 3$$

$$\Rightarrow a - b = -5 \tag{1}$$

Applying the second condition, we get $\frac{a-2}{b+3} = \frac{3}{8}$

$$\Rightarrow 8a - 16 = 3b + 9$$

$$\Rightarrow 8a - 3b = 25 \quad (2)$$

Solving Eqs. (1) and (2), using any of the methods discussed earlier, we get $a = 8$ and $b = 13$.

$\therefore$ The fraction is $\frac{8}{13}$.

EXAMPLE 3.16

In a box, the total number of ₹2 coins and ₹5 coins is 20. If the total coins amount to ₹76, find the number of coins of each denomination.

SOLUTION

Let the number of ₹2 coins and ₹5 coins be x and y respectively.

Given, $x + y = 20$ (1)

$2x + 5y = 76$ (2)

Solving Eqs. (1) and (2), we get

$x = 8$ and $y = 12$.

That is, The number of ₹2 coins = 8 and number of ₹5 coins = 12.

EXAMPLE 3.17

Four years ago, the age of a person was thrice that of his son. Eight years later, the age of the person will be twice that of his son. Find the present ages of the person and his son.

SOLUTION

Let the present ages of the person and his son be x years and y years respectively.

Given,

$$x - 4 = 3(y - 4)$$

$$\Rightarrow x - 4 = 3y - 12$$

$$\Rightarrow x - 3y = -8 \quad (1)$$

Also,

$$x + 8 = 2(y + 8)$$

$$\Rightarrow x + 8 = 2y + 16$$

$$\Rightarrow x - 2y = 8 \quad (2)$$

Solving Eqs. (1) and (2), we get

$x = 40$ and $y = 16$.

EXAMPLE 3.18

For what value of k do the set of equations $4x - (3k + 2)y = 20$ and $(11k - 3)x - 10y = 40$ have infinite solutions?

SOLUTION

Given equations are,

$$4x - (3k + 2)y = 20 \quad (1)$$

$$(11k - 3)x - 10y = 40 \quad (2)$$

System of Eqs. (1) and (2) have infinite solutions, if $\frac{4}{11k-3} = \frac{-(3k+2)}{-10} = \frac{-20}{-40}$

$$\Rightarrow \frac{4}{11k-3} = \frac{1}{2}$$

$$\Rightarrow 8 = 11k - 3$$

$$\Rightarrow 11k = 11$$

$$\Rightarrow k = 1.$$

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short answer Type Questions

1. If $99x + 101y = 400$ and $101x + 99y = 600$, then $x + y$ is ________.

2. The number of common solutions for the system of linear equations $5x + 4y + 6 = 0$ and $10x + 8y = 12$ is ________.

3. Sum of the heights of A and B is 320 cm and the difference of heights of A and B is 20 cm. The height of B can be ________. (140 cm/145 cm/150 cm).

4. If $a : b = 7 : 3$ and $a + b = 20$, then $b =$ ________.

5. If $\frac{1}{x} + \frac{1}{y} = k$ and $\frac{1}{x} - \frac{1}{y} = k$, then the value of y ________. (is 0/does not exist).

6. If $p + q = k$, $p - q = n$ and $k > n$, then q is ________ (positive/negative).

7. If $a + b = x$, $a - b = y$ and $x < y$, then b is ________ (positive/negative).

8. If $3a + 2b + 4c = 26$ and $6b + 4a + 2c = 48$, then $a + b + c =$ ________.

9. Sum of the ages of X and Y, 12 years ago, was 48 years and sum of the ages of X and Y, 12 years hence will be 96 years. Present age of X is ________.

10. If the total cost of 3 chairs and 2 tables is ₹1200 and the total cost of 12 chairs and 8 tables is ₹4800, then the cost of each chair must be ₹200 and each table must be ₹300. (True/False)

11. If the total cost of 2 apples and 3 mangoes is ₹22, then the cost of each apple and each mango must be ₹5 and ₹4 respectively, (where cost of each apple and mango is an integer). (True/False)

12. Number of non-negative integral solutions for the equation $2x + 3y = 12$ is ________.

13. Two distinct natural numbers are such that the sum of one number and twice the other number is 6. The two numbers are ________.

14. If $2x + 3y = 5$ and $3x + 2y = 10$, then $x - y =$ ________.

15. If $a + b = p$ and $ab = p$, then find the value of p. (where a and b are positive integers).

Short Answer Type Questions

16. Solve: $331a + 247b = 746$ and $247a + 331b = 410$.

17. Six gallery seats and three balcony seats for a play were sold for ₹162. Four gallery seats and five balcony seats were sold for ₹180. Find the price of a gallery seat and the price of a balcony seat.

18. If the numerator of a fraction is increased by 2 and the denominator is decreased by 4, then it becomes 2. If the numerator is decreased by 1 and the denominator is increased by 2, then it becomes $\frac{1}{3}$. Find the fraction.

19. Solve: $\frac{1}{x} + \frac{1}{y} = 6$, $\frac{1}{y} + \frac{1}{z} = 7$ and $\frac{1}{z} + \frac{1}{x} = 5$.

20. Solve: $\frac{2}{x+y} - \frac{1}{x-y} = 11$ and $\frac{5}{x+y} + \frac{4}{x-y} = 8$.

21. Jaydeep starts his job with a certain monthly salary and earns a fixed increment in his monthly salary at the middle of every year, starting from the first year. If his monthly salary was ₹78000 at the end of 6 years of service and ₹84000 at the end of 12 years of service, find his initial salary and annual increment.

22. Alok was asked to find, $\frac{6}{7}$ of a number but instead he multiplied it by $\frac{7}{6}$. As a result he got an answer,which was more than the correct answer by 299. What was the number?

23. Shriya has certain number of 25 paise and 50 paise coins in her purse. If the total number of coins is 35 and their total value is ₹15.50, find the number of coins of each denomination.

24. For what value of k, will the following pair of linear equations have no solution?
$2x + 3y = 1$ and $(3k - 1)x + (1 - 2k)\ y = 2k + 3$.

25. Solve: $\frac{x}{a} + \frac{y}{b} = a^2 + b^2$ and $\frac{x}{a^2} + \frac{y}{b^2} = a + b$.

Essay Type Questions

26. Solve: $x - 2y + z = 0$, $9x - 8y + 3z = 0$ and $2x + 3y + 5z = 36$.

27. Four friends P, Q, R and S have some money. The amount with P equals the total amount with the others. The amount with Q equals one-third of the total amount with the others. The amount with R equals one-fifth of the total amount with the others. The amount with S equals one-eleventh of the total amount with the others. The sum of the smallest and the largest amounts with them is ₹210. Find the sum of the amounts with the other two (in ₹).

28. The population of a town is 25000. If in the next year the number of males were to increase by 5% and that of females by 3%, the population would grow to 26010. Find the number of males and females in the town at present.

29. What is the solution set of

$$\frac{12}{2x+3y} + \frac{5}{3x-2y} = -7 \text{ and}$$

$$\frac{8}{2x+3y} + \frac{6}{3x-2y} = -10?$$

30. Tito purchased two varieties of icecream cups, vanilla and strawberry—spending a total amount of ₹330. If each vanilla cup costs ₹25 and each strawberry cup costs ₹40, then in how many different combinations could he have purchased the icecream cups?

CONCEPT APPLICATION

Level 1

1. For what value of k do the equations $3(k - 1)x + 4y = 24$ and $15x + 20y = 8(k + 13)$ have infinite solutions?

   (a) 1 (b) 4
   (c) 3 (d) 2

2. If the system of equations $4x + py = 21$ and $px - 2y = 15$ has unique solution, then which of the following could be the value of p?

   (A) 103 (B) 105
   (C) 192 (D) 197

   (a) Both (A) and (B)
   (b) Both (C) and (D)
   (c) (A), (B) and (D)
   (d) All of (A), (B), (C) and (D)

3. If the system of equations $2x - 3y = 3$ and $-4x + qy = \frac{p}{2}$ is inconsistent, which of the following cannot be the value of p?

   (a) −24 (b) −18
   (c) −12 (d) −36

4. The semi-perimeter of a triangle exceeds each of its side by 5, 3 and 2 respectively. What is the perimeter of the triangle?

   (a) 12 (b) 10
   (c) 15 (d) 20

5. If (p, p) is the solution of system of equations $ax + by + (t - s) = 0$ and $bx + ay + (s - r) = 0$, $(a \neq b)$, then which of the following must be true?

   (a) $2r = s + t$ (b) $2t = r + s$
   (c) $2s = r + t$ (d) $r + s + t = 0$

6. If $173x + 197y = 149$ and $197x + 173y = 221$, then find (x, y).

   (a) (3, −2) (b) (2, 1)
   (c) (1, −2) (d) (2, −1)

7. Mallesh has some cows and some hens in his shed. The total number of legs is 92 and the total number of heads is 29. Find the number of cows in his shed.

   (a) 12 (b) 14
   (c) 17 (d) 19

8. Total cost of 14 pens and 21 books is ₹130 and the total cost of 6 pens and p books is ₹90. Which of the following cannot be the value of p?

PRACTICE QUESTIONS

(a) 8 (b) 9

(c) 10 (d) 11

9. If an ordered pair satisfying the equations $2x - 3y = 18$ and $4x - y = 16$ also satisfies the equation $5x - py - 23 = 0$, then find the value of p.

(a) 1 (b) 2

(c) −1 (d) −2

10. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are such that a_1, b_1, c_1, a_2, b_2 and c_2 are consecutive integers in the same order, then find the values of x and y.

(a) 1, −2 (b) 2, −3

(c) 1, 3 (d) −1, 1

11. If we increase the length by 2 units and the breadth by 2 units, then the area of rectangle is increased by 54 square units. Find the perimeter of the rectangle (in units).

(a) 44 (b) 50

(c) 56 (d) 52

12. A mother said to her son, 'the sum of our present ages is twice my age 12 years ago and nine years hence, the sum of our ages will be thrice my age 14 years ago'. What is her son's present age? (in years)

(a) 8 (b) 12

(c) 15 (d) 10

13. A told B, 'when I was as old as you are now, then your age was four years less than half of my present age'. If the sum of the present ages of A and B is 61 years, what is B's present age? (in years)

(a) 9 (b) 25

(c) 43 (d) 36

14. If the system of equations $4x - 5y = 6$ and $-12x + ay = b$ is inconsistent, which of the following cannot be the value of b?

(a) −16 (b) −18

(c) −20 (d) −22

15. A sum of ₹400 was distributed among the students of a class. Each boy received ₹8 and each girl received ₹4. If each girl had received ₹10, then each boy would have received ₹5. Find the total number of students of the class.

(a) 40 (b) 50

(c) 60 (d) 70

Level 2

16. Dheeraj has twice as many sisters as he has brothers. If Deepa, Dheeraj's sister has the same number of brothers as she has sisters, then Deepa has how many brothers?

(a) 2 (b) 3

(c) 4 (d) 6

17. The total cost of six books, five pencils and seven sharpeners is ₹115 and that of eight books, ten pencils and fourteen sharpeners is ₹190, then which of the following article's cost can be found uniquely?

(a) Book (b) Pencil

(c) Sharpener (d) None of these

18. The sum of the speeds of a boat in still water and the speed the current is 10 kmph. If the boat takes 40% of the time to travel downstream when compared to that upstream, then find the difference of the speeds of the boat when travelling upstream and down stream.

(a) 3 kmph (b) 6 kmph

(c) 4 kmph (d) 5 kmph

19. In a fraction, if the numerator is decreased by 1 and the denominator is increased by 1, then the fraction becomes $\frac{1}{2}$. Instead, if the numerator is increased by 1 and the denominator is decreased by 1, then the fraction becomes $\frac{4}{5}$. Find the numerator of the fraction.

(a) 5 (b) 6

(c) 7 (d) 8

20. Ram has 18 coins in the denominations of ₹1, ₹2 and ₹5. If their total value is ₹54 and the number of ₹2 coins are greater than that of ₹5 coins, then find the number of ₹1 coins with him.

PRACTICE QUESTIONS

(a) 2 (b) 3

(c) 4 (d) 1

21. If the ordered pair ($\sin\theta$, $\cos\theta$) satisfies the system of equations $mx + ny + a + b = a - b$ and $nx + my + 2b = 0$, then find the value of θ where $0 \le \theta \le 90°$. ($m \ne n$)

(a) 30° (b) 45°

(c) 50° (d) 60°

22. Swaroop can row 16 km downstream and 8 km upstream in 6 hours. He can row 6 km upstream and 24 km downstream in 6 hours. Find the speed of Swaroop in still water.

(a) 6 kmph (b) 8 kmph

(c) 3 kmph (d) 5 kmph

23. A two-digit number is formed by either subtracting 17 from nine times the sum of the digits or by adding 21 to 13 times the difference of the digits. Find the number.

(a) 37 (b) 73

(c) 71 (d) 72

24. Swathi starts her job with certain monthly salary and earns a fixed increment every year. If her salary was ₹22500 per month after 6 years of service and ₹30000 per month after 11 years of service. Find her salary after 8 years of service (in ₹).

(a) 24000 (b) 25500

(c) 26000 (d) 24500

25. A three digit number *abc* is 459 more than the sum of its digits. What is the sum of the 2-digit number *ab* and the 1-digit number *a*?

(a) 71 (b) 61

(c) 51 (d) 41

26. The following sentences are the steps involved in solving the inequation $|5x + 4| > 5x - 4$. Arrange in sequential order from first to last.

(A) $10x < 0$

(B) $5x + 4 < -5x + 4$ or $5x + 4 > 5x - 4$

(C) $x < 0$

(D) $5x + 4 < -5x + 4$

(a) BDAC (b) DBAC

(c) ABDC (d) BADC

27. The following sentences are the steps involved in solving the inequation $5x + 2 > 7x - 4$. Arrange them in sequential order from first to last.

(A) Solution set = {0, 1, 2} ($\because x \in W$)

(B) $-2x > -6$

(C) $5x - 7x > -4 - 2$

(D) $x < 3$

(a) CBAD (b) BCDA

(c) CBDA (d) BCAD

28. The total cost of 6 erasers and 9 pens is at least ₹102 and the cost of each eraser is at most ₹5. Find the minimum possible cost (in rupees) of a pen. The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) Let the cost of each eraser be ₹x and cost of each pen be ₹y.

(B) $6x + 9y \ge 102$ and $y \le 5$.

(C) $6 \times 5 + 9y \ge 102 \Rightarrow 9y \ge 72 \Rightarrow y \ge 8$.

(D) The minimum possible cost of a pen is ₹8.

(a) ABDC (b) ABCD

(c) DABC (d) ACBD

Level 3

29. An examination consists of 100 questions. Two marks are awarded for every correct option. If one mark is deducted for every wrong option and half mark is deducted for every question left, then a person scores 135. Instead, if half mark is deducted for every wrong option and one mark is deducted for every question left, then the person scores 133. Find the number of questions left unattempted by the person.

(a) 14 (b) 16

(c) 10 (d) 12

30. Ram, Shyam, Tarun and Varun together had a total amount of ₹240 with them. Ram had half

of the total amount with the others. Shyam had one-third of the total amount with the others. Tarun had one-fourth of the total amount with the others. Find the amount with Varun (in ₹).

(a) 64 (b) 70

(c) 52 (d) 58

31. Ramu had 13 notes in the denominations of ₹10, ₹50 and ₹100. The total value of the notes with him was ₹830. He had more of ₹100 notes than that of ₹50 notes with him. Find the number of ₹10 notes with him.

(a) 4 (b) 3

(c) 2 (d) 5

32. A, B, C and D share a certain amount amongst themselves. B sees that the other three get 3 times what he himself gets. C sees that the other three get 4 times what he gets, while D sees that the other three get 5 times what he gets. If the sum of the largest and smallest shares is 99, what is the sum of the other two shares?

(a) 99 (b) 81

(c) 64 (d) 54

33. If $3|x| + 5|y| = 8$ and $7|x| - 3|y| = 48$, then find the value of $x + y$.

(a) 5 (b) −4

(c) 4 (d) The value does not exist

34. The cost of 2 puffs, 14 cups of coffee and 5 pizzas is ₹356. The cost of 20 puffs, 7 cups of coffee and 15 pizzas is ₹830. Find the cost of 38 puffs and 25 pizzas. (in ₹)

(a) 1296 (b) 1104

(c) 1304 (d) 1034

35. A father's present age is 6 times his son's present age. Thirty years hence the father's age will be ten years less than twice the son's age. After how many years will the son's age be half of the father's present age?

(a) 20 (b) 30

(c) 10 (d) 15

36. Ramesh had a total of 30 coins in his purse of denominations ₹5 and ₹2. If the total amount with him is ₹120, then find the number of ₹5 coins.

(a) 20 (b) 10

(c) 4 (d) 12

37. Mukesh has some goats and hens in his shed. Upon counting, Mukesh found that the total number of legs is 112 and the total number of heads is 40. Find the number of hens in his shed.

(a) 18 (b) 20

(c) 22 (d) 24

38. If the length and breadth of a room are increased by 1 m each, its area would increase by 31 m^2. If the length is increased by 1 m and breadth is decreased by 1 m, the area would decrease by 9 m^2. Find the area of the floor of the room, in m^2.

(a) 200 (b) 209

(c) 250 (d) 199

39. A hybrid mango tree, whose life span is 10 years, starts giving fruits from the first year onwards. In the nth year it produces $11n$ raw mangoes. But during the first half of the tree's life, every year, a certain number, which is constant, fail to ripen into fruits. During the second half of the tree's life, every year,the number of raw fruits that fail to ripen is half the corresponding number in the first half of the tree's life. In the fourth year of the tree's life, it produces 36 ripe mangoes. How many mangoes ripen during the 9th year of the tree's life?

(a) 100 (b) 96

(c) 95 (d) 86

40. A teacher wanted to distribute 900 chocolates among the students of a class. Each boy received 12 chocolates and each girl received 6 chocolates. If each girl had been given 10 chocolates, then each boy would have received 5 chocolates. Find the number of students of the class.

(a) 80 (b) 90

(c) 100 (d) 110

41. Sridevi purchased cakes of two varieties of soap, Lux and Dove—spending a total ₹360. If each Lux costs ₹30 and each Dove costs ₹40, then in how many different combinations could she have purchased the cakes?

(a) 3 (b) 4

(c) 5 (d) 2

42. Venu has as many sisters as he has brothers. If Karuna, Venu's sister has thrice as many brothers as shehas sisters, then Venu has how many sisters?

(a) 1 (b) 2
(c) 3 (d) 4

43. The cost of two pencils, five erasers and eight sharpeners is ₹47. The cost of three pencils, three erasers and seven sharpeners is ₹42. Find the cost of twelve pencils, three erasers and eighteen sharpeners. (in ₹)

(a) 37 (b) 92
(c) 138 (d) 111

44. Total cost of 15 erasers and 25 pencils is ₹185 and the total cost of 9 erasers and x pencils is ₹106. Which of the following cannot be the value of x?

(a) 12 (b) 10
(c) 13 (d) 15

45. In a fraction, if the numerator is decreased by 1 and the denominator is increased by 1, then the resulting fraction is $\frac{1}{4}$. Instead, if the numerator is increased by 1 and the denominator is decreased by 1, then the resulting fraction is $\frac{2}{3}$. Find the difference of the numerator and the denominator of the fraction.

(a) 2 (b) 3
(c) 4 (d) 5

46. A two-digit number is such that, it exceeds the sum of the number formed by reversing the digits and sum of the digits by 4. Also, the original number exceeds the reversed number by 18. Find the product of the digits.

(a) 48 (b) 36
(c) 42 (d) 56

47. Bhanu has a total of 40 coins of denominations 30 paise and 10 paise. The total amount with him is ₹9. Find the number of 10 paise coins with him.

(a) 25 (b) 35
(c) 15 (d) 20

48. A father's present age is seven years less than 30 times of what his son's age was 20 years ago. Also, the father's present age is 31 years more than his son's present age. Find the sum of their present ages, in years.

(a) 75 (b) 74
(c) 73 (d) 72

49. A and B, have some coins. If A gives 100 coins to B, then B will have twice the number of coins left with A. Instead, if B gives 40 coins to A, then A will have thrice the number of coins left with B. How many more coins does A have than B?

(a) 64 (b) 88
(c) 75 (d) 96

50. Snehal can row 28 km downstream and 12 km upstream in 5 hours. He can row 21 km downstream and 10 km upstream in 4 hours. Find the speed of Snehal in still water.

(a) 9 kmph (b) 8 kmph
(c) 6 kmph (d) 5 kmph

51. The ratio of monthly incomes of Mr X and Mr Y is 3 : 4 and the ratio of their monthly expenditures is 5 : 7. If the ratio of their monthly savings is 3 : 2 and Mr X saves ₹500 more than Mr Y per month, then find the monthly income of Mr Y.

(a) ₹35000 (b) ₹32000
(c) ₹26000 (d) ₹22000

52. A two-digit number is seven times the sum of its digits. The number formed by reversing the digits is 6 more than half of the original number. Find the difference of the digits of the given number.

(a) 2 (b) 3
(c) 4 (d) 5

53. Sanjana travels 660 km, partly by train and partly by car. If she covers 300 km by train and the rest by car, it takes 13.5 hours. But, if she travels 360 km by train and the rest by car, she takes 30 minutes longer. Find the time taken by Sanjana if she travels 660 km by car. (in hours)

(a) 13 (b) 14
(c) 12 (d) 11

54. In a test of 50 questions, each correct answer fetches two marks and each wrong answer fetches $-\frac{1}{2}$ marks. A candidate attempted all the questions

PRACTICE QUESTIONS

and scored 40 mark. How many questions did he attempt correctly?

(a) 24 (b) 26

(c) 22 (d) 20

55. The average weight of the students of a class is 60 kg. If eight new students of average weight 64 kg join the class, the average weight of the entire class becomes 62 kg. How many students were there in the class initially?

(a) 12 (b) 10

(c) 8 (d) 14

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 5
2. zero
3. 150 cm
4. 6
5. Does not exist
6. Positive
7. Negative
8. 10
9. Cannot be determined
10. False
11. False
12. 3
13. 4 and 1
14. 5
15. 4

Short Answer Type Questions

16. $a = 3$ and $b = -1$
17. ₹15, ₹24
18 $\frac{4}{7}$
19. $x = \frac{1}{2}, y = \frac{1}{4}$
20. $x = \frac{-1}{24}, y = \frac{7}{24}$
21. ₹72000, ₹1000
22. 966
23. Number of 25 paise coins is 8 and number of 50 paise coins is 27.
24. $\frac{5}{13}$
25. $x = a^3$, $y = b^3$

Essay Type Questions

26. $x = 1$, $y = 3$, $z = 5$
27. $x = 4$ and $y = 5$
28. 13000, 12000
29. $\left(\frac{1}{2}, 1\right)$
30. 2

CONCEPT APPLICATION

Level 1

1. (d) **2.** (d) **3.** (c) **4.** (d) **5.** (c) **6.** (d) **7.** (c) **8.** (b) **9.** (b) **10.** (a)
11. (b) **12.** (b) **13.** (b) **14.** (b) **15.** (c)

Level 2

16. (b) **17.** (a) **18.** (b) **19.**(c) **20.** (b) **21.** (b) **22.** (d) **23.** (b) **24.** (b) **25.** (c)
26. (a) **27.** (c) **28.** (b)

Level 2

29. (a) **30.** (c) **31.** (b) **32.** (b) **33.** (d) **34.** (c) **35.** (c) **36.** (a) **37.** (d) **38.** (b)
39. (c) **40.** (d) **41.** (d) **42.** (b) **43.** (d) **44.** (d) **45.** (c) **46.** (a) **47.** (c) **48.** (a)
49. (b) **50.** (a) **51.** (d) **52.** (c) **53.** (d) **54.** (b) **55.** (c)

CONCEPT APPLICATION

Level 1

1. Condition for infinite solutions is, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
2. For unique solution, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.
3. If two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2x + c_2 = 0$ are inconsistent, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.
4. (i) Semi perimeter $(s) = \frac{a+b+c}{2}$.

   (ii) $a + 5 = s$; $b + 3 = s$; $c + 2 = s$.
5. If $x = y$, then $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$.
6. Add two equations.
7. Each cow has 4 legs and each hen has 2 legs.
8. Frame the linear equation and write $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.
9. Solve the first two equations and substitute(x, y) in the third equation.
10. Take any consecutive integers for a_1, b_1, c_1, a_2, b_2 and c_2 and solve.

    Or,

    Substitute any convenient consecutive integers for a_1, b_1, c_1, a_2, b_2 and c_2 and solve.
11. Frame linear equation by taking length and breadth as l and b respectively.
12. Frame linear equations and solve.
13. (i) Let the present ages of A and B be x years and y years respectively.

    (ii) $y - (x - y) = \frac{x}{2} - 4$.

    (iii) $x + y = 61$.
14. If the system of equations $a_1x + b_1y + c_1$, $a_2x + b_2y + c_2 = 0$ is inconsistent, $\frac{a_1}{a_2} = \frac{b_1}{b_2} + \frac{c_1}{c_2}$.
15. Let the cost of each book and each pencil and each sharpener be b, p and s respectively. Then frame the equations.

Level 2

16. (i) Let the number of sons and daughters that Dheeraj's parents have be b and d respectively.

    (ii) Form the equations and solve them.
17. Let the cost of each book and each pencil and each sharpener be b, p and s respectively. Then frame the equations.
19. Let the fraction be $\frac{n}{a}$ and frame the linear equations.
20. (i) Let the number of ₹1, ₹2 and ₹5 coins be x, y and z respectively.

    (ii) From the information given, obtain an equation in y and z and proceed.
21. If (p, q) is a solution of $ax + by + c = 0$ and $bx + ay + c = 0$, then $p = q$.
23. (i) $10x + y = 9(x + y) - 17$.

    (ii) $10x + y = 13(x - y) + 21$.

    (iii) Solve the above equations.
24. Frame linear equations and solve.
25. (i) Let the three digit number be $100x + 10y + z$.

    (ii) Frame the equations and solve them.
26. BDAC are in sequential order from first to last.
27. CBDA are in sequential order from first to last.
28. ABCD is the required sequential order.

Level 3

29. (i) $x + y + z = 100$, $2x - y - \frac{1}{2}z = 135$; $2x - \frac{y}{2} - z = 133$

    where x, y, and z are the number of correct answers, wrong answers and unattempted questions respectively.

    (ii) Solve the above equations.

30. (i) Let r, s, t, v, be the amounts, with Ram, Shyam, Tarun, Varun respectively.

(ii) $r = \frac{1}{2}(s + t + v)$, $s = \frac{1}{3}(r + t + v)$,

$t = \frac{1}{4}(r + s + v)$, $v + s + t + v = 240$.

(iii) Solve the above equation set to get v.

32. Let the total amounts be $4B$ or $5C$ or $6D$ and $B + D = 99$.

33. **(i)** Let $|x| = a$ and $|y| = b$ and solve linear equations which are in a and b.

(ii) Let $|x| = k_1$ and $|y| = k_2$ and solve them.

(iii) $|x| = \pm x$, $|y| = \pm y$.

34. Let the cost of each puff be p.

Let the cost of each coffee be c.

Let the cost of each pizza be z.

Then $2p + 14c + 5z = 356$ (1)

Also, $20p + 7c + 15z = 830$ (2)

Multiply Eq. (2) by 2, we have

$40p + 14c + 30z = 1660$ (3)

Subtracting Eq. (1) from Eq. (3), we have

$38p + 25z = 1304$.

35. Let the present ages of father and son be f

and s respectively.

Then, $f = 6s$ (1)

The second condition gives

$f + 30 = 2(s + 30) - 10 \Rightarrow f = 2s + 20$

$\therefore 6s - 2s = 20$ ($\because$ from Eq. (1))

$\Rightarrow s = 5$ and $f = 30$

Half the father's present age is 15. After 10 years, the son's age will be 15.

36. Let the number of 5-rupee coins be x.

Let the number of 2-rupee coins be y.

$\therefore x + y = 30$ (1)

Also $5x + 2y = 120$ (2)

Multiplying Eq. (1) by 2, we have

$2x + 2y = 60$ (3)

subtracting Eq. (3) from Eq. (2),

$3x = 60$

$x = 20$

37. Let the number of goats and hens, be x and y respectively.

Given, $4x + 2y = 112$ (1)

$x + y = 40$ (2)

Solving Eqs. (1) and (2), we get

$x = 16$ and $y = 24$

$\therefore$ Number of hens = 24

38. Let the length of the room = 1 m

Breadth of the room = b m

Area of the room = lb m^2.

The length and breadth are increased by 1 m each and the area increases by 31 m^2

$(l + 1)(b + 1) = lb + 31$

$\Rightarrow lb + l + b + 1 = lb + 31$

$\Rightarrow l + b = 30$ (1)

If length increased by 1 m and breadth decreased by 1 m, then the area decreases by 9 m^2, i.e.,

$(l + 1)(b - 1) = lb - 9$

$\Rightarrow lb - l + b - 1 = lb - 9$

$\Rightarrow l - b = 8$ (2)

On solving Eq. (1) and Eq. (2), we get

$l = 19$, $b = 11$

$\Rightarrow$ Area of the floor = lb = 19 (11)

= 209 m^2

39. Every year in the first half of the tree's life, the number of mangoes that cannot become ripen fruit is y (say).

The number of fruits produced in the xth year = $11x - y$

Number of fruits produced in 4th year = 36

$\Rightarrow 11(4) - y = 36 \Rightarrow y = 8$

Number of fruits produced in xth year of the second half-life = $11x - \frac{y}{2}$

Number of fruits produced in the 9th year

$= 11(9) - \frac{8}{2} = 99 - 4 = 95.$

HINTS AND EXPLANATION

40. Let there be x boys and y girls in the class.

Case 1:

$12x + 6y = 900$

$2x + y = 150$ (1)

Case 2:

$5x + 10y = 900$

$x + 2y = 180$ (2)

Solving (1) and (2), we get $x = 40$ and $y = 70$.

$\therefore$ Number of students in the class $= 70 + 40 = 110$.

41. Let the number of Lux cakes be x and Dove cakes be y.

Given $30x + 40y = 360$

$\Rightarrow 3x + 4y = 36$

$\Rightarrow x = \dfrac{36 - 4y}{3}$

$\Rightarrow x = 12 - \dfrac{4y}{3}$

As x and y are positive integers, $y = 3$ or 6

$\Rightarrow x = 8, 1$

There are two combinations.

42.

No. of Girls	No. of Boys
x	$(x + 1)$ (including Venu)
$(x - 1)$ (without Karuna)	$3(x - 1)$ (including Venu)

$\Rightarrow 3(x - 1) = x + 1$

$\Rightarrow 2x = 4 \Rightarrow x = 2.$

43. Let the cost of each pencils be P.

Let the cost of each sharpner be s.

Let the cost of each eraser be e.

Then, $2p + 5e + 8s = 47$ (1)

$3p + 3e + 7s = 42$ (2)

Multiply Eq. (2) by 2 and subtracting Eq. (1) from Eq. (2), we have

$4p + e + 6s = 37$ (3)

Multiply Eq. (3) by 3, we get

$12p + 3e + 18s = 111.$

44. Let the cost of each eraser and each pencil be ₹e and ₹p respectively.

$\Rightarrow 15e + 25p = 185$ and $9e + xp = 106$

As the cost of these articles is unique,

$\dfrac{15}{9} \neq \dfrac{25}{x}$

$\Rightarrow x \neq 15.$

45. Let the fraction be $\dfrac{x}{y}$.

Case 1:

$\dfrac{x-1}{x+1} = \dfrac{1}{4} \quad \Rightarrow \quad 4x - 4 = y + 1$

$\Rightarrow 4x - y = 5$ (1)

Case 2:

$\dfrac{x+1}{y-1} = \dfrac{2}{3}$

$\Rightarrow 3x + 3 = 2y - 2$

$\Rightarrow 3x - 2y + 5 = 0$ (2)

Solving Eqs. (1) and (2), we get $x = 3$ and $y = 7$.

$\therefore$ The required difference is 4.

46. Let x, y be the tens and units digits respectively.

Sum of the digits $= x + y$

Value of the number $= 10x + y$

$10x + y = x + y + 4 + 10y + x$

$8x - 10y = 4$ or $4x - 5y = 2$ (1)

Also given that

$10x + y = 10y + x + 18$

$x - y = 2$ (2)

Multiply Eq. (2) by 4, we have

$4x - 4y = 8$ (3)

Subtract Eq. (1) from Eq. (3), we get

$y = 6$

Substitute $y = 6$ in Eq. (2), we have $x = 8$

The product of the digits $= 48$

HINTS AND EXPLANATION

47. Let the total number of 30 paise coins with him = x.

Let the total number of 10 paise coins with him = y.

Then $x + y = 40$

Also, $30x + 10y = 900$

Solving for x and y we have $y = 15$.

48. Let the present age of father be f years.

Let son's present age be s years.

Then $f = 30(s - 20) - 7$ (1)

$f = s + 31$ (2)

$s + 31 = 30s - 600 - 7$

$638 = 29s$

$22 = s$

$\Rightarrow f = 53$

Sum of their ages = 75 years.

49. Let A and B have x and y coins respectively.

Case 1:

If A gives 100 coins to B.

$y + 100 = 2(x - 100)$

$2x - y = 300$ (1)

Case 2:

If B gives 40 coins to A

$x + 40 = 3(y - 40)$

$\Rightarrow x - 3y + 160 = 0$ (2)

Solving Eqs. (1) and (2), we get

$x = 212$ and $y = 124$

$\therefore$ The required difference = 212 − 124 = 88.

50. Let Snehal's speed and the speed of the stream be x kmph and y kmph respectively.

Given, $\dfrac{28}{x+y} + \dfrac{12}{x-y} = 5$ (1)

and, $\dfrac{21}{x+y} + \dfrac{10}{x-y} = 4$ (2)

Let $\dfrac{1}{x+y} = a$ and $\dfrac{1}{x-y} = b$, then

Eq. (1) $\Rightarrow 28a + 12b = 5$ (3)

and Eq. (2) $\Rightarrow 21a + 10b = 4$ (4)

Solving Eqs. (3) and (4), we get

$a = \dfrac{1}{14}$ and $b = \dfrac{1}{4}$

$\Rightarrow x + y = 14$ and $x - y = 4$

$\Rightarrow x = 9$ and $y = 5$

$\therefore$ Snehal's speed in still water = 9 kmph

51. Let the monthly income of X and Y be $3x$, $4x$ and monthly expenditures be $5y$, $7y$

Savings of $X = 3x - 5y$

Savings of $Y = 4x - 7y$

Given, $\dfrac{3x-5y}{4x-7y} = \dfrac{3}{2}$ and

$(3x - 5y) - (4x - 7y) = 500$

$\Rightarrow 6x - 10y = 12x - 21y$

$\Rightarrow 6x - 11y = 0$ (1)

and $-x + 2y = 500$ (2)

Solving Eqs. (1) and (2), we get $y = 3000$ and $x = 5500$

$\therefore$ Monthly income of Mr Y = ₹$4x$ = ₹22000.

52. Let the two-digit number be $10x + y$

Given, $10x + y = 7(x + y)$

$\Rightarrow 3x = 6y \Rightarrow x = 2y$ (1)

Also, $10y + x = \dfrac{1}{2}(10x + y) + 6$

$\Rightarrow -8x + 19y = 12$

$\Rightarrow -16y + 19y = 12$ ($\because$ from Eq. (1))

$\Rightarrow y = 4$ and $x = 8$ $\therefore$ $x - y = 4$.

53. Let the speed of the train and car be t kmph and c kmph respectively.

$\therefore \dfrac{300}{t} + \dfrac{360}{c} = 13.5$ (1)

$\dfrac{360}{t} + \dfrac{300}{c} = 14$ (2)

On solving Eqs. (1) and (2), we get

$t = 40$ km/h and $c = 60$ kmph

The time taken by car to travel 660 km

$= \dfrac{660}{60} = 11$ hours.

54. Let the number of questions which are correct be c and the questions which are wrong be w.

$$c + w = 50 \quad (1)$$

$$2c - \frac{w}{2} = 40$$

$$4c - w = 80 \quad (2)$$

From Eqs. (1) and (2), we have $\Rightarrow 5c = 130 \Rightarrow c = 26$

55. Let there be n students in the class. The total weight of the students after the eight new students join the class.

$$60n + 64(8) = 62(n + 8)$$

$$\Rightarrow 60n + 64(8) = 62n + 62(8)$$

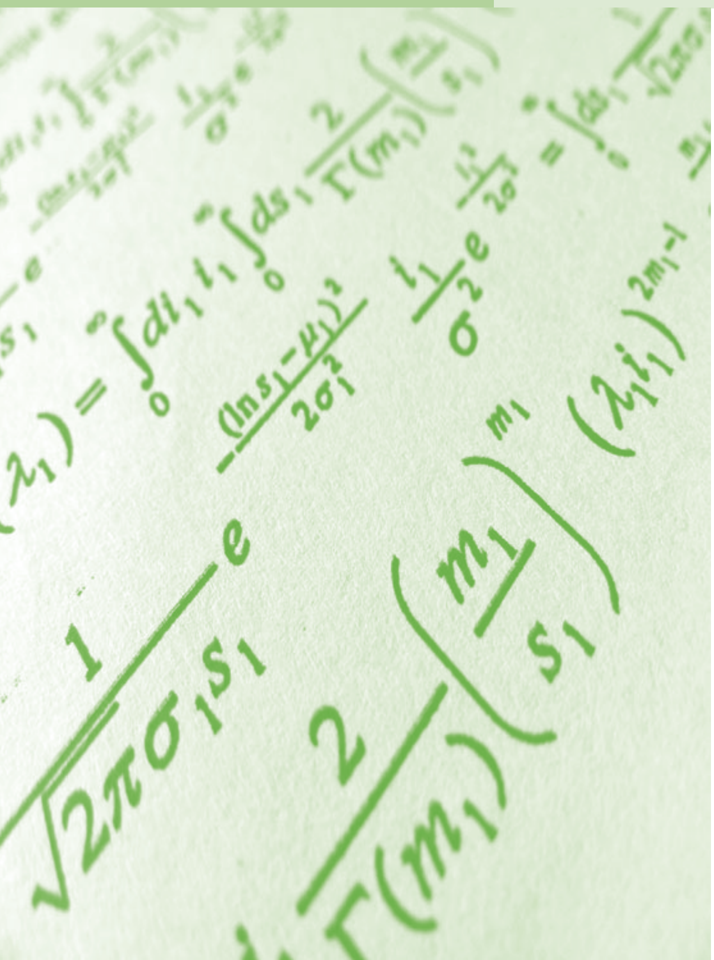
$$\Rightarrow 62n - 60n = 64(8) - 62(8)$$

$$\Rightarrow 2n = 8(2) \Rightarrow n = 8$$

$\therefore$ Initially, there were 8 students in the class.

Chapter 4

Quadratic Equations and Inequalities



REMEMBER

Before beginning this chapter, you should be able to:

- Know the terms, such as quadratic expression, zero in a equation
- Understand reciprocal equation and maximum or minimum value of a quadratic equation

KEY IDEAS

After completing this chapter, you would be able to:

- Find the roots of a quadratic equation by factorization, using a formula, and graphical method
- Study nature and signs of the roots
- Construct a quadratic equation
- Solve word problems on quadratic equations

INTRODUCTION

Very often we come across many equations involving several powers of one variable. If the indices of all these powers are integers then the equation is called a polynomial equation. If the highest index of a polynomial equation in one variable is two, then it is a quadratic equation.

A quadratic equation is a second degree polynomial in x usually equated to zero. In other words, for an equation to be a quadratic, the coefficient of x^2 should not be zero and the coefficients of any higher power of x should be 0.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real). The following are some examples of quadratic equations.

1. $x^2 - 5x + 6 = 0$ (1)
2. $x^2 - x - 6 = 0$ (2)
3. $2x^2 + 3x - 2 = 0$ (3)
4. $2x^2 + x - 3 = 0$ (4)

ROOTS OF THE EQUATION

Just as a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x has two values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or complex.

The roots of the four quadratic equations given above are:

For Eq. (1), $x = 2$ and $x = 3$

For Eq. (2), $x = -2$ and $x = 3$

For Eq. (3), $x = \frac{1}{2}$ and $x = -2$

For Eq. (4), $x = 1$ and $x = \frac{-3}{2}$

In general, the roots of a quadratic equation can be found in two ways:

1. By factorizing the expression on the left hand side.
2. By using the standard formula.

   All the expressions may not be easy to factorize, whereas applying the formula is simple and straightforward.

Finding the Roots by Factorization

If the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is written in the form, $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first express it in the form of $(x - \alpha)(x - \beta) = 0$, i.e., the left hand side, $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorized.

For this purpose, we should go through the following steps with the help of above equations.

Consider Equation (1)

Step 1: The equation is, $x^2 - 5x + 6 = 0$. Here $a = 1$, $b = -5$ and $c = 6$.

First write down b (the coefficient of x) as the sum of two quantities whose product is equal to ac. In this case, -5 has to be written as the sum of two quantities whose product is 6. We write -5 as $(-3) + (-2)$, because the product of (-3) and (-2) is equal to 6.

Step 2: Now, rewrite the equation. In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$.

Step 3: Consider the first two terms and rewrite them together after taking out their common factor. Similarly, the third and the fourth terms should be rewritten after taking out their common factor. In this process, we should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and fourth terms (after removing their common factor).

In this case, the equation can be rewritten as $x(x - 3) -2(x - 3) = 0$; now $(x - 3)$ is a common factor.

Step 4: If we take out $(x - 3)$ as the common factor, we can rewrite the given equation as $(x - 3)(x - 2) = 0$.

We know that if α and β are the roots of the given quadratic equation $(x - \alpha)(x - \beta) = 0$.

Hence, the roots of the given equation are 3 and 2.

Consider Equation (2)

The equation is, $x^2 - x - 6 = 0$. Here, the coefficient of x is -1 which can be rewritten as $(-3) + (+2)$, because the product of (-3) and 2 is -6, which is equal to 'ac' (1 multiplied by -6). Then, we can rewrite the equation as $(x - 3)(x + 2) = 0$ to get the roots as 3 and -2.

Consider Equation (3)

The equation is, $2x^2 + 3x - 2 = 0$. Here, the co-efficient of x is 3, which can be rewritten as $(+4) + (-1)$ so that their product is -4, which is the value of 'ac' (-2 multiplied by 2). Then, we can rewrite the equation as $(2x - 1)(x + 2) = 0$, obtaining the roots as $\frac{1}{2}$ and -2.

Consider Equation (4)

The equation is, $2x^2 + x - 3 = 0$. Here, the coefficient of x is 1, which can be rewritten as $(+3) + (-2)$ because their product is -6, which is equal to 'ac' (2 multiplied by -3). Then we can rewrite the given equation as $(x - 1)(2x + 3) = 0$ to get the roots as 1 and $\frac{-3}{2}$.

Finding the Roots by Using the Formula

For the quadratic equation $ax^2 + bx + c = 0$, we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Sum and Product of Roots of a Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$, let α and β be the roots, then

the sum of the roots, $(\alpha + \beta) = \frac{-b}{a}$

the product of the roots, $(\alpha\beta) = \frac{c}{a}$.

These two rules will be very helpful in solving problems on quadratic equation.

Nature of the Roots

We have already learnt that the roots of a quadratic equation with real coefficients can be real or complex. When the roots are real, they can be rational or irrational and, also, they can be equal or unequal.

Consider the expression $b^2 - 4ac$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the DISCRIMINANT of the quadratic equation.

A quadratic equation has real roots only if $b^2 - 4ac \geq 0$.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation are complex conjugates.

The following table gives us a clear idea about the nature of the roots of a quadratic equation, when a, b and c are all rational.

Condition	Nature of Roots
when $b^2 - 4ac < 0$	the roots are complex conjugates
when $b^2 - 4ac = 0$	the roots are rational and equal.
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal.
when $b^2 - 4ac > 0$ and not a perfect square	the roots are irrational and unequal.

Notes

1. Whenever the roots of the quadratic equation are irrational, (a, b, c being rational), are of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e., whenever $a + \sqrt{b}$ is one root of a quadratic equation, $a - \sqrt{b}$ is the other root of the quadratic equation and vice-versa. In other words, if the roots of a quadratic equation are irrational, then they are conjugate to each other.
2. If the sum of the coefficients of a quadratic equation, say $ax^2 + bx + c = 0$, is zero, then its roots are 1 and $\frac{c}{a}$.

That is, if $a + b + c = 0$, then the roots of $ax^2 + bx + c = 0$ are 1 and $\frac{c}{a}$.

Signs of the Roots

We can comment on the signs of the roots, i.e., whether the roots are positive or negative, based on the sign of the sum of the roots and the product of the roots of the quadratic equation. The following table indicates the signs of the roots when the signs of the sum and the product of the roots are given.

Sign of Product of the Roots	Sign of Sum of the Roots	Sign of the Roots
+ve	+ve	Both the roots are positive.
+ve	−ve	Both the roots are negative.
−ve	+ve	One root is positive and the other negative. The numerically greater root is positive.
−ve	−ve	One root is positive and the other negative. The numerically greater root is negative.

CONSTRUCTING A QUADRATIC EQUATION

We can build a quadratic equation in the following cases:

1. When the roots of the quadratic equation are given.
2. When the sum of the roots and the product of the roots of the quadratic equation are given.

Case 1: If the roots of the quadratic equation are α and β, then its equation can be written as $(x - \alpha)(x - \beta) = 0$, i.e., $x^2 - x\,(\alpha + \beta) + \alpha\beta = 0$.

Case 2: If p is the sum of the roots of the quadratic equation and q is their product, then the equation can be written as $x^2 - px + q = 0$.

Constructing a New Quadratic Equation by Changing the Roots of a Given Quadratic Equation

If we are given a quadratic equation, we can build a new quadratic equation by changing the roots of this equation in the manner specified to us.

For example, consider the quadratic equation $ax^2 + bx + c = 0$ and let its roots be α and β respectively. Then, we can build new quadratic equations as per the following points:

1. A quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, i.e., the roots are reciprocal to the roots of the given quadratic equation, can be obtained by substituting $\left(\frac{1}{x}\right)$ for x in the given equation, which gives us $cx^2 + bx + a = 0$, i.e., we get the equation required by interchanging the coefficient of x^2 and the constant term.
2. A quadratic equation whose roots are $(\alpha + k)$ and $(\beta + k)$ can be obtained by substituting $(x - k)$ for x in the given equation.
3. A quadratic equation whose roots are $(\alpha - k)$ and $(\beta - k)$ can be obtained by substituting $(x + k)$ for x in the given equation.
4. A quadratic equation whose roots are $(k\alpha)$ and $(k\beta)$ can be obtained by substituting $\left(\frac{x}{k}\right)$ for x in the given equation.
5. A quadratic equation whose roots are $\left(\frac{\alpha}{k}\right)$ and $\left(\frac{\beta}{k}\right)$ can be obtained by substituting (kx) for x in the given equation.
6. A quadratic equation whose roots are $(-\alpha)$ and $(-\beta)$ can be obtained by replacing x by $(-x)$ in the given equation.

Finding the Roots of a Quadratic Equation by Graphical Method

First Method

First, let us learn how to draw the graph of $y = x^2$.

We assume certain real values for x, i.e., we substitute some values for x in $y = x^2$. We can find the corresponding values of y. We tabulate the values, as shown in next page:

x	5	4	3	2	1	0	−1	−2	−3	−4	−5
$y = x^2$	25	16	9	4	1	0	1	4	9	16	25

Plotting the points corresponding to the ordered pairs (5, 25), (4, 16), (3, 9), (2, 4), (1, 1), (0, 0) (−1, 1), (−2, 4), (−3, 9), (−4, 16) and (−5, 25) on the graph paper and joining them with a smooth curve we obtain the graph of $y = x^2$, as shown in the Fig. 4.1.

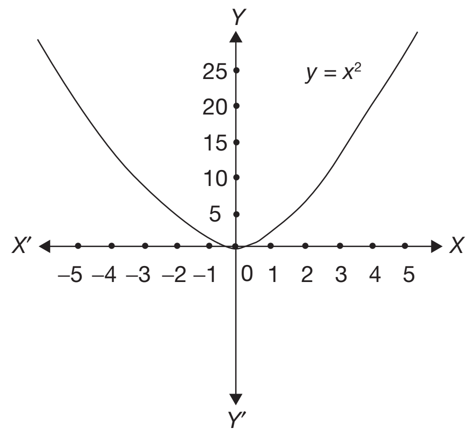


Figure 4.1

We observe the following about the graph of $y = x^2$.

1. It is a U shaped graph and it is called a parabola. The arms of the 'U' spread outwards.
2. For every value of x ($\neq 0$) we notice that y is always positive. Hence, the graph lies entirely in the first and second quadrants.
3. When $x = 0$, $y = 0 \Rightarrow y = x^2$ passes through origin.
4. The graph is symmetric about the Y-axis.
5. Using the graph of $y = x^2$, we can find the square of any real number as well as the square root of any non-negative real number.
   (i) for any given value of x, the corresponding value of y on the graph is its square and
   (ii) for any given value of y (≥ 0), the corresponding value of x on the graph is its square root.
6. The graph of $y = kx^2$, when $k > 0$ lies entirely in Q_1 and Q_2 and when $k < 0$ the graph lies entirely in Q_3 and Q_4 (see Fig. 4.2).

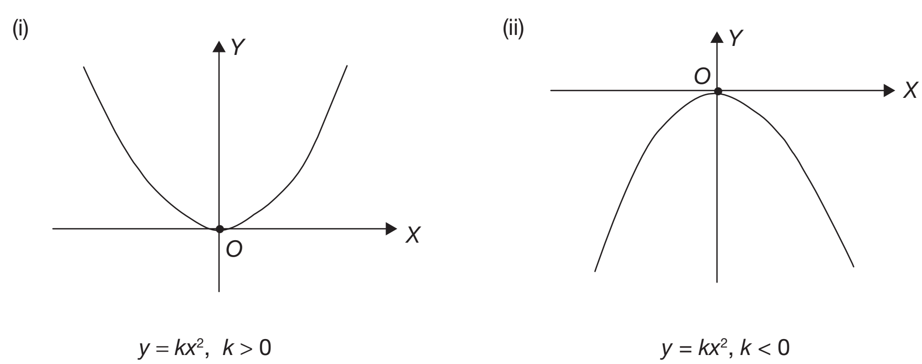


Figure 4.2

The method of solving the quadratic equation of the form $px^2 + qx + r = 0$, whose roots are real is shown in the following example.

EXAMPLE 4.1

Solve $x^2 - 5x + 6 = 0$ using the graphical method.

SOLUTION

Let $y = x^2 - 5x + 6$

Prepare the following table by assuming different values for x.

x	0	1	2	3	−1	− 2	4	5
x^2	0	1	4	9	1	4	16	25
$5x$	0	5	10	15	−5	−10	20	25
$y = x^2 - 5x + 6$	6	2	0	0	12	20	2	6

Plot the points (0, 6), (1, 2), (2, 0), (3, 0), (−1, 12), (−2, 20), (4, 2) and (5, 6) on the graph and join the points with a smooth curve, as shown in the Fig. 4.3.

Here, we notice that the given graph (parabola) intersects the X-axis at (2, 0) and (3, 0).

The roots of the given quadratic equation $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

∴ The roots of the given equation are the x-coordinates of the points of intersection of the curve with X-axis.

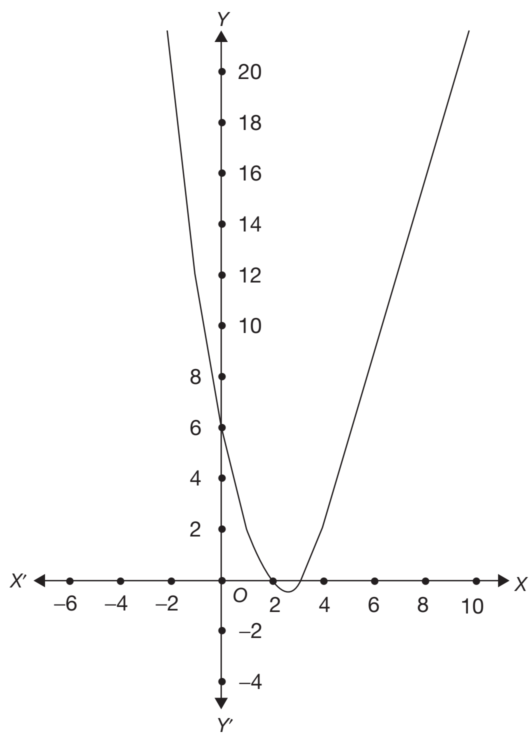


Figure 4.3

Notes

1. If the graph meets the X-axis at two distinct points, then the roots of the given equation are real and distinct (see Fig. 4.4).

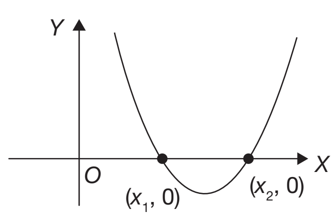


Figure 4.4

2. If the graph touches the X-axis at only one point, then the roots of the quadratic equation are real and equal (see Fig. 4.5).

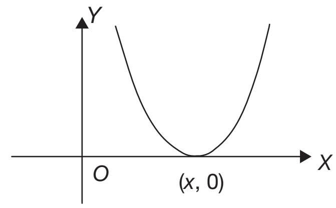


Figure 4.5

3. If the graph does not meet the X-axis, then the roots of the quadratic equation are not real, i.e., they are complex (see Fig. 4.6).

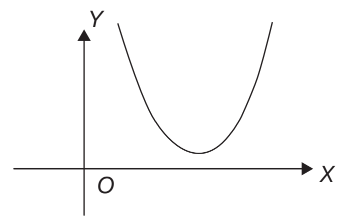


Figure 4.6

Second Method

We can also solve the quadratic equation $px^2 + qx + r = 0$ by considering the following equations:

$$y = px^2 \tag{1}$$

and

$$y = -qx - r \tag{2}$$

Clearly, $y = px^2$ is a parabola and $y = -qx - r$ is a straight line.

Step 1: Draw the graph of $y = px^2$ and $y = -qx - r$ on the same graph paper.

Step 2: Draw perpendiculars from the points of intersection of parabola and the straight line onto the X-axis. Let the points of intersection on the X-axis be $(x_1, 0)$ and $(x_2, 0)$.

Step 3: The x-coordinates of the points in Step (2), i.e., x_1 and x_2 are the two distinct roots of $px^2 + qx + r = 0$.

EXAMPLE 4.2

Solve $2x^2 - x - 3 = 0$.

SOLUTION

We know that the roots of $2x^2 - x - 3 = 0$ are the x-coordinates of the points of intersection of the parabola, $y = 2x^2$ and the straight line, $y = x + 3$.

(1) $y = 2x^2$

x	0	1	2	−1	−2
$y = 2x^2$	0	2	8	2	8

(2) $y = x + 3$

x	0	1	2	−1	−2	−3
$y = x + 3$	3	4	5	2	1	0

Draw the graph of $y = 2x^2$ and $y = x + 3$ (see Fig. 4.7).

Clearly, the perpendiculars drawn from the points of intersection of parabola and the line meet the X-axis at $\left(\frac{3}{2}, 0\right)$ and $(-1, 0)$.

$\therefore$ The roots of the given quadratic equation $2x^2 - x - 3 = 0$ are $\frac{3}{2}$ and -1.

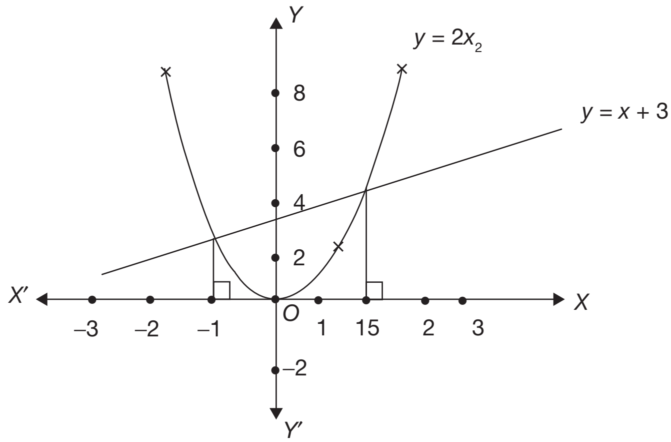


Figure 4.7

Notes

1. If the line meets the parabola at two points, then the roots of the quadratic equation are real and distinct.
2. If the line touches the parabola at only one point, then the quadratic equation has real and equal roots (see Fig. 4.8).

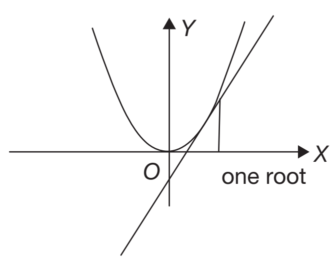


Figure 4.8

3. If the line does not meet the parabola, i.e., when the line and the parabola have no points in common, then the quadratic equation has no real roots. In this case, the roots of the quadratic equation are imaginary (see Fig. 4.9).

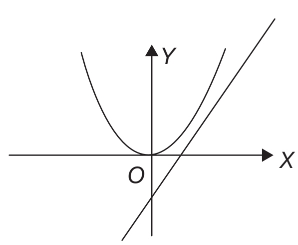


Figure 4.9

Equations of Higher Degree

The index of the highest power of x in the equation is called the degree of the equation. For example, if the highest power of x in the equation is x^3, then the degree of the equation is 3. An equation whose degree is 3 is called a cubic equation. A cubic equation will have three roots.

Note An equation whose degree is n will have n roots.

Maximum or Minimum Value of a Quadratic Expression

The quadratic expression $ax^2 + bx + c$ takes different values, as x takes different values.

$$\begin{aligned}
& ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + 2\left(\frac{b}{2a}\right)x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}
\end{aligned}$$

For all the values of x, as x varies from $-\infty$ to $+\infty$, (i.e., when x is real), the quadratic expression $ax^2 + bx + c$

1. has a minimum value if $a > 0$ (i.e., a is positive). The minimum value of the quadratic expression is $\frac{(4ac - b^2)}{4a}$ and it occurs at $x = \frac{-b}{2a}$.
2. has a maximum value if $a < 0$ (i.e., a is negative). The maximum value of the quadratic expression is $\frac{(4ac - b^2)}{4a}$ and it occurs at $x = \frac{-b}{2a}$.

EXAMPLE 4.3

Find the roots of the equation $x^2 + 3x - 4 = 0$.

SOLUTION

$$\begin{aligned}
& x^2 + 3x - 4 = 0 \\
\Rightarrow\ & x^2 - x + 4x - 4 = 0 \\
\Rightarrow\ & x(x - 1) + 4(x - 1) = 0 \\
\Rightarrow\ & (x + 4)(x - 1) = 0.
\end{aligned}$$

$\therefore\ x = -4$ or $x = 1$.

EXAMPLE 4.4

Find the roots of the equation $4x^2 - 13x + 10 = 0$.

SOLUTION

$$4x^2 - 13x + 10 = 0$$
$$\Rightarrow 4x^2 - 8x - 5x + 10 = 0$$
$$\Rightarrow 4x(x - 2) - 5(x - 2) = 0$$
$$\Rightarrow (4x - 5)(x - 2) = 0.$$

$\therefore\ x = \frac{5}{4}$ or $x = 2$.

EXAMPLE 4.5

Find the roots of the equation $26x^2 - 43x + 15 = 0$.

SOLUTION

We have to write 43 as the sum of two parts whose product should be equal to (26×15).

$$26 \times 15 = 13 \times 30 \text{ and } 13 + 30 = 43$$
$$\therefore\ 26x^2 - 43x + 15 = 0$$
$$\Rightarrow 26x^2 - 13x - 30x + 15 = 0$$
$$\Rightarrow (13x - 15)(2x - 1) = 0$$
$$\Rightarrow x = \frac{15}{13} \text{ or } x = \frac{1}{2}$$

We can also find the roots of the equation by using the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{43 \pm \sqrt{(43)^2 - (1560)}}{52}$$
$$= \frac{43 \pm \sqrt{(1849) - (1560)}}{52}$$
$$= \frac{43 \pm \sqrt{289}}{52} \Rightarrow x = \frac{43 \pm 17}{52}$$
$$\Rightarrow x = \frac{43 + 17}{52} \text{ or } \frac{43 - 17}{52} = \frac{60}{52} \text{ or } \frac{26}{52}$$
$$\therefore x = \frac{15}{13} \text{ or } \frac{1}{2}.$$

EXAMPLE 4.6

Discuss the nature of the roots of the equation $4x^2 - 2x + 1 = 0$.

SOLUTION

Discriminant $= (-2)^2 - 4(4)(1) = 4 - 16 = -12 < 0$.

Since the discriminant is negative, the roots are imaginary.

EXAMPLE 4.7

If the sum of the roots of the equation $kx^2 - 3x + 9 = 0$ is $\frac{3}{11}$, then find the product of the roots of that equation.

SOLUTION

Sum of roots of the equation $= \frac{3}{k} = \frac{3}{11}$ (given)

$\therefore k = 11.$

In the given equation, product of the roots $= \frac{9}{k}$.

As $k = 11$, product of the roots $= \frac{9}{11}$.

EXAMPLE 4.8

Form the quadratic equation whose roots are 2 and 7.

SOLUTION

Sum of the roots $= 2 + 7 = 9$

Product of the roots $= 2 \times 7 = 14$

We know that if p is the sum of the roots and q is the product of the roots of a quadratic equation, then its equation is $x^2 - px + q = 0$.

Hence, the required equation is $x^2 - 9x + 14 = 0$.

EXAMPLE 4.9

Form a quadratic equation with rational coefficients, one of whose roots is $3+\sqrt{5}$.

SOLUTION

If $(3+\sqrt{5})$ is one root, then the other root is $(3-\sqrt{5})$.

Sum of the roots $= 6$

Product of the roots $= 4$

Thus, the required equation is $x^2 - 6x + 4 = 0$.

EXAMPLE 4.10

A person can buy 15 books less for ₹900, when the price of each book goes up by ₹3. Find the original price and the number of copies he could buy at the initial price.

SOLUTION

Let the number of books bought initially for ₹900 be 'x'. The original price of the each book was $\frac{900}{x}$. Now the price of the each book is increased by ₹3.

That is, the new price of each book is ₹$\left(\frac{900}{x}\right) + 3$.

And the number of books bought is reduced by 15, i.e., $(x - 15)$.

Since the total amount spent is still ₹900, the product of the price of each book and the number of books are still 900.

$$\left[\left(\frac{900}{x}\right) + 3\right](x - 15) = 900$$

$$\Rightarrow (900 + 3x)(x - 15) = 900x$$

$$\Rightarrow 3x^2 + 855x - 13500 = 900x$$

$$\Rightarrow 3x^2 - 45x - 13500 = 0$$

$$\Rightarrow x^2 - 15x - 4500 = 0$$

$$\Rightarrow x^2 - 75x + 60x - 4500 = 0$$

$$\Rightarrow x(x - 75) + 60(x - 75) = 0$$

$$\Rightarrow (x - 75)(x + 60) = 0$$

$$\Rightarrow x = 75 \text{ or } -60.$$

Since x cannot be negative, $x = 75$.

Thus, the original price of the book $= \frac{900}{75} =$ ₹12.

EXAMPLE 4.11

If α and β are the roots of the equation $x^2 - 6x + 8 = 0$, then find the values of

(a) $\alpha^2 + \beta^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta}$

(c) $\alpha - \beta$ $(\alpha > \beta)$.

SOLUTION

From the given equation, we get $\alpha + \beta = 6$ and $\alpha\beta = 8$.

(a) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (6)^2 - 2(8) = 20$

(b) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{6}{8} = \frac{3}{4}$

(c) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$\Rightarrow (\alpha - \beta) = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \sqrt{6^2 - 4(8)}$$

$$\Rightarrow (\alpha - \beta) = \pm 2.$$

$$\therefore \alpha - \beta = 2, \quad (\because \alpha > \beta).$$

EXAMPLE 4.12

Solve for x: $3^{x+1} + 3^{2x+1} = 270$.

SOLUTION

$$3^{x+1} + 3^{2x+1} = 270$$
$$\Rightarrow 3 \cdot 3^x + 3^{2x} \cdot 3 = 270$$
$$\Rightarrow 3^x + 3^{2x} = 90.$$

Substituting $3^x = a$, we get,

$$a + a^2 = 90$$
$$\Rightarrow a^2 + a - 90 = 0$$
$$\Rightarrow a^2 + 10a - 9a - 90 = 0$$
$$\Rightarrow (a + 10)(a - 9) = 0$$
$$\Rightarrow a = 9 \text{ or } a = -10.$$

If $3^x = 9$, then $x = 2$.

If $3^x = -10$, which is not possible.

$\therefore\ x = 2$.

EXAMPLE 4.13

Solve $|x|^2 - 7|x| + 12 = 0$.

SOLUTION

Given equation is $|x|^2 - 7|x| + 12 = 0$

$$\Rightarrow (|x| - 3)(|x| - 4) = 0$$
$$\Rightarrow |x| = 3 \text{ or } |x| = 4$$
$$\Rightarrow x = \pm 3 \text{ or } x = \pm 4.$$

EXAMPLE 4.14

Solve $|x|^2 + 7|x| + 10 = 0$.

SOLUTION

Given equation is $|x|^2 + 7|x| + 10 = 0$

$$\Rightarrow (|x| + 2)(|x| + 5) = 0$$
$$\Rightarrow |x| = -2 \text{ or } |x| = -5.$$

But, the absolute value of any number can never be negative.

$\therefore$ No roots are possible for the given equation.

EXAMPLE 4.15

In writing a quadratic equation of the form $x^2 + bx + c = 0$, a student writes the coefficient of x incorrectly and finds the roots as -6 and 7. Another student makes a mistake in writing the constant term and finds the roots as 4 and 11. Find the correct quadratic equation.

(a) $x^2 + 15x - 42 = 0$ **(b)** $x^2 + x + 44 = 0$ **(c)** $x^2 - 15x - 42 = 0$ **(c)** $x^2 - x + 44 = 0$

HINTS

(i) Use sum of roots $= \frac{-b}{a}$, product of the roots $= \frac{c}{a}$.

(ii) Identify that the first student got the correct product and the second student got the correct sum.

EXAMPLE 4.16

A man bought 50 dozen fruits consisting of apples and bananas. An apple is cheaper than a banana. The number of dozens of apples he bought is equal to the cost per dozen of bananas in rupees and vice versa. If he had spent a total amount of ₹1050, find the number of dozens of apples and bananas he bought, respectively.

(a) 12 and 38 **(b)** 14 and 36 **(c)** 15 and 35 **(d)** 18 and 32

HINT

Form a quadratic equation in terms of apples or bananas.

EXAMPLE 4.17

If $3 \cdot 2^{2x+1} - 5 \cdot 2^{x+2} + 16 = 0$ and x is an integer, find the value of x.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4

HINTS

(i) Let $2^x = P$.

(ii) Frame the quadratic equation in terms of P and solve it.

EXAMPLE 4.18

If $(x + 1)\ (x + 3)\ (x + 5)\ (x + 7) = 5760$, find the real values of x.

(a) 5, −13 **(b)** −5, 13 **(c)** −5, −13 **(d)** 5, 13

HINTS

(i) $(a)\ (b)\ (c)\ (d) = e$ can be written as $(a\ d)\ (b\ c) = e$.

(ii) Write the equation in the form of a quadratic equation.

WORKOUT

For what value of x: $-3x^2 + 5x - 12$ has maximum value?

(a) $-\frac{5}{3}$ **(b)** $\frac{5}{6}$ **(c)** $-\frac{5}{6}$ **(d)** $\frac{5}{3}$

QUADRATIC INEQUATIONS

Consider the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$, where a, b and c are real numbers.

The quadratic inequations related to $ax^2 + bx + c = 0$ are $ax^2 + bx + c < 0$ and $ax^2 + bx + c > 0$.

Assume that $a > 0$.

The following cases arise:

Case 1: If $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has real and unequal roots.

Let α and $\beta(\alpha < \beta)$ be the roots.

Then,

$\therefore\ ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

1. If $x < \alpha$, then $(x - \alpha) < 0$ and $(x - \beta) < 0$.

   $\therefore\ ax^2 + bx + c > 0$.

2. If $\alpha < x < \beta$, then $(x - \alpha) > 0$ and $(x - \beta) < 0$

   $\therefore\ ax^2 + bx + c < 0$.

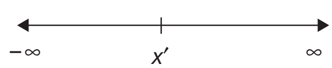
3. If $x > \beta$, then $x - \alpha > 0$ and $x - \beta > 0$.

   $\therefore\ ax^2 + bx + c > 0$.

Case 2: If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has real and equal roots.

Let x' be the equal root.

$\Rightarrow ax^2 + bx + c = a(x - x')(x - x')$



1. If $x < x'$. Then $x - x' < 0$.

   $\therefore\ ax^2 + bx + c > 0$.

2. If $x > x'$, then $x - x' > 0$.

   $\therefore\ ax^2 + bx + c > 0$.

Case 3: If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has imaginary roots.

In this case, $ax^2 + bx + c > 0$, $\forall\ x \in R$.

The above concept can be summarized as:

1. If $\alpha < x < \beta$, then $(x - \alpha)(x - \beta) < 0$ and vice-versa.
2. If $x < \alpha$ and $x > \beta$ $(\alpha < \beta)$, then $(x - \alpha)(x - \beta) > 0$ and vice-versa.

Note If $a < 0$ and $b^2 - 4ac < 0$, then the solution for $ax^2 + bx + c > 0$ does not exist.

EXAMPLE 4.19

Solve the inequation $x^2 + x - 6 < 0$.

SOLUTION

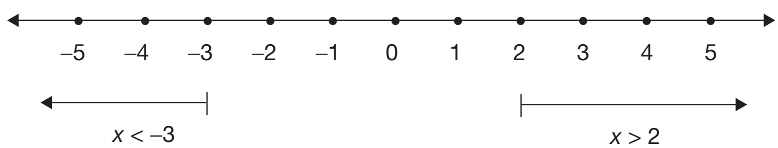
Given inequation is $x^2 + x - 6 < 0$.

$$\Rightarrow (x + 3)(x - 2) < 0$$

$$\Rightarrow (x + 3) < 0,\ (x - 2) > 0 \text{ or } (x + 3) > 0,\ (x - 2) < 0$$

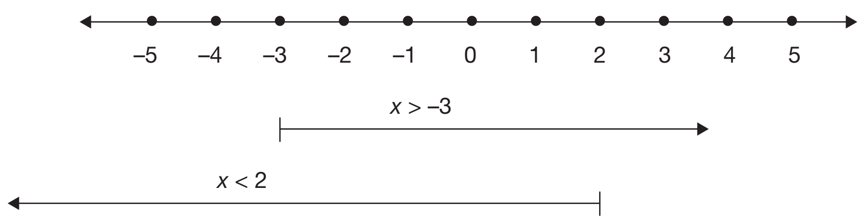
$$\Rightarrow x < -3,\ x > 2 \text{ (Case 1) (or) } x > -3,\ x < 2 \text{ (Case 2)}$$

Case 1: $x < -3$ and $x > 2$



There exists no value of x so that $x < -3$ and $x > 2$ (as there is no overlap of the regions). Hence, in this case no value of x satisfies the given inequation.

Case 2: $x > -3$ and $x < 2$



All the points in the overlapping region, i.e., $-3 < x < 2$, satisfy the inequation. Hence, the solution set of the inequation. $x^2 + x - 6 < 0$ is $\{x/-3 < x < 2\}$ or $(-3, 2)$.

EXAMPLE 4.20

Solve for x: $x^2 - 4x + 3 \geq 0$.

SOLUTION

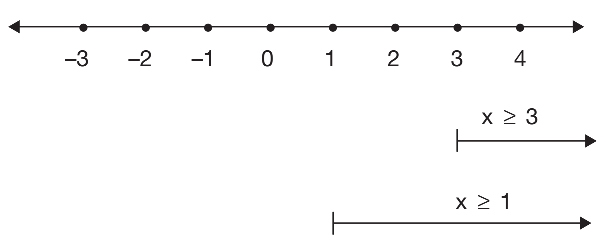
Given inequation is $x^2 - 4x + 3 \geq 0$.

$$\Rightarrow (x - 1)(x - 3) \geq 0$$

$$\Rightarrow x - 1 \geq 0;\ x - 3 \geq 0 \text{ or } x - 1 \leq 0;\ x - 3 \leq 0$$

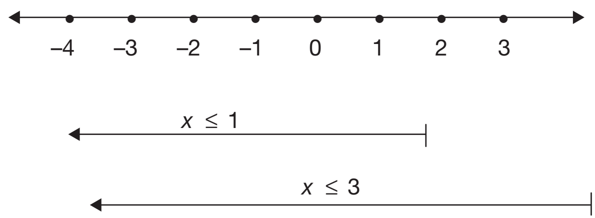
$$\Rightarrow x \geq 1;\ x \geq 3 \text{ (Case 1) (or) } x \leq 1;\ x \leq 3 \text{ (Case 2)}$$

Case 1: $x \geq 1$ and $x \geq 3$.



All the points in the overlapping region, i.e., $x \geq 3$, satisfy the given inequation.

Case 2: $x \leq 1$ and $x \leq 3$.



All the points in the overlapping region, i.e., $x \leq 1$, satisfy the given inequation.

Hence, the solution for the given inequation is $x \in (-\infty, 1] \cup [3, \infty)$.

EXAMPLE 4.21

Solve $x^2 + 6x + 13 > 0$.

SOLUTION

Given inequation is $x^2 + 6x + 13 > 0$.

Here, factorization is not possible.

Rewriting the given inequation we get,

$$(x^2 + 6x + 9) + 4 > 0$$

$$\Rightarrow (x + 3)^2 + 4 > 0.$$

We know that $(x + 3)^2 \geq 0 \; \forall \; x \in R$,

$$(x + 3)^2 + 4 \geq 4 > 0 \; \forall \; x \in R.$$

$\therefore$ The required solution is the set of all real numbers, i.e., $(-\infty, \infty)$.

EXAMPLE 4.22

Solve $\dfrac{x^2 + 5x + 3}{x + 2} < x$.

SOLUTION

$$\frac{x^2 + 5x + 3}{x + 2} < x$$

$$\Rightarrow \frac{x^2 + 5x + 3}{x + 2} - x < 0$$

$$\Rightarrow \frac{x^2 + 5x + 3 - x^2 - 2x}{x + 2} < 0$$

$$\Rightarrow \frac{3x + 3}{x + 2} < 0$$

$$\Rightarrow \frac{x + 1}{x + 2} < 0. \qquad (1)$$

The solution of inequation (1) is same as $(x + 1)(x + 2) < 0$.

We know that, $(x-\alpha)(x-\beta)<0$

$$\Rightarrow \alpha < x < \beta \qquad \text{where } (\alpha < \beta)$$

$\therefore -2 < x < -1.$

Thus, the required solution is $-2 < x < -1$.

EXAMPLE 4.23

Solve $\dfrac{1}{x-1} < \dfrac{-2}{1-2x}$.

SOLUTION

$$\frac{1}{x-1} < \frac{-2}{1-2x}$$

$$\Rightarrow \frac{1}{x-1} + \frac{2}{1-2x} < 0$$

$$\Rightarrow \frac{1-2x+2x-2}{(x-1)(1-2x)} < 0$$

$$\Rightarrow \frac{-1}{(x-1)(x-2x)} < 0$$

$$\Rightarrow \frac{1}{(x-1)(1-2x)} > 0. \tag{1}$$

Inequation (1) holds good if, $(x-1)(1-2x) > 0 \Rightarrow (x-1)(2x-1) < 0$.

We know that, $(x-\alpha)(x-\beta) < 0 \Rightarrow \alpha < x < \beta$, where $(\alpha < \beta)$.

$\therefore$ The solution of the given inequation is $\dfrac{1}{2} < x < 1$, i.e., $x \in \left(\dfrac{1}{2}, 1\right)$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. For a quadratic equation $ax^2 + bx + c = 0$, if $a + b + c = 0$, then $x = 1$. (True/False)
2. If $a = c$, then the roots of $ax^2 + bx + c = 0$ are _____ to each other.
3. If $a^2 - b^2 > 0$, then $a > b$ or $a < -b$. (True/False)
4. If $b = 0$, then $ax^2 + bx + c = 0$ is called _____ quadratic equation.
5. If the sum of the roots of a quadratic equation is positive and the product of the roots is negative, then the numerically greater root is _____.
6. If we divide 8 into two parts such that their product is 15, then the parts are _____ and _____.
7. One side of a rectangle exceeds its other side by 2 cm. If its area is 24 cm^2, then the length and breadth are _____ and _____ respectively.
8. If the sum of a number and its reciprocal is 2, then the number is _____.
9. If we divide 4 into two parts such that the sum of their squares is 8, then the parts are _____ and _____.
10. If the product of two consecutive positive numbers is 20, then the numbers are _____ and _____.
11. The area of a right triangle is 32 sq. units. If its base is 4 times the altitude, then the altitude is _____.
12. The diagonal of a rectangle is 5 units. If one side is twice the other side, then the length is _____.
13. If α and β are the roots of the quadratic equation, such that $\alpha > \beta$, then for $(x - \alpha)(x - \beta) > 0$, the solution set of x is _____.
14. The positive number which is less than its square by 12 is _____.
15. If the product of two successive even numbers is 48, then the numbers are _____ and _____.
16. The point $(1, -6)$ satisfies the equation $y = x^2 - x - 6$. Is the statement true?
17. The line $y = 5x - 6$ meets the parabola $y = x^2$ in two points, where x-coordinates are _____ and _____.
18. The solution set of $x^2 - 4x + 4 < 0$ does not exist. (True/False)
19. If the product of Shiva's age 3 years ago and his age three years later is 16, then Shiva's present age is _____.
20. The solution set of $x^2 - 10x + 25 \geq 0$ exists. (True/False)
21. The sides of two squares are x cm and $(x + 2)$ cm. If the sum of their areas is 20 cm^2, then the sides of the squares are _____ and _____.
22. The solution set of $x^2 - 3x - 4 \leq 0$ is _____.
23. The solution set of $x^2 - 8x + 15 \geq 0$ is _____.
24. The solution set of $x^2 - 5x + 6 > 0$ is _____.
25. Find the positive number which is less than its square by 42.
26. If $mx^2 < nx$, where m and n are positive, then find the range of x.
27. The solution set of $x^2 - x - 6 < 0$ is _____.
28. Find the dimensions of a rectangular hall, if its length exceeds its breadth by 7 m and the area of the hall being 228 m^2.
29. If $(x + 1)^2 > mx$, and $x \in Z^+$, then the possible values of m are _________.
30. If $y = \sqrt{k\sqrt{k\sqrt{k\sqrt{k\ldots\infty}}}}$, where $k > 0$; find the value of y.

Short Answer Type Questions

31. The equation $(m + 1)x^2 + 2mx + 5x + m + 3 = 0$ has equal roots. Find the value of m.
32. The denominator of a fraction exceeds the numerator by 3. The sum of the fraction and its multiplicative inverse is $2\frac{9}{28}$. Find the fraction.
33. Find the value of q^2 in terms of p and r, so that the roots of the equation $px^2 + qx + r = 0$ are in the ratio of 3 : 4.
34. A number consists of two digits whose product is 18. If 27 is added to the number, the number

formed will have the digits in reverse order, when compared to the original number. Find the number.

35. Find the minimum value of $x^2 + 12x$.

36. If the sides of a right triangle are x, $3x + 3$, and $3x + 4$, then find the value of x.

37. Find the roots of the equation $\frac{1}{x} - \frac{1}{x-a} = \frac{1}{b} - \frac{1}{b-a}$, where $a \neq 0$.

38. Find the values of x which satisfy the inequation, $x - 5 < x^2 - 3x - 50$.

39. In a boys' hostel, there are as many boys in each room as the number of rooms. If the number of rooms is doubled and the number of boys in each room is reduced by 10, then the number of boys in the hostel becomes 1200. Find the number of rooms in the hostel.

40. If the equations $x^2 + 3x + 2 = 0$ and $x^2 + kx + 6 = 0$ have a common root, then find the values of k.

41. If $x^4 - 17x^2 + 16 = 0$, then find the sum of the squares of the roots.

42. If the quadratic equation $x^2 - mx - 4x + 1 = 0$ has real and distinct roots, then find the values of m.

43. $\sqrt{y+1} - \sqrt{y-1} = \sqrt{4y-1}$. Find the value of y.

44. Umesh and Varun are solving an equation of form $x^2 + bx + c = 0$. In doing so, Umesh commits a mistake in noting down the constant term and finds the roots as -3 and -12. And Varun commits a mistake in noting down the coefficient of x and finds the roots as -27 and -2. If so, find the roots of original equation.

45. Solve: $\frac{1}{4x+4} > \frac{2}{4x-2}$.

Essay Type Questions

46. If $-(4x + 27) < (x + 6)^2 < -4(6 + x)$, then find all the integral values of x.

47. Determine the values of x which satisfy the simultaneous inequations, $x^2 + 5x + 4 > 0$ and $-x^2 - x + 42 > 0$.

48. Solve $x^2 - 2x + 1 = 0$ graphically.

49. Draw the graph of $y = x^2 - x - 12$.

50. Solve $x^2 - x - 42 = 0$ graphically.

CONCEPT APPLICATION

Level 1

1. The roots of the equation $3x^2 - 2x + 3 = 0$ are
   (a) real and distinct.
   (b) real and equal.
   (c) imaginary.
   (d) irrational and distinct.

2. Find the sum and the product of the roots of the equation $\sqrt{3}x^2 + 27x + 5\sqrt{3} = 0$.
   (a) $-9\sqrt{3}, 5$ (b) $9\sqrt{3}, 5$
   (c) $6\sqrt{3}, -5$ (d) $6\sqrt{3}, 5$

3. If α and β are the roots of the equation $x^2 - 12x + 32 = 0$, then find the value of $\frac{\alpha^2 + \beta^2}{\alpha + \beta}$.
   (a) $\frac{-8}{3}$ (b) $\frac{8}{3}$
   (c) $\frac{-20}{3}$ (d) $\frac{20}{3}$

4. Find the maximum or minimum value of the quadratic expression, $x^2 - 3x + 5$ whichever exists.
   (a) The minimum value is $\frac{9}{10}$.
   (b) The minimum value is $\frac{11}{4}$.
   (c) The maximum value is $\frac{9}{10}$.
   (d) The maximum value is $\frac{11}{4}$.

5. Find the values of x which satisfy the equation, $\sqrt{3x+7} - \sqrt{2x+3} = 1$.

(a) 2, −2 (b) 4, 3

(c) 5, −1 (d) 3, −1

6. If one of the roots of a quadratic equation having rational coefficients is $\sqrt{7} - 4$, then the quadratic equation is ______.

(a) $x^2 - 2\sqrt{7}x - 9 = 0$.

(b) $x^2 - 8x + 9 = 0$.

(c) $x^2 + 8x + 9 = 0$.

(d) $x^2 - 2\sqrt{7}x + 9 = 0$.

7. If the quadratic equation $px^2 + qx - r = 0$ $(p \neq 0)$ is to be solved by the graphical method, then which of the following graphs have to be drawn?

(a) $y = x^2$, $y = r - qx$

(b) $y = px^2$, $y = qx - r$

(c) $y = x^2$, $qx + py - r = 0$

(d) $y = x^2$, $qx - py = r$

8. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, then the quadratic equation whose roots are $-\alpha$ and $-\beta$ is ______.

(a) $ax^2 - bx - c = 0$

(b) $ax^2 - bx + c = 0$

(c) $ax^2 + bx - c = 0$

(d) $ax^2 - bx + 2c = 0$

9. Find the sum and the product of the roots of the quadratic equation $-x^2 - \frac{25}{3}x + 25 = 0$.

(a) $\frac{25}{3}$, 25 (b) $\frac{-25}{3}$, 25

(c) $\frac{25}{3}$, −25 (d) $\frac{-25}{3}$, −25

10. For what value of k, if one root of the quadratic equation $9x^2 - 18x + k = 0$ is double of the other?

(a) 36 (b) 9

(c) 12 (d) 8

11. The sum of a number and its square is greater than 6, then the number belongs to ______.

(a) $(-\infty, 2) \cup (3, \infty)$

(b) $(-\infty, -3) \cup (2, \infty)$

(c) $(2, 3)$

(d) $[2, 3]$

12. For which of the following intervals of x is $x^2 > \frac{1}{x^2}$?

(a) $(-\infty, -1) \cup (1, \infty)$

(b) $(-\infty, -1) \cup (1, \infty)$

(c) $(-1, 1)$

(d) $[-1, 1]$

13. If x and y are two successive multiples of 2 and their product is less than 35, then find the range of x.

(a) $\{2, 4, 0\}$

(b) $\{-6, -4, -2, 2, 4, 6\}$

(c) $\{-6, -4, -2, 0, 2, 4\}$

(d) $\{-6, -4, -2, 0, 2, 4, 6\}$

14. If $x^2 < n$, and $n \in (-\infty, 0)$, then x

(a) is any real number.

(b) is only positive number.

(c) has no value.

(d) is any negative number.

15. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$ such that x does not lie between α and β, then

(a) $a > 0$ and $ax^2 + bx + c < 0$.

(b) $ax^2 + bx + c < 0$ and $a < 0$.

(c) $a > 0$ and $ax^2 + bx + c > 0$.

(d) Both (b) and (c)

16. The condition for the sum and the product of the roots of the quadratic equation $ax^2 - bx + c = 0$ to be equal, is ______.

(a) $b + c = 0$ (b) $b - c = 0$

(c) $a + c = 0$ (d) $a + b + c = 0$

17. The quadratic equation having rational coefficients and one of the roots as $4 + \sqrt{15}$, is ______.

PRACTICE QUESTIONS

(a) $x^2 - 8x + 1 = 0$

(b) $x^2 + x - 8 = 0$

(c) $x^2 - x + 8 = 0$

(d) $x^2 + 8x + 8 = 0$

18. If α and β are the zeros of the quadratic polynomial $ax^2 + bx + c$ and x lies between α and β, then which of the following is true?

(a) If $a < 0$ then $ax^2 + bx + c > 0$.

(b) If $a > 0$ then $ax^2 + bx + c < 0$.

(c) If $a > 0$ then $ax^2 + bx + c > 0$.

(d) Both (a) and (b)

19. Find the nature of the roots of the equation $4x^2 - 2x - 1 = 0$.

(a) Real and equal

(b) Rational and unequal

(c) Irrational and unequal

(d) Imaginary

20. The solution of the inequation, $15x^2 - 31x + 14 < 0$ is given by ______.

(a) $x \in \left(\frac{7}{5}, \infty\right)$ (b) $\frac{2}{3} < x < \frac{7}{5}$

(c) $x \in \left(\frac{7}{5}, \infty\right)$ (d) $x \in R$

21. If $mx^2 < nx$ such that m and n have opposite signs, then which of the following can be true?

(a) $x \in \left(\frac{n}{m}, \infty\right)$ (b) $x \in \left(-\infty, \frac{n}{m}\right)$

(c) $x \in \left(\frac{n}{m}, 0\right)$ (d) None of these

22. Find the range of values of x which satisfy the inequation, $(x + 1)^2 + (x - 1)^2 < 6$.

(a) $(-\sqrt{2}, \sqrt{2})$

(b) $(-1, 1)$

(c) $(-\infty, -2) \cup (2, \infty)$

(d) $(-\infty, -1) \cup (1, \infty)$

23. If the sum of the squares of three consecutive odd natural numbers is 155, then their product will be equal to ______.

(a) 99 (b) 105

(c) 693 (d) 315

24. If $x^2 > 0$, then find the range of the values that x can take.

(a) $x = 0$ (b) $x \in R$

(c) $x \in (0, \infty)$ (d) $x \in R - \{0\}$

25. Find the range of the values of x which satisfy the inequation, $x^2 - 7x + 3 < 2x + 25$.

(a) $(-2, 11)$

(b) $(2, 11)$

(c) $(-\infty, -1) \cup (2, 11)$

(d) $(-8, -2) \cup [11, \infty)$

26. If A and B are the roots of the quadratic equation $x^2 - 12x + 27 = 0$, then $A^3 + B^3$ is ________.

(a) 27 (b) 729

(c) 756 (d) 64

27. By drawing which of the following graphs can the quadratic equation $4x^2 + 6x - 5 = 0$ be solved by graphical method?

(a) $y = x^2$, $3x - 2y - 5 = 0$

(b) $y = 4x^2$, $6x - 2y - 5 = 0$

(c) $y = x^2$, $6x - y - 5 = 0$

(d) $y = 2x^2$, $6x + 2y - 5 = 0$

28. If the quadratic equation $(a^2 - b^2)x^2 + (b^2 - c^2)x + (c^2 - a^2) = 0$ has equal roots, then which of the following is true?

(a) $b^2 + c^2 = a^2$ (b) $b^2 + c^2 = 2a^2$

(c) $b^2 - c^2 = 2a^2$ (d) $a^2 = b^2 + 2c^2$

29. Which of the following are the roots of the equation $|x|^2 + |x| - 6 = 0$?

(A) 2 (B) −2

(C) 3 (D) −3

(a) Both (A) and (B)

(b) Both (C) and (D)

(c) Both (A) and (C)

(d) (A), (B), (C) and (D)

30. What are the values of x which satisfy the equation, $\sqrt{5x-6} + \frac{1}{\sqrt{5x-6}} = \frac{10}{3}$?

(a) 3 (b) $4, \frac{11}{9}$

(c) $\frac{11}{9}$ (d) $3, \frac{11}{9}$

Level 2

31. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β, then the equation whose roots are α^2 and β^2 is ______.

(a) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

(b) $a^2x^2 + b^2x + c^2 = 0$

(c) $a^2x^2 + (b^2 + 2ac)x + c^2 = 0$

(d) $a^2x^2 - (b^2 + 2ac)x + c^2 = 0$

32. If the roots of the equation $3ax^2 + 2bx + c = 0$ are in the ratio 2 : 3, then ______.

(a) $8ac = 25b$

(b) $8ac = 9b^2$

(c) $8b^2 = 9ac$

(d) $8b^2 = 25ac$

33. Find the roots of the equation $l^2(m^2 - n^2)x^2 + m^2(n^2 - l^2)x + n^2(l^2 - m^2) = 0$.

(a) $1, \dfrac{n^2(l^2 - m^2)}{l^2(m^2 - n^2)}$ (b) $1, \dfrac{-m^2(l^2 - n^2)}{l^2(m^2 - n^2)}$

(c) $1, \dfrac{n^2(l^2 + m^2)}{l^2(m^2 - n^2)}$ (d) $1, \dfrac{-m^2(l^2 + n^2)}{l^2(m^2 - n^2)}$

34. Comment on the sign of the quadratic expression $x^2 - 5x + 6$ for all $x \in R$.

(a) $x^2 - 5x + 6 \geq 0$ when $2 \leq x \leq 3$ and $x^2 - 5x + 6 < 0$ when $x < 2$ or $x > 3$.

(b) $x^2 - 5x + 6 \leq 0$ when $2 \leq x \leq 3$ and $x^2 - 5x + 6 > 0$ when $x < 2$ or $x > 3$.

(c) $x^2 - 5x + 6 \leq 0$ when $-1 \leq x \leq 6$ and $x^2 - 5x + 6 > 0$ when $x < -1$ or $x > 6$.

(d) $x^2 - 5x + 6 \geq 0$ when $-1 \leq x \leq 6$ and $x^2 - 5x + 6 < 0$ when $x < -1$ or $x > 6$.

35. If $a - b$, $b - c$ are the roots of $ax^2 + bx + c = 0$, then find the value of $\dfrac{(a-b)(b-c)}{c-a}$.

(a) $\dfrac{b}{c}$ (b) $\dfrac{c}{b}$

(c) $\dfrac{ab}{c}$ (d) $\dfrac{bc}{a}$

36. The values of x for which $\dfrac{x+3}{x^2 - 3x - 54} \geq 0$ are ________.

(a) $(-6, -3) \cup (9, \infty)$

(b) $[-6, -3] \cup [9, \infty]$

(c) $(-6, -3) \cup (9, \infty)$

(d) $(-6, \infty)$

37. In a right triangle, the base is 3 units more than the height. If the area of the triangle is less than 20 sq. units, then the possible values of the base lie in the region ________.

(a) (4, 6) (b) (3, 8)

(c) (6, 8) (d) (5, 8)

38. The values of x for which $-2x - 4 \leq (x + 2)^2 \leq -2x - 1$ is satisfied are ______.

(a) $[-5, -1]$

(b) $[-5, 0]$

(c) $[-5, -4] \cup [-2, -1]$

(d) $[-5, -4] \cup [-2, -1]$

39. If $\dfrac{x^2 + x - 12}{x^2 - 3x + 2} < 0$, then x lies in _________.

(a) $(-4, 3)$

(b) $(-4, 2)$

(c) $[-4, 1] \cup [2, 3]$

(d) $(-4, 1) \cup (2, 3)$

40. For all real values of x, $\dfrac{x^2 - \left(\dfrac{x}{2}\right) + 1}{x^2 + 1} - \dfrac{5}{4}$ is ______.

(a) equal to 1

(b) non-negative

(c) greater than $\dfrac{1}{4}$

(d) non-positive

41. If $x^2 - 4x + 3 > 0$ and $x^2 - 6x + 8 < 0$, then ______.

(a) $x > 3$ (b) $x < 4$

(c) $3 < x < 4$ (d) $1 < x < 2$

42. Find the product of the roots of $x^2 + 8x - 16 = 0$.

(a) 8 (b) −8

(c) 16 (d) −16

PRACTICE QUESTIONS

Level 3

43. If the roots of the equation $2x^2 + 7x + 4 = 0$ are in the ratio $p : q$, then find the value of $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}$.

(a) $\pm\frac{7}{\sqrt{7}}$ (b) $\pm 7\sqrt{2}$

(c) $\pm\frac{7\sqrt{2}}{16}$ (d) $\pm\frac{7\sqrt{2}}{4}$

44. In a forest, a certain number of apes equal to the square of one-eighth of the total number of their group are playing and having great fun. The rest of them are twelve in number and are on an adjoining hill. The echo of their shrieks from the hills frightens them. They come and join the apes in the forest and play with enthusiasm. What is the total number of apes in the forest?

(a) 16 (b) 48

(c) 16 or 48 (d) 64

45. If the roots of the quadratic equation $x^2 - 2kx + 2k^2 - 4 = 0$ are real, then the range of the values of k is _________.

(a) $[-2, 2]$ (b) $[-\infty, -2] \cup [2, \infty]$

(c) $[0, 2]$ (d) None of these

46. Find the values of x for which the expression $x^2 - (\log_5 2 + \log_2 5)\, x + 1$ is always positive.

(a) $x > \log_2 5$ or $x < \log_5 2$

(b) $\log_5 2 < x < \log_2 5$

(c) $-\log_5 2 < x < \log_2 5$

(d) $x < -\log_5 2$ or $x > \log_2 5$

47. Find the values of x which satisfy the quadratic inequation $|x|^2 - 2\,|x| - 8 \le 0$.

(a) $[-4, 4]$ (b) $[0, 4]$

(c) $[-4, 0]$ (d) $[-4, 2]$

48. The roots of the equation $x^2 - px + q = 0$ are consecutive integers. Find the discriminant of the equation.

(a) 1 (b) 2

(c) 3 (d) 4

49. Rohan and Sohan were attempting to solve the quadratic equation, $x^2 - ax + b = 0$. Rohan copied the coefficient of x wrongly and obtained the roots as 4 and 12. Sohan copied the constant term wrongly and obtained the roots as -19 and 3. Find the correct roots.

(a) -2 and -24 (b) 2 and 24

(c) 4 and 12 (d) -4 and -12

50. If $(x + 2)(x + 4)(x + 6)(x + 8) = 945$ and x is an integer, then find x.

(a) -1 or -11 (b) 1 or -11

(c) -1 or 11 (d) 1 or 11

51. The difference of the roots of $2y^2 - ky + 16 = 0$ is $\frac{1}{3}$. Find k.

(a) $\pm\frac{32}{3}$ (b) $\pm\frac{34}{3}$

(c) $\pm\frac{38}{3}$ (d) $\pm\frac{40}{3}$

52. Find the condition to be satisfied by the coefficients of the equation $px^2 + qx + r = 0$, so that the roots are in the ratio $3 : 4$.

(a) $12q^2 = 49pr$ (b) $12q^2 = -49pr$

(c) $49q^2 = 12\,pr$ (d) $49q^2 = -12pr$

53. If the roots of the equation $3x^2 + 9x + 2 = 0$ are in the ratio $m : n$, then find $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}}$.

(a) $\frac{-3\sqrt{3}}{\sqrt{2}}$ (b) $\frac{3\sqrt{2}}{2}$

(c) $\frac{3\sqrt{3}}{\sqrt{2}}$ (d) $\frac{-3\sqrt{3}}{2}$

54. If $(p^2 - q^2)\, x^2 + (q^2 - r^2)\, x + r^2 - p^2 = 0$ and $(p^2 - q^2)\, y^2 + (r^2 - p^2)\, y + q^2 - r^2 = 0$ have a common root for all real values of p, q and r, then find the common root.

(a) -1 (b) 1

(c) 2 (d) -2

55. Which of the following are the roots of $|y|^2 - |y| - 12 = 0$?

(A) 4 (B) -4

(C) 3 (D) -3

(a) Both (A) and (B)

(b) Both (C) and (D)

(c) Both (A) and (C)

(d) (A), (B), (C) and (D)

56. Find the value of $\sqrt{30+\sqrt{30+\sqrt{30+\cdots\infty}}}$.

(a) 6 (b) −5

(c) Either (a) or (b) (d) Neither (a) nor (b)

57. If $y^2 + 6y - 3m = 0$ and $y^2 - 3y + m = 0$ have a common root, then find the possible values of m.

(a) $0, -\frac{27}{16}$ (b) $0, -\frac{81}{16}$

(c) $0, \frac{81}{16}$ (d) $0, \frac{27}{16}$

58. The students of a class contributed for a programme. Each student contributed the same amount. Had there been 15 more students in the class and each student had contributed ₹40 less, the total amount contributed would have increased from ₹3000 to ₹3200. Find the strength of the class.

(a) 25 (b) 15

(c) 10 (d) 20

59. The graphs of $y = 2x^2$ and $y = ax + b$ intersect at two points (2, 8) and (6, 72). Find the quadratic equation in x whose roots are $a + 2$ and $\frac{b}{4} - 1$.

(a) $x^2 + 11x - 126 = 0$

(b) $x^2 - 11x + 126 = 0$

(c) $x^2 + 11x + 126 = 0$

(d) $x^2 - 11x - 126 = 0$

60. The equation $9y^2(m + 3) + 6(m - 3)y + (m + 3) = 0$, where m is real, has real roots. Which of the following is true?

(a) $m = 0$

(b) $m < 0$

(c) Either (a) or (b)

(d) Neither (a) nor (b)

61. Find the values of y which satisfy the quadratic inequalities below $y^2 + 5y + 4 \leq 0$ and $y^2 - 2y - 15 \geq 0$.

(a) $-1 \leq y \leq 5$

(b) $-4 \leq y \leq -3$

(c) $-1 \geq y \geq 5$

(d) $-4 \geq y \geq -3$

62. If $\frac{y^2+y-6}{y^2+y-2} < 0$, then which of the following is true?

(a) $1 < y < 2$

(b) $-3 < y < -2$

(c) Either (a) or (b)

(d) Neither (a) nor (b)

63. The product of two consecutive even numbers exceeds twice their sum by more than 20. Which of the following is the range of values that the smaller of the numbers can take?

(a) $x > 4$ or $x < -6$

(b) $-6 < x < 4$

(c) $x > 6$ or $x < -4$

(d) $-4 < x < 6$

64. Which of the following statements about the sign of the quadratic expression $E = y^2 - 12y + 20$ is true?

(a) $E \leq 0$ when $20 \leq y \leq 10$ and $E > 0$ when $y < 2$ or $y > 10$.

(b) $E \geq 0$ when $2 \leq y \leq 10$ and $E < 0$ when $y < 2$ or $y > 10$.

(c) $E \leq 0$ when $-10 \leq y \leq -2$ and $E > 0$ when $y < -10$ or $y > -2$.

(d) $E \geq -2$ and $E < 0$ when $y < -10$ or $y > -2$.

65. If $|y|^2 - 4|y| - 60 \leq 0$, then which of the following is the range of y?

(a) [−6, 6] (b) [0, 6]

(c) [0, 10] (d) [−10, 10]

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. True
2. reciprocal
3. True
4. pure
5. positive
6. 3, 5
7. 6 cm and 4 cm
8. 1
9. 2, 2
10. 4 and 5
11. 4 units
12. $2\sqrt{5}$
13. $x < \beta$ or $x > \alpha$
14. 4
15. 6 and 8
16. Yes
17. 3, 2
18. True
19. 5 years
20. True
21. 2, 4
22. $[-1, 4]$
23. $(-\infty, 3) \cup (5, \infty)$
24. $x \in (-\infty, 2) \cup (3, \infty)$
25. 7
26. $0 < x < \frac{n}{m}$
27. $-2 < x < 3$
28. Length = 19 m, breadth = 12 m
29. $(-\infty, 4)$
30. k

Short Answer Type Questions

31. $\frac{-13}{4}$
32. $\frac{4}{7}$
33. $q^2 = \frac{49pr}{12}$
34. 36
35. −36
36. $x = 7$
37. b and $(a - b)$
38. $\{x/-7 < x < 9\}$
39. $\{x/x < -12 \text{ or } x > -8\}$
40. 5 or 7
41. 34
42. $m \in (-\infty, -6) \cup (-2, \infty)$
43. There is no solution for the given equation.
44. −9, −6
45. $x \in \left(-\infty, \frac{-5}{2}\right) \cup \left(-1, \frac{1}{2}\right)$

Essay Type Questions

46. ϕ
47. $x \in (-7, -4) \cup (-1, 6)$
48. $(1, 1)$
50. $\{-6, 7\}$

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (a)	**3.** (d)	**4.** (b)	**5.** (d)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (d)	**10.** (d)
11. (b)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (d)	**19.** (c)	**20.** (b)
21. (c)	**22.** (a)	**23.** (d)	**24.** (d)	**25.** (a)	**26.** (c)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (d)

Level 2

31. (a)	**32.** (d)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (a)	**37.** (b)	**38.** (d)	**39.** (d)	**40.** (d)
41. (c)	**42.** (d)	**47.** (b)	**46.** (d)	**47.** (c)					

Level 3

43. (a)	**44.** (a)	**45.** (a)	**48.** (a)	**49.** (d)	**50.** (b)	**51.** (b)	**52.** (a)	**53.** (a)	**54.** (b)
55. (a)	**56.** (a)	**57.** (d)	**58.** (a)	**59.** (d)	**60.** (c)	**61.** (b)	**62.** (c)	**63.** (c)	**64.** (a)
65. (d)									

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. Find the discriminant.
2. Use $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.
3. Find the roots α and β or write $\alpha^2+\beta^2$ as $(\alpha+\beta)^2-2\alpha\beta$ and $\alpha+\beta=$ sum of the roots and $\alpha\beta=$ product of the roots.
4. Use the formula and check the coefficient of x^2.
5. Square both sides.
6. If $a+\sqrt{b}$ is one root then $a-\sqrt{b}$ is other root.
7. Take the first degree expression and the constant on the other side.
8. Use $x^2-(\alpha+\beta)x+\alpha\beta=0$.
9. Use $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.
10. Simplify and solve by finding $\alpha+\beta$ and $\alpha\beta$. Where $\beta=2\alpha$.
11. Let the number be x and frame the inequations.
12. Simplify and solve the inequation.
13. Assume the two successive multiples as n and $n+2$.
14. x^2 is always positive and n is always negative.
16. Find the sum and the product of the roots then equate them.
17. If one of the roots is $a+\sqrt{b}$, this the other root is $a-\sqrt{b}$.
19. Use the formula of discriminant.
20. Factorise the LHS of the inequation.
21. Take nx on the left hand side and solve.
22. Simplify and solve the inequation.
23. Let the three consecutive odd natural numbers be $x-2$, x and $x+2$.
24. Observe the options.
25. Simplify and solve the inequation.
26. Find the roots or write A^3+B^3 in terms of $A+B$ and AB.
27. Take $6x-5$ on other side.
28. Observe the coefficient of each term and guess one root.
29. (i) Put $|x|=y$ and frame the equation.

    (ii) Solve for y.
30. Find the LCM and then square on both sides.

Level 2

31. Use $x^2-(\alpha+\beta)x+\alpha\beta=0$.
32. If the roots of $ax^2+bx+c=0$ are in the ratio $m:n$, then $(m+n)^2\,a\cdot c=mnb^2$.
33. In the equation $ax^2+bx+c=0$ when $a+b+c=0$, then the roots are 1 and $\frac{c}{a}$.
34. $(x-a)(x-b)\le 0$ when $a\le x\le b$.

    $(x-a)(x-b)\ge 0$ when $x\le a$ or $x\ge b$.
35. (i) $(a-b)\,(b-c)=$ product of the roots $=\frac{c}{a}$.

    (ii) $c-a=-(a-b+b-c)=-$(sum of the roots) $=\frac{b}{a}$.
36. (i) Express the quadratic equations given in the numerator and the denominator as the product of two linear factors.

    (ii) Form the various regions by using critical points.

    (iii) Identify the regions in which the inequality holds good.
37. (i) Let the height of the triangle be x. Hence base $=x+3$.

    (ii) Form the quadratic inequation.
38. Form two different inequalities and find the common solution to them.
39. (i) Express the quadratic equations given in the numerator and the denominator as the product of two linear factors.

(ii) Form the various regions by using critical points.

(iii) Identify the regions in which the given inequality holds good.

40. Simplify the expression and identify whether it is non-positive or non-negative.

41. (i) If $(x-a)(x-b) > 0$, then $x \in (-\infty, a) \cup (b, \infty)$.

If $(x-a)(x-b) < 0$, then $x \in (a, b)$.

(ii) Take the common range of both the inequations.

42. $x^2 + 8x - 16 = 0$.

The product of the roots $= \frac{c}{a} = -16$.

Level 3

43. (i) Let the roots be pk and qk.

(ii) Use sum of roots $= \frac{-b}{a}$ and product of roots $= \frac{c}{a}$.

44. (i) Let the number of apes be equal to n.

(ii) The number of apes which are on the adjoining hills is $n - \frac{n^2}{64}$, which is equal to 12.

45. Discriminant ≥ 0.

46. (i) $x^2 - (\log_5 2 + \log_2 5)\, x + 1 > 0$.

(ii) $x^2 - (\log_5 2 + \log_2 5)\, x + 1$

$= (x - \log_5 2)\,(x - \log_2 5)$ and proceed.

47. (i) Put $|x| = y$.

(ii) Form a quadratic inequation in terms of y.

(iii) If $(x-a)\,(x-b) \leq 0$ then $x \in (a, b)$.

48. Let the roots be α and $\alpha + 1$.

$\alpha + (\alpha + 1) = p$, i.e., $2\alpha + 1 = p$.

$(\alpha)\,(\alpha + 1) = \frac{q}{1} = q$.

Discriminant $= p^2 - 4q = (2\alpha + 1)^2 - 4\,(\alpha)\,(\alpha + 1) = 4\alpha^2 + 4\alpha + 1 - 4\alpha^2 - 4\alpha = 1$.

49. Rohan copied only the coefficient of x wrongly.

$\therefore$ He must have copied the constant term correctly.

$\therefore$ Correct product of the roots $= \frac{b}{1} = 4(12)$

$\Rightarrow b = 48$

Sohan copied only the constant term wrongly.

$\therefore$ He must have copied the coefficient of x correctly.

$\therefore$ Correct sum of the roots $= a = -19 + 3 = -16$

Correct equation is $x^2 - (-16)x + 48 = 0$

$\Rightarrow x^2 + 16x + 48 = 0$

$\Rightarrow (x + 4)(x + 12) = 0$

$\Rightarrow x = -4$ or -12.

$\therefore$ Correct roots are -4 and -12.

50. As x is an integer.

$\therefore$ $x + 2$, $x + 4$, $x + 6$ and $x + 8$ are four consecutive odd/even integers. Their product = 945, which is odd.

$\therefore$ $x + 2$, $x - 4$, $x + 6$ and $x + 8$ must be odd.

Factorizing 945 as a product of 4 consecutive odd integers.

We have, $945 = 3(5)(7)(9) = (-3)(-5)(-7)(-9)$

$\therefore$ The smallest, i.e., $x + 2 = 3$ or -9.

$\therefore$ $x = 1$ or $x = -11$.

51. Let the roots of $2y^2 - ky + 16 = 0$ be α and β, where with $\alpha \geq \beta$. $\alpha + \beta = \frac{k}{2}$ and $\alpha\beta = 8$.

Difference of its roots > 0. $\alpha \neq \beta$ $\therefore$ $\alpha > \beta$

$\therefore$ Difference of its roots $= \alpha - \beta = \frac{1}{3}$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$\left(\frac{1}{3}\right)^2 = \left(\frac{k}{2}\right)^2 - 4(8)$$

$$\left(\frac{k}{2}\right)^2 = \frac{289}{9} = \left(\frac{17}{3}\right)^2$$

$$\Rightarrow \frac{k}{2} = \pm\frac{17}{3} \Rightarrow k = \pm\frac{34}{3}.$$

52. Let the common factor for the roots be y.

The roots are $3y$ and $4y$, $3y + 4y = \frac{-q}{p}$

$$\Rightarrow y = \frac{-q}{7p}$$

$$(3y)(4y) = \frac{r}{p} \Rightarrow y^2 = \frac{r}{12p}$$

$$y^2 = \left(\frac{-q}{7p}\right)^2 = \frac{r}{12p} 12q^2 = 49pr.$$

53. Let the common factor for the roots be y.

The roots are my and ny.

$my + ny = -3 \Rightarrow (m + n)\ y = -3$

We can take m, n as positive and y as negative

$$(my)(ny) = \frac{2}{3}.$$

Now, $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} = \frac{m+n}{\sqrt{mn}}$

Multiplying the numerator denominator by y, we have

$$\frac{m+n}{\sqrt{mn}} = \frac{y(m+n)}{y\sqrt{mn}} = \frac{y(m+n)}{\sqrt{(my)(ny)}}$$

$$= \frac{-3}{\sqrt{\frac{2}{3}}} = \frac{-3\sqrt{3}}{\sqrt{2}}.$$

54. $(p^2 - q^2)\ x^2 + (q^2 - r^2)\ x + (r^2 - p^2) = 0$ (1)

The sum of the coefficients $(p^2 - q^2) + (q^2 - r^2) + (r^2 - p^2) = 0$

$\therefore\ x = 1$ is a root of Eq. (1)

$(p^2 - q^2)\ y^2 + (r^2 - p^2)\ y + q^2 - r^2 = 0$ (2)

$p^2 - q^2 + r^2 - p^2 + q^2 - r^2 = 0$

$\therefore\ y = 1$ is a root of Eq. (2)

$\therefore$ 1 is the common root of Eqs. (1) and (2).

55. $|y|^2 - |y| - 12 = 0$

$(|y| - 4)\ (|y| + 3) = 0$

$|y| = 4$ or -3.

But $|y|$ must be non-negative.

$\therefore\ |y| = 4$, i.e., $y = \pm 4$.

56. $x = \sqrt{30 + x}$

Squaring on both sides, we get $x^2 = 30 + x$

$x^2 - x - 30 = 0$

$(x - 6)(x + 5) = 0$

$x = 6$ or -5.

But x must be positive.

$\therefore\ x = 6$.

57. Let the common root be c.

$c^2 + 6c - 3m = 0$ and $c^2 - 3c + m = 0$

That is, $3m = c^2 + 6c$ and $m = 3c - c^2$

$\therefore\ 3m = c^2 + 6c = 3\ (3c - c^2)$

$c^2 + 6c = 9c - 3c^2$

$4c^2 - 3c = 0$

$c = 0$ or $\frac{3}{4}$

If $c = 0$, $m = 0$

If $c = \frac{3}{4}$, $m = \frac{27}{16}$.

58. Let the strength of the class be x.

Let the amount each student contributed be ₹y.

$xy = 3000$ (1)

$(x + 15)(y - 40) = 3200$

That is, $xy + 15y - 40x - 600 = 3200$

From Eq. (1),

$$\Rightarrow 15\left(\frac{3000}{x}\right) - 40x - 800 = 0$$

$\Rightarrow x^2 + 20x - 1125 = 0$

$\Rightarrow (x + 45)\ (x - 25) = 0$

$x = -45$ or 25.

But $x > 0$ $\quad\therefore\ x = 25$.

59. At (2, 8) and (6, 72), $y = 2x^2 = ax + b$

$8 = 2a + b$ and $72 = 6a + b$.

Solving for a and b, $a = 16$ and $b = -24$.

The required equation is that whose roots are 18 and -7.

Sum of its roots = 11

Product of its roots = -126

$\therefore$ The Required equation is $x^2 - 11x - 126 = 0$.

60. Discriminant $(6(m-3))^2 - 4[9(m+3)(m+3)]$

$= 36[(m-3)^2 - (m+3)^2]$

$= 36((m^2 - 6m + 9 - (m^2 + 6m + 9))$

$= 36(-12m)$. This must be non-negative for the roots to be real.

$36(-12m) > 0$, i.e., $-2m \geq 0$, i.e., $m \leq 0$.

61. $y^2 + 5y + 4 \leq 0$

$\Rightarrow (y+1)\ (y+4) \leq 0$ of $y + 1$ and $y + 4$

The expression with the smaller value is $y + 1$ and the one with the greater value is $y + 4$. If the product is negative, the smaller has to be negative and the greater is positive.

$\Rightarrow -4 \leq y \leq 1$ (1)

Among $y - 5$ and $y + 3$, the smaller is $y - 5$, the greater, i.e., $y + 3$. If the product is positive either the smaller is positive or the greater is negative.

$y^2 - 2y - 15 \geq 0 \Rightarrow (y-5)(y+3) \geq 0$

$\Rightarrow y - 5, 3 \geq 0$ or $y + 3 \leq 0$

$\Rightarrow y \geq 5$

or $y \leq -3$. (2)

From Inequations (1) and (2), we get

$\Rightarrow -4 \leq y \leq -3$.

62. $\dfrac{y^2 + y - 6}{y^2 + y - 2} < 0$

$\dfrac{(y+3)(y-2)}{(y+2)(y-1)} < 0$

$\dfrac{(y+3)(y-2)(y+2)(y-1)}{(y+2)^2(y-1)^2} < 0$

$(y+3)\ (y-2)\ (y+2)\ (y-1) < 0$,

i.e., $(y-2)\ (y-1)\ (y+2)\ (y+3) < 0$

$-\infty$		-3		-2		1		2
	×		✓		×		✓	

$1 < y < 2$ or $-3 < y < -2$.

63. Let the smaller of the numbers be x. The greater number = $x + 2$.

$x(x+2) > [2(x + x + 2)] + 20$

$\Rightarrow x^2 + 2x > 4x + 4 + 20$

$\Rightarrow x^2 - 2x - 24 > 0\ (x-6)(x+4) > 0$.

The smaller is positive as the greater is negative.

That is, $x > 6$ or $x < -4$.

64. $E = y^2 - 12y + 20 = (y-2)(y-10)$

We consider the following possibilities.

(1) $y < 2$. In this case, $y - 2 < 10$,

$y - 10 < 0$. $\therefore E > 0$

(2) $y = 2$ or 10. In this case $E = 0$

(3) $2 < y < 10$. In this case, $y - 2 > 0$ and

$y - 10 < 0$. $\therefore$ $E < 0$

(4) $y > 10$. In this case, $y - 2 > 0$,

$y - 10 > 0$. $\therefore$ $E > 0$

(2) and (3) $\Rightarrow 2 \leq y \leq 10 \Rightarrow E \leq 0$. Only choice (1) is correct.

65. $|y|^2 - 4|y| - 60 \leq 0$

$\Rightarrow x^2 - 4x - 60 \leq 0$

where $x = |y|$

$(x-10)(\ x+6) \leq 0$

$\Rightarrow x < 10$ and $x \geq -6$

$\therefore -6 \leq |y| \leq 10$.

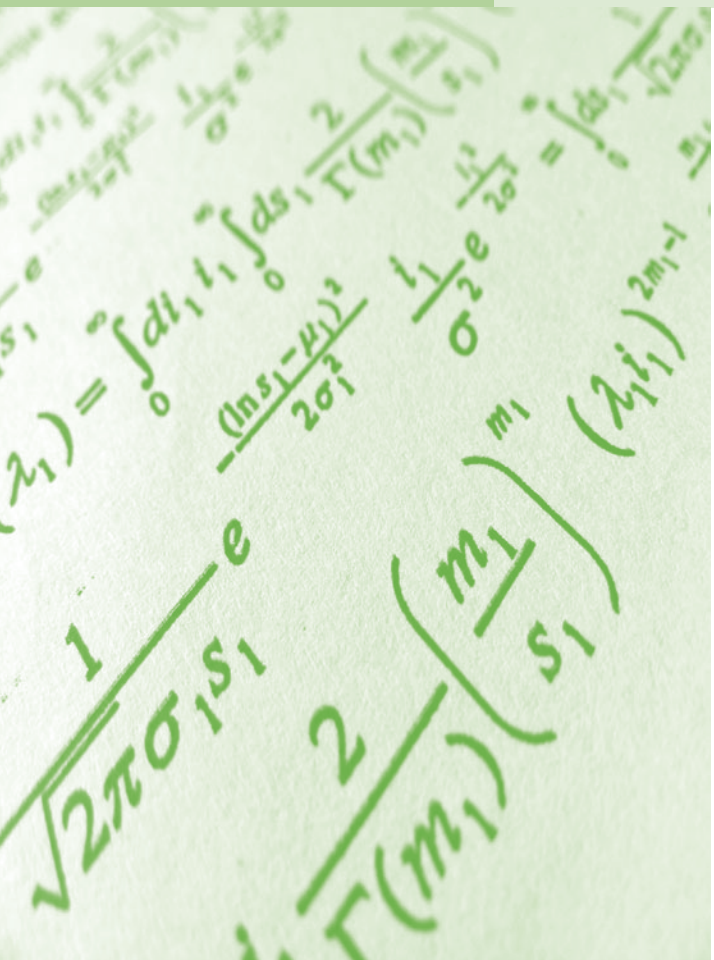
But $|y| \geq 0$.

$\therefore\ 0 \geq |y| \leq 10$

That is, $-10 \leq y \leq 10$.

Chapter 5

Statements



REMEMBER

Before beginning this chapter, you should be able to:

- Have knowledge of different types of numbers
- Apply fundamental operations on numbers

KEY IDEAS

After completing this chapter, you would be able to:

- Understand statements and truth tables of different compound statements
- Study laws of algebra of statements
- Learn methods of proof and disproof to explain validity of a statement
- Understand application to switching networks, etc.

INTRODUCTION

In this chapter, we will learn about statements, truth tables of different compound statements, laws of algebra of statements, application of truth tables to switching networks, etc.

STATEMENT

A sentence which can be judged either true or false but not both is called a **statement**. Statements are denoted by lower case letters like p, q, r, etc.

Examples:

1. p: 2 is a prime number. This statement is true.
2. q: $2 + 3 = 6$. This statement is false.

Truth Value

The truthness or falsity of a statement is called its truth value.

Truthness of a statement is denoted by T, while its falsity is denoted by F.

Examples:

1. The truth value of the statement, p: The sun rises in the east, is True.
2. The truth value of the statement q: All odd numbers are prime, is False.

Negation of a Statement

The denial of a statement is called its negation. Negation of a statement p is denoted by $\sim p$ and read as *not p* or *negation p*.

Truth Table

p	$\sim p$
T	F
F	T

Examples:

1. p: $2 + 4 = 6$

   $\sim p$: $2 + 4 \neq 6$
2. p: 3 is a factor of 10

   $\sim p$: 3 is not a factor of 10
3. p: Charminar is in Delhi

   $\sim p$: Charminar is not in Delhi

Compound Statement

A statement obtained by combining two or more simple statements using connectives is called a compound statement.

Examples:

Consider the two statements.
p: 2 is a prime number and
q: 2 is an even number.

Some compound statements that can be formed by using the statements p and q are:

1. 2 is a prime number and 2 is an even number.
2. 2 is a prime number or 2 is an even number.
3. 2 is neither a prime number nor an even number.

Let us look at some basic compound statements.

Conjunction

If p and q are two simple statements, the compound statement p and q is called the conjunction of p and q. It is denoted by $p \wedge q$.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

We observe that $p \wedge q$ is true only when both p and q are true.

Examples:

1. Let p: 4 is a perfect square and q: 2 is an odd number.

   $p \wedge q$: 4 is a perfect square and 2 is an odd number.

   As p is T and q is F, the truth value of $p \wedge q$ is F.
2. Let p: $3 > 2$ and q: $\sqrt{2}$ is an irrational number.

   Then, $p \wedge q$: $3 > 2$ and $\sqrt{2}$ is an irrational number.

   The truth value of $p \wedge q$ is true as both p and q are true.

Disjunction

If p and q are two simple statements, then the compound statement p or q is called the disjunction of p and q. It is denoted by $p \vee q$.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

We observe that, $p \vee q$ is false only when both p and q are false.

Examples:

1. Let p: The set of even primes is an empty set.

   q: 1 is a factor of every natural number.

   $p \vee q$: The set of even primes is an empty set or 1 is a factor of every natural number.

   As p is false and q is true, truth value of $p \vee q$ is true.

2. Let p: 5 is a factor of 18.

   q: 12 divides 6.

   Then, $p \vee q$: 5 is factor of 18 or 12 divides 6.

   The truth value of $p \vee q$ is false as both p and q are false.

Implication or Conditional

If p and q are two statements, the compound statement *if p then q*, is called a conditional statement. It is denoted by $p \Rightarrow q$.

The statement p is called the hypothesis (or given) and the statement q is called the conclusion (or result).

Truth Table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

We observe that, a true statement cannot imply a false statement.

Examples:

1. Let p: Every set is a subset of itself.

   q: $3 + 5 = 8$

   $p \Rightarrow q$: If every set is a subset of itself, then $3 + 5 = 8$.

   As p is true and q is true, the truth value of $p \Rightarrow q$ is true.

2. Let p: ABC is a right triangle if $\angle A = 100°$.

   q: $\angle A + \angle B + \angle C = 180°$

   $p \Rightarrow q$: ABC is a right triangle if $\angle A = 100°$, then $\angle A + \angle B + \angle C = 180°$.

   As p is false and q is true, the truth value of $p \Rightarrow q$ is true.

Bi-conditional or Bi-implication

If p and q are two statements, then the compound statement p *if and only if* q is called the bi-conditional of p and q. It is denoted by $p \Leftrightarrow q$.

Truth Table

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We observe that, $p \Leftrightarrow q$ is true if both p and q have the same truth values.

Examples:

1. Let p: $2 \times 3 = 6$
   q: $2 + 8 = 10$
   $p \Leftrightarrow q$: $2 \times 3 = 6$ if and only if $2 + 8 = 10$.
   Since both p and q are true, the truth value of $p \Leftrightarrow q$ is true.
2. Let p: Every triangle is equilateral.
   q: Charminar is in Hyderabad.
   $p \Leftrightarrow q$: Every triangle is equilateral if and only if Charminar is in Hyderabad.
   As p is false and q is true, the truth value of $p \Leftrightarrow q$ is false.

Converse, Inverse and Contrapositive of a Conditional

Let $p \Rightarrow q$ or *if p then q* be a conditional,

1. If q then p, i.e., $q \Rightarrow p$, is called the converse of $p \Rightarrow q$.
2. If not p then not q, i.e., $\sim p \Rightarrow \sim q$, is called the inverse of $p \Rightarrow q$.
3. If not q then not p, i.e., $\sim q \Rightarrow \sim p$ is called the contrapositive of $p \Rightarrow q$.

Truth Table

p	q	$\sim p$	$\sim q$	Conditional $p \Rightarrow q$	Converse $q \Rightarrow p$	Inverse $\sim p \Rightarrow \sim q$	Contrapositive $\sim q \Rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

EXAMPLE 5.1

Write the converse, inverse and contrapositive of the conditional ‘If x is odd then x^2 is odd’.

SOLUTION

Conditional: If x is odd, then x^2 is odd.

Converse: If x^2 is odd, then x is odd.

Inverse: If x is not odd, then x^2 is not odd.

Contrapositive: If x^2 is not odd, then x is not odd.

EXAMPLE 5.2

Write the converse, and the contrapositive of the conditional, ‘If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$’.

SOLUTION

Conditional: If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$.

Converse: If $\angle A + \angle B + \angle C = 180°$, then ABC is a triangle.

Contrapositive: If $\angle A + \angle B + \angle C \neq 180°$, then ABC is not a triangle.

Let us now look at the truth tables of some compound statements:

EXAMPLE 5.3

Write the truth table of $p \vee \sim q$.

SOLUTION

Truth Table

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

EXAMPLE 5.4

Write the truth table of $\sim p \vee (p \wedge q)$.

SOLUTION

Truth Table

p	q	$\sim p$	$p \wedge q$	$\sim p \vee (p \wedge q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

EXAMPLE 5.5

Write the truth table of $\sim p \Rightarrow p \vee q$.

SOLUTION

Truth Table

p	q	$\sim p$	$p \vee q$	$\sim p \Rightarrow p \vee q$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

Tautology

A compound statement which always takes **True** as its truth value is called a tautology.

Example: The truth table of $p \vee \sim p$ is

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

We observe that $p \vee {\sim}p$ always takes T as its truth value. So, $p \vee {\sim}p$ is a tautology.

EXAMPLE 5.6

Show that the compound statement $p \Rightarrow p \vee q$ is a tautology.

SOLUTION

Truth Table

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

We observe that $p \Rightarrow p \vee q$ is always True.

Hence, $p \Rightarrow p \vee q$ is a tautology.

Contradiction

A compound statement which always takes **False** as its truth value is called a contradiction.

Example:

The truth table of $p \wedge {\sim}p$ is

p	${\sim}p$	$p \wedge {\sim}p$
T	F	F
F	T	F

We observe that $p \wedge {\sim}p$ always takes F as its truth value. So $p \wedge {\sim}p$ is a contradiction.

EXAMPLE 5.7

Show that the compound statement $(p \vee {\sim}p) \Rightarrow (q \wedge {\sim}q)$ is a contradiction.

SOLUTION

Truth Table

p	q	${\sim}p$	${\sim}q$	$p \vee {\sim}p$	$q \wedge {\sim}q$	$p \vee {\sim}p \Rightarrow q \wedge {\sim}q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

We observe that $p \vee {\sim}p \Rightarrow q \wedge {\sim}q$ is always False.

Hence, $p \vee {\sim}p \Rightarrow q \wedge {\sim}q$ is a contradiction.

Contingency

A compound statement which is neither a tautology nor a contradiction is called a contingency.

Example: $p \vee q \Rightarrow {\sim}p$

Truth Table

p	q	$\sim p$	$p \vee q$	$p \vee q \Rightarrow \sim p$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Logically Equivalent Statements

Two statements r and s are said to be logically equivalent, if the last columns of their truth tables are identical.

(OR)

Two statements r and s are said to be logically equivalent if $r \Leftrightarrow s$ is a tautology. Generally, r and s will be compound statements.

If the statements r and s are logically equivalent, then we denote this as $r \equiv s$.

Note $r \Leftrightarrow s$ is always true only if both r and s have same truth values.

EXAMPLE 5.8

Show that $p \wedge q \equiv q \wedge p$.

SOLUTION

Truth Table

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

We observe that $p \wedge q$ and $q \wedge p$ have the same truth values. Hence, $p \wedge q \equiv q \wedge p$.

EXAMPLE 5.9

Show that $p \Rightarrow q \equiv \sim p \vee q$.

SOLUTION

Truth Table

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that, $p \Rightarrow q$ and $\sim p \vee q$ have the same truth values.

Hence, $p \Rightarrow q \equiv \sim p \vee q$.

Laws of Algebra of Statements

Some logical equivalences are listed under the following laws:

1. **Commutative Laws:**

   (i) $p \vee q \equiv q \vee p$

   (ii) $p \wedge q \equiv q \wedge p$

2. **Associative Laws:**

   (i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

   (ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. **Distributive Laws:**

   (i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   (ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

4. **Idempotent Laws:**

   (i) $p \vee p \equiv p$

   (ii) $p \wedge p \equiv p$

5. **De Morgan's Laws:**

   (i) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

   (ii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

6. **Identity Laws:**

   (i) $p \vee f \equiv p$, $p \vee t \equiv t$

   (ii) $p \wedge f \equiv f$, $p \wedge t \equiv p$

7. **Complement Laws:**

   (i) $p \vee (\sim p) \equiv t$

   (ii) $p \wedge (\sim p) \equiv f$

   (iii) $\sim(\sim p) \equiv p$

   (iv) $\sim t \equiv f$

   (v) $\sim f \equiv t$

List of Equivalences Based on Implications

1. $p \Rightarrow q \equiv \sim p \vee q$
2. $\sim(p \Rightarrow q) \equiv p \wedge \sim q$
3. $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

   (i.e., a conditional and its contrapositive are logically equivalent)

4. $q \Rightarrow p \equiv \sim q \Rightarrow \sim p$

   (i.e., converse and inverse of a conditional are logically equivalent)

5. $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$
6. $\sim(p \Leftrightarrow q) \equiv (\sim p) \Leftrightarrow q$ or $p \Leftrightarrow (\sim q)$

OPEN SENTENCE

A sentence involving one or more variables is called an open sentence, it becomes TRUE or FALSE when the variables are replaced by some specific values from the given set. The set from which the values of a variable can be considered is called the replacement set or domain of the variable.

Examples:

1. $x + 2 = 9$ is an open sentence.

   For $x = 7$, it becomes True and for other real values of x it becomes False.

2. $x^2 + 1 > 0$ is an open sentence.

   For all real values of x it is True.

Quantifiers

A quantifier is a word or phrase which quantifies a variable in the given open sentence.

There are two types of quantifiers.

1. Universal quantifier
2. Existential quantifier

Universal Quantifier

The quantifiers like *for all*, *for every*, *for each* are called universal quantifiers. A universal quantifier is denoted by '$\forall$'.

Examples:

1. Consider an open sentence, $|x| \geq 0$.

   This is true for all $x \in R$. So, we write $|x| \geq 0, \forall x \in R$.

2. Consider the sentence,

   $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \forall n \in N.$$

Existential Quantifier

The quantifiers like *for some*, *not all*, *there is/exists at least one* are called existential quantifiers.

An existential quantifier is denoted by '$\exists$'.

Examples:

1. Not all prime numbers are odd.
2. $\exists\, x \in R$ such that $x + 4 = 11$.

Negation of Statements Involving Quantifiers

1. p: All odd numbers are prime.

   $\sim p$: Not all odd numbers are prime

   (or)

Some odd numbers are not prime.
(or)
There is an odd number which is not prime.

2. p: All questions are difficult.
$\sim p$: Not all questions are difficult.
(or)
Some questions are not difficult.
(or)
There is at least one question which is not difficult.

3. p: All birds can fly.
$\sim p$: Not all birds can fly.
(or)
There are some birds which cannot fly.
(or)
There is at least one bird which cannot fly.

METHODS OF PROOF

Statements in mathematics are usually examined for their validity. The various steps involved in the process is referred as proof.

There are two important types of proofs.

Direct Proof

In this method, we begin with the hypothesis and end up with the desired result through a logical sequence of steps.

Example:
If x is odd, then x^2 is odd.
Given, x is an odd number.
Conclusion: x^2 is an odd number.

Proof:
$\Rightarrow x = 2k + 1$ (for some $k \in Z$)
$\Rightarrow x^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2m + 1$, where $m = 2k^2 + 2k \in Z$ ($\because k \in Z \Rightarrow 2k^2 + 2k \in Z$).
$\therefore x^2 = 2m + 1, m \in Z$
$\Rightarrow x^2$ is an odd number.
Hence, if x is an odd number, then x^2 is an odd number.

Indirect Proof

In this method, we proceed by assuming that the conclusion is false. Then we arrive at a contradiction. This implies that the desired result must be true.

Example:
If $a + b = 0$, then $(a + b)^2 = 0$. where $a, b \in Z - \{0\}$
Given: $a + b = 0$
Conclusion: $(a + b)^2 = 0$

Proof: Let us assume that $(a + b)^2 \neq 0$
$\Rightarrow (a + b)^2 > 0$ (As $(a + b)^2$ cannot be negative)
$\Rightarrow a + b > 0$ or $a + b < 0$
Which is a contradiction to the hypothesis, i.e., $a + b = 0$.
$\therefore$ Our assumption, i.e., $(a + b)^2 \neq 0$ is false.
Hence, if $a + b = 0$, then $(a + b)^2 = 0$.

METHODS OF DISPROOF

To disprove a given statement there are two methods.

Counter Example Method

In this method, we look for a counter example which disproves the given statement.

Examples:

1. Every odd number is a prime number.

   This statement is false, as 9 is an odd number but it is not a prime.

2. $x^2 - x - 6 = 0$, for all real values of x.

   This statement is false, as for $x = 2$, $x^2 - x - 6 = (2)^2 - 2 - 6 = -4 \neq 0$.

   $\therefore$ $x = 2$ is a counter example here.

Method of Contradiction

In this method, we assume that the given statement is true. Then we arrive at a contradiction. This implies that the given statement is false.

EXAMPLE 5.10

Disprove the statement, ‘There can be two right angles in a triangle’.

SOLUTION

Let ABC be a triangle.

If possible, let $\angle A = 90°$ and $\angle B = 90°$

We know that, the sum of the three internal angles of a triangle is 180°.

That is, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow 90° + 90° + \angle C = 180°$

$\Rightarrow \angle C = 0°$ which is a contradiction.

Hence, there cannot be two right angles in a triangle.

APPLICATION TO SWITCHING NETWORKS

Now we consider the statements p and p' as switches with the property that if one is on, then the other is off and vice versa.

Further, a switch allows only two possibilities.

1. It is either open (F), in which case there is no flow of current.

   (Or)

2. It is closed (T), in which case there is a flow of current.

Hence, every switch has two truth values T or F only.

Let p and q denote two switches. We can connect p and q by using a wire in a series or parallel combination as shown in Fig. 5.1.

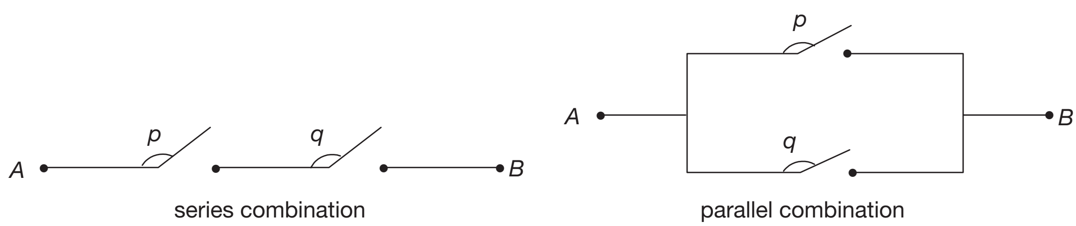


Figure 5.1

Note $p \wedge q$ denote the series combination and $p \vee q$ denote the parallel combination.

Switching Network

A switching network is a repeated arrangement of wires and switches in series and parallel combinations.

So, such a network can be described by using the connectives $\wedge$ and $\vee$.

EXAMPLE 5.11

Describe the behaviour of flow of current from A to B in the following circuit network (see Fig. 5.2).

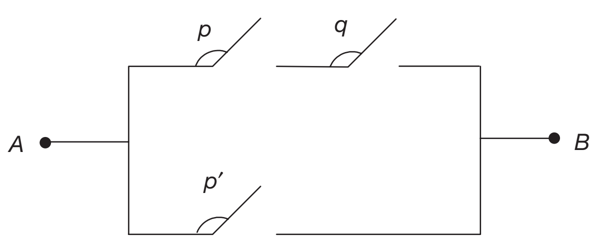


Figure 5.2

SOLUTION

The given network can be described by the compound statement $(p \wedge q) \vee p^1$. Truth table of $(p \wedge q) \vee p'$ is:

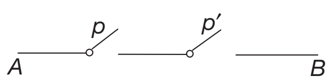
p	q	p'	$p \wedge q$	$(p \wedge q) \vee p'$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

So, current flows from A to B if:

1. p is closed, q is closed
2. p is open, q is closed
3. p is open, q is open.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If truth value of $(\sim p)$ is T, then truth value $\sim(\sim p)$ is ________.
2. $\sim(p \vee \sim q) \equiv$ ________.
3. Truth value of 'If x is even, then x^2 is even' is ________.
4. Counter example which disproves 'all primes are odd' is ________.
5. The negation of 'no dog barks' is ________.
6. The truth value of 'Hyderabad is the capital city of AP' is ________.
7. $\sim(\sim p \Rightarrow q) = \sim(p \vee q)$ (True/False)
8. If p is F and q is T, then $\sim q \Rightarrow p$ is ________.
9. The quantifier used in negation of 'every planet in the solar system has a satellite'. ($\forall$ or $\exists$)
10. Truth value of '$3 \times 7 = 28$, iff $3 + 7 = 12$' is ________.
11. The negation of '$a > b$' is $a = b$. (True/False)
12. 

    The current ________ (does/does not) flow from A to B.
13. The quantifier to be used to describe the statement, 'not all isosceles triangles are equilateral is ________. ($\forall$ or $\exists$)'.
14. If $p \vee \sim q$ is F, then q is ________.
15. The connective used in the negation of 'if the grass is green, then sky is blue' is ________.
16. Write the conjunction and implication of the statement: He is smart; He is intelligent.
17. Write the suitable quantifier for the sentence: there exists a real number x such that $x + 2 = 3$.
18. Find the truth value of 'Are you attending the meeting tomorrow?'.
19. The symbolic form of the statement, 'If p, then neither q nor r' is ______.
20. Find the inverse of the conditional, 'If I am tired, then I will take rest'.
21. The converse of converse of the statement $p \Rightarrow \sim q$ is ______.
22. Find the truth value of the statement, 'The sum of any two odd numbers is an odd number'.
23. Find the negation of the statement, 'Some odd numbers are not prime'.
24. What is the truth value of the statement, '$2 \times 3 = 6$ or $5 + 8 = 10$'?
25. $p \Rightarrow q$ is logically equivalent to _____.
26. The counter example of the statement, 'All odd numbers are primes', is _____.
27. Find the converse of the statement, 'If $ABCD$ is square, then it is a rectangle'.
28. Write the compound statement, 'If p, then q and if q, then p' in symbolic form.
29. The negation of the statement, 'I go to school everyday', is _____.
30. The contrapositive of the statement $p \Rightarrow \sim q$ is. _____.

Short Answer Type Questions

31. What is the converse of the statement $p \Rightarrow p \vee q$?
32. Write the converse, inverse and contrapositive of the conditional, 'If she is rich, then she is happy'.
33. Show that $p \Rightarrow p \vee q$ is a tautology.
34. Write the converse, inverse and contrapositive, of the statement 'In a ΔABC, if $AB \neq AC$, then $\angle B \neq \angle C$'.
35. Show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
36. Write the truth table of $(p \vee q) \vee \sim r$.
37. Show that $p \Rightarrow q \equiv \sim p \vee q$.

38. Write the truth table of $p \wedge \sim q$.

39. Show that $(p \wedge q) \vee \sim q \equiv p \vee \sim q$.

40. Write the truth table of $p \Rightarrow (p \wedge q)$.

41. Write the suitable quantifier for the following sentences:

(a) $x + 1 > x$ for all real values of x.

(b) There is no real number x such that $x^2 + 2x + 2 = 0$.

42. Prove that $(\sim p \wedge q) \wedge q$ is neither a tautology nor a contradiction.

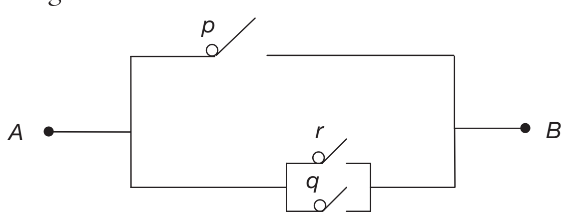
43. Write the truth table of $\sim p \vee (p \wedge q)$.

44. Show that $(p \wedge \sim p) \wedge (p \vee q)$ is a contradiction.

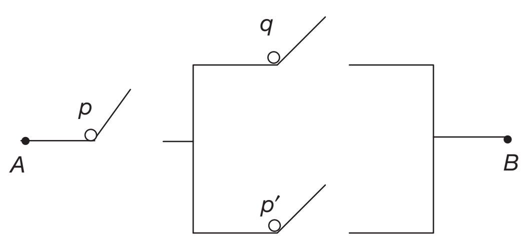
45. Negation of the compound statement $[(p \wedge q) \vee (p \wedge \sim q)]$ is ________.

Essay Type Questions

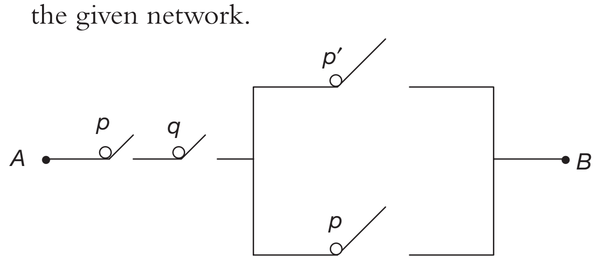
46. Discuss when does the current flow from A to B in the given network.



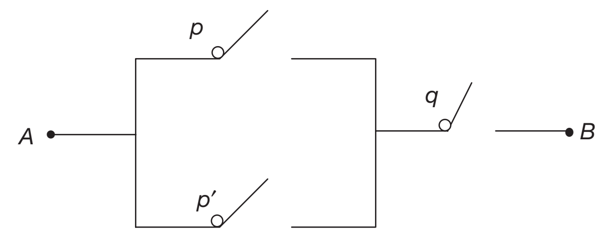
47. Discuss when does the current flow from A to B in the given network.



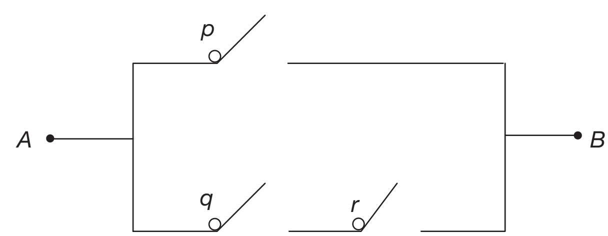
48. Discuss when does the current flow from A to B in the given network.



49. Discuss when does the current flow from A to B in the given network.



50. Discuss when does the current flow from the points A to B in the given network.



CONCEPT APPLICATION

Level 1

1. For which of the following cases does the statement $p \wedge \sim q$ take the truth value as true?

(a) p is true, q is true

(b) p is false, q is true

(c) p is false, q is false

(d) p is true, q is false

2. Which of the following sentences is a statement?

(a) Ramu is a clever boy.

(b) What are you doing?

(c) Oh! It is amazing.

(d) Two is an odd number.

3. Which of the following laws does the connective $\wedge$ satisfy?
(a) Commutative law
(b) Idempotent law
(c) Associative law
(d) All of these

4. The truth value of the statement, 'We celebrate our Independence day on August 15th', is
(a) T
(b) F
(c) Neither T nor F
(d) Cannot be determined

5. When does the inverse of the statement $\sim p \Rightarrow q$ results in T?
(a) $p = T$, $q = T$
(b) $p = T$, $q = F$
(c) $p = F$, $q = F$
(d) Both (b) and (c).

6. Which of the following is a contradiction?
(a) $p \vee q$ (b) $p \wedge q$
(c) $p \vee \sim p$ (d) $p \wedge \sim p$

7. In which of the following cases, $p \Leftrightarrow q$ is true?
(a) p is true, q is true
(b) p is false, q is true
(c) p is true, q is false
(d) p is false, q is false

8. Find the counter example of the statement 'Every natural number is either prime or composite'.
(a) 5 (b) 1
(c) 6 (d) None of these

9. Which of the following pairs are logically equivalent?
(a) Conditional, Contrapositive
(b) Conditional, Inverse
(c) Contrapositive, Converse
(d) Inverse, Contrapositive

10. The property $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ is called
(a) associative law.
(b) commutative law.
(c) distributive law.
(d) idempotent law.

11. Which of the following is contingency?
(a) $p \vee \sim p$ (b) $p \wedge q \Rightarrow p \vee q$
(c) $p \wedge (\sim q)$ (d) None of these

12. Which of the following pairs are logically equivalent?
(a) Converse, Contrapositive
(b) Conditional, Converse
(c) Converse, Inverse
(d) Conditional, Inverse

13. Which of the following connectives can be used for describing a switching network?
(a) $(p \wedge q) \vee p'$
(b) $(p \vee q) \wedge q'$
(c) Both (a) and (b)
(d) None of these

14. Find the quantifier which best describes the variable of the open sentence $x + 3 = 5$.
(a) Universal
(b) Existential
(c) Neither (a) nor (b)
(d) Cannot be determined

15. Which of the following is equivalent to $p \Leftrightarrow q$?
(a) $p \Rightarrow q$
(b) $q \Rightarrow p$
(c) $(p \Rightarrow q) \wedge (q \Rightarrow p)$
(d) $(p \Rightarrow q) \vee (q \Rightarrow p)$

16. The property $\sim(p \wedge q) \equiv \sim p \vee \sim q$ is called ________.
(a) associative law
(b) De Morgan's law
(c) commutative law
(d) idempotent law

17. The statement $p \vee q$ is

(a) a tautology.

(b) a contradiction.

(c) Neither (a) nor (b)

(d) Cannot say

18. Which of the following compound statement represents a series network?

(a) $p \vee q$ (b) $p \Rightarrow q$

(c) $p \wedge q$ (d) $p \Leftrightarrow q$

19. Which of the following is a tautology?

(a) $p \wedge q$ (b) $p \vee q$

(c) $p \vee \sim p$ (d) $p \wedge \sim p$

20. Find the truth value of the compound statement, 'If 2 is a prime number, then hockey is the national game of India'.

(a) T

(b) F

(c) Neither (a) nor (b)

(d) Cannot be determined

21. Find the truth value of the compound statement, '4 is the first composite number and $2 + 5 = 7$'.

(a) T

(b) F

(c) Neither (a) nor (b)

(d) Cannot be determined

22. Find the inverse of the statement, 'If ΔABC is equilateral, then it is isosceles'.

(a) If ΔABC is isosceles, then it is equilateral.

(b) If ΔABC is not equilateral, then it is isosceles.

(c) If ΔABC is not equilateral, then it is not isosceles.

(d) If ΔABC is not isosceles, then it is not equilateral.

23. The statement $p \Rightarrow p \vee q$ is

(a) a tautology.

(b) a contradiction.

(c) Both (a) and (b)

(d) Neither (a) nor (b)

24. What is the truth value of the statement 'Two is an odd number, iff 2 is a root of $x^2 + 2 = 0$'?

(a) T

(b) F

(c) Neither (a) nor (b)

(d) Cannot be determined

25. Which of the following connectives satisfy commutative law?

(a) $\wedge$ (b) $\vee$

(c) $\Leftrightarrow$ (d) All of these

26. $\sim[\sim p \wedge (p \Leftrightarrow q)] \equiv$ ________.

(a) $p \vee q$ (b) $q \wedge p$

(c) T (d) F

27. Write the negation of the statement 'If the switch is on, then the fan rotates'.

(a) 'If the switch is not on, then the fan does not rotate'.

(b) 'If the fan does not rotate, then the switch is not on'.

(c) 'The switch is not on or the fan rotates'.

(d) 'The switch is on and the fan does not rotate'.

28. If p: The number of factors of 20 is 5 and q: 2 is an even prime number, then the truth values of inverse and contrapositive of $p \Rightarrow q$ respectively are

(a) T, T. (b) F, F.

(c) T, F. (d) F, T.

29. 'No square of a real number is less than zero' is equivalent to

(a) for every real number a, a^2 is non-negative.

(b) $\forall\, a \in R$, $a^2 \geq 0$.

(c) Either (a) or (b)

(d) None of these

30. If a compound statement r is contradiction, then find the truth value of $(p \Rightarrow q) \wedge (r) \wedge [p \Rightarrow (\sim r)]$.

(a) T (b) F

(c) T or F (d) None of these

PRACTICE QUESTIONS

Level 2

31. Which of the following is/are counter example(s) of the statement $x^2 - 7x + 10 > 0$, for all real x?

(A) 2 (B) 3

(C) 4 (D) 5

(a) Only (A) and (D)
(b) Only (B) and (C)
(c) All (A), (B), (C) and (D)
(d) None of these

32. If p: 3 is an odd number and q: 15 is a prime number, then the truth value of $[\sim(p \Leftrightarrow q)]$ is equivalent to that of ________.

(A) $p \Leftrightarrow (\sim q)$

(B) $(\sim p) \Leftrightarrow q$

(C) $\sim(p \wedge q)$

(a) Only (A)
(b) Only (C)
(c) Both (A) and (B)
(d) All (A), (B) and (C)

33. The compound statement, 'If you want to top the school, then you do not study hard' is equivalent to

(a) 'If you want to top the school, then you need to study hard'.

(b) 'If you will not top the school, then you study hard'.

(c) 'If you study hard, then you will not top the school'.

(d) 'If you do not study hard, then you will top the school'.

34. If p: 25 is a factor of 625 and q: 169 is a perfect square, then $\sim (p \Rightarrow q)$ is equivalent to

(a) $p \wedge q$. (b) $(\sim p) \wedge q$.

(c) $p \wedge (\sim q)$. (d) Both (b) and (c)

35. The compound statement, 'If you won the race, then you did not run faster than others' is equivalent to

(a) 'If you won the race, then you ran faster than others'.

(b) 'If you ran faster than others, then you won the race'.

(c) 'If you did not win the race, then you did not run faster than others'.

(d) 'If you ran faster than others, then you did not win the race'.

36. 'If x is a good actor, then y is bad actress' is

(a) a tautology. (b) a contradiction.

(c) a contingency. (d) None of these

37. Which of the following is negation of the statement 'All birds can fly'.

(a) 'Some birds cannot fly'.

(b) 'All the birds cannot fly'.

(c) 'There is at least one bird which can fly'.

(d) All of these

38. What are the truth values of $(\sim p \Rightarrow \sim q)$ and $\sim(\sim p \Rightarrow q)$ respectively, when p and q always speak true in any argument?

(a) T, T (b) F, F

(c) T, F (d) F, T

39. If p: 4 is an odd number and q: 4^3 is an even number, then which of the following is equivalent to $\sim(p \Rightarrow q)$?

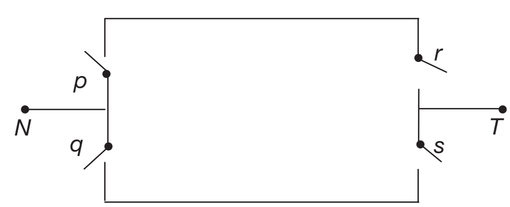
(a) '4 is an odd number and 4^3 is an even number'.

(b) 'The negation of the statement '4 is not an odd number or 4^3 is not an even number'.

(c) Both (a) and (b)

(d) None of these

40.



In the above network, current flows from N to T, when

(a) p is closed, q is closed, r is open and s is open.

(b) p is closed, q is open, s is closed and r is open.

(c) q is closed, p is open, r is open and s is closed.

(d) p is open, q is open, r is closed and s is closed.

41. If p: In a triangle, the centroid divides each median in the ratio 1 : 2 from the vertex and q: In an equilateral triangle, each median is perpendicular bisector of one of its sides. The truth values of inverse and converse of $p \Rightarrow q$ are respectively

(a) T, T. (b) F, F.

(c) T, F. (d) F, T.

42. If p always speaks against q, then $p \Rightarrow p \vee \sim q$ is

(a) a tautology. (b) contradiction.

(c) contingency. (d) None of these

43. If p: Every equilateral triangle is isosceles and q: Every square is a rectangle, then which of the following is equivalent to $\sim(p \Rightarrow q)$?

(a) The negation of 'Every equilateral triangle is not isosceles or every square is rectangle'.

(b) 'Every equilateral triangle is not isosceles, then every square is not a rectangle'.

(c) 'Every equilateral triangle is isosceles, then every square is a rectangle'.

(d) None of these

44. When does the value of the statement $(p \wedge r) \Leftrightarrow (r \wedge q)$ become false?

(a) p is T, q is F

(b) p is F, r is F

(c) p is F, q is F and r is F

(d) None of these

45. If the truth value of $p \vee q$ is true, then truth value $\sim p \wedge q$ is

(a) false if p is true.

(b) true if p is true.

(c) false if q is true.

(d) true if q is true.

46. The compound statement, 'If you want to win the gold medal in Olympics, then you need to work hard' is equivalent to

(a) 'If you will not win the gold medal in Olympics, then you need not to work hard.'

(b) 'If you do not work hard, then you will not win the gold medal in Olympics.'

(c) Both (a) and (b)

(d) None of these

47. If p: sum of the angles in a triangle is 180° and q: every angle in a triangle is more than 0°, then $(p \Rightarrow q) \vee p$ is equivalent to

(a) $p \vee \sim q$. (b) $(p \wedge q) \Rightarrow p$.

(c) $\sim p \wedge \sim q$. (d) Both (a) and (b)

48. Which of the following is the negation of the statement, 'All animals are carnivores'?

(a) Some animals are not carnivores.

(b) Not all animals are carnivores.

(c) There is atleast one animal which is not carnivores.

(d) All of these

49. $(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow \sim q)$ is a

(a) tautology. (b) contradiction.

(c) contingency. (d) None of these

50. Which of the following is a counter example of $x^2 - 6x + 8 \leq 0$?

(a) $x = 2$ (b) $x = 3$

(c) $x = 4$ (d) $x = 5$

Level 3

51. If p and q are two statements, then $p \vee \sim(p \Rightarrow \sim q)$ is equivalent to

(a) $p \wedge \sim q$ (b) p

(c) q (d) $\sim p \wedge q$

52.

In the above network, current does not flow, when

(a) p is open, q is open and r is open.

(b) p is closed, q is open and r is open.

(c) p is open, q is closed and r is closed.

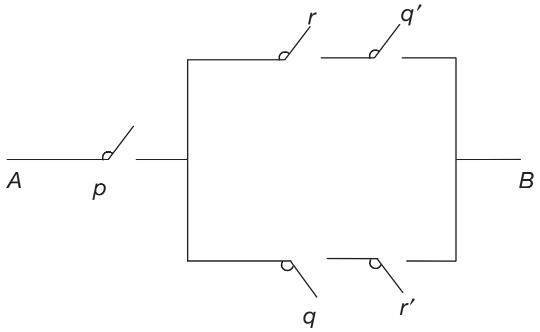
(d) None of these

53. The contrapositive of the statement $\sim p \Rightarrow (p \wedge \sim q)$ is

(a) $p \Rightarrow (\sim p \vee q)$ (b) $p \Rightarrow (p \wedge q)$

(c) $p \Rightarrow (\sim p \wedge q)$ (d) $(\sim p \vee q) \Rightarrow p$

54.



In the above circuit, the current flows from A to B when

(a) p is closed, q is open and r is open.

(b) p is closed, q is closed and r is open.

(c) p is closed, q is closed and r is closed.

(d) All of these

55. If p: $5x + 6 = 8$ is an open sentence and q: 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$, then which of following is equivalent to $\sim[\sim p \vee q]$?

(a) The negation of 'If $5x + 6 = 8$ is an open sentence, then 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$'.

(b) $5x + 6 = 8$ is an open sentence or 3, 4 are not roots of the equation $x^2 - 7x + 12 = 0$.

(c) $5x + 6 = 8$ is not an open sentence and 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$.

(d) None of these

56. If the truth value of $p \vee q$ is T, then the truth value $p \wedge q$ is

(a) T

(b) F

(c) Neither (a) nor (b)

(d) Cannot be determined

57. $(p \wedge \sim q) \wedge q$ is equivalent to

(a) $p \vee q$ (b) $p \wedge q$

(c) $p \vee \sim q$ (d) $p \wedge \sim q$

58. Which of the following is a tautology?

(a) $p \Rightarrow (q \vee \sim q)$ (b) $p \Leftrightarrow (\sim q \wedge p)$

(c) $p \Leftrightarrow (p \wedge \sim p)$ (d) $(p \wedge \sim p) \wedge q$

59. $\sim[\sim p \vee (\sim p \Leftrightarrow q)]$ is equivalent to

(a) $p \wedge (p \Leftrightarrow q)$.

(b) $p \wedge q$.

(c) Both (a) and (b)

(d) None of these

60. When does the truth value of the statement $(p \Rightarrow q) \vee (r \Leftrightarrow s)$ become true?

(a) p is true (b) q is true

(c) r is true (d) s is true

61. Given that p: x is a prime number and q: x^2 is a composite number. Then $\sim(\sim p \Rightarrow q)$ is equivalent to

(a) x is not a prime number and x^2 is not a composite number.

(b) x is not a prime number and x^2 is a composite number.

(c) Both (a) and (b)

(d) None of these

62. Which of the following is a contradiction?

(a) $(p \vee q) \Leftrightarrow (p \wedge q)$

(b) $(p \vee q) \Rightarrow (p \wedge q)$

(c) $(p \Rightarrow q) \vee (q \Rightarrow p)$

(d) $(\sim q) \wedge (p \wedge q)$

63. When does the truth value of the statement $[(p \Leftrightarrow q) \Rightarrow (q \Leftrightarrow r)] \Rightarrow (r \Leftrightarrow p)$ become false?

(a) p is true, q is false, r is true.

(b) p is false, q is true, r is true.

(c) p is false, q is false, r is false.

(d) p is true, q is true, r is true.

64. If p is negation of q, then $(p \Rightarrow q) \vee (q \Rightarrow p)$ is a

(a) tautology.

(b) contradiction.

(c) contingency.

(d) None of these

PRACTICE QUESTIONS

65. Which of the following is a contingency?

(a) $\sim p \wedge q$ (b) $p \vee \sim q$

(c) $\sim p \wedge \sim q$ (d) All of these

66. Which among the following is negation of $\sim(p \wedge q) \vee r$?

(a) $(p \wedge q) \vee r$

(b) $(p \vee q) \vee \sim r$

(c) $(p \vee q) \vee r$

(d) $(p \wedge q) \wedge \sim r$

67. The inverse of a conditional statement is, 'If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$'. Then the contrapositive of the conditional statement is

(a) 'If $\angle A + \angle B + \angle C = 180°$, then ABC is a triangle'.

(b) 'If $\angle A + \angle B + \angle C \neq 0$, then ABC is not a triangle'.

(c) 'If ABC is not a triangle, then $\angle A + \angle B + \angle C \neq 180°$'.

(d) 'If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$'.

68. 'There is atleast one prime number which is not odd number'. The quantifier used in the above statement is _______.

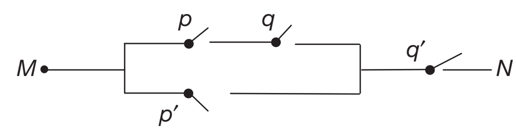
(a) universal quantifier

(b) existential quantifier

(c) Both (a) and (b)

(d) No quantifier is used

69.



In the above network, current flows from M to N, when

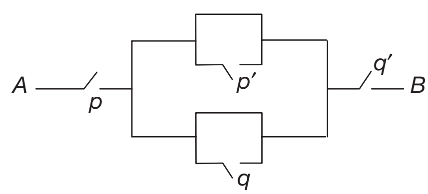
(a) p is closed, q is closed.

(b) p is closed, q is open.

(c) p is open, q is closed.

(d) p is open, q is open.

70.



In the above network, current flows from A to B, when

(a) p is open, q is open.

(b) p is closed, q is closed.

(c) p is closed, q is open.

(d) p is open, q is closed.

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. False
2. $\sim p \wedge q$
3. True
4. 2 is even prime
5. some dogs bark.
6. True
7. True
8. True
9. $\exists$
10. True
11. False
12. does not
13. $\exists$
14. True
15. and
16. He is smart and he is intelligent.
    If he is smart, then he is intelligent.
17. $\exists$
18. Neither true nor false
19. $p \Rightarrow \sim q \wedge \sim r$.
20. 'If I am not tired, then I will not take rest'.
21. $p \Rightarrow \sim q$
22. False
23. All odd numbers are primes.
24. True
25. $\sim p \vee q$
26. 9
27. If $ABCD$ is a rectangle, then it is square.
28. $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
29. Some days I do not go to school.
30. $q \Rightarrow \sim p$

Short Answer Type Questions

31. $p \vee q \Rightarrow p$
32. Converse: If she is happy, then she is rich.
    Inverse: If she is not rich, then she is not happy.
    Contrapositive: If she is not happy, then she is not rich.
34. Converse: In a ΔABC, if $\angle B \neq \angle C$, then $AB \neq AC$.
    Inverse: In a ΔABC, if $AB = AC$, then $\angle B = \angle C$.
    Contrapositive: In a ΔABC, if $\angle B = \angle C$, then $AB = AC$.
41. $\forall$
    $\forall$
45. $\sim p \vee [-q \vee q]$

Essay Type questions

46. (i) p is closed, q is closed, r is closed
    (ii) p is closed, q is closed, r is open
    (iii) p is closed, q is open, r is closed
    (iv) p is open, q is closed, r is closed
    (v) p is closed, q is open, r is open
    (vi) p is open, q is closed, r is open
    (vii) p is open, q is open, r is closed
47. p is closed, q is closed
48. p is closed, q is closed.
49. (i) p is closed, q is closed
    (ii) p is open, q is closed
50. (i) p is closed, q is closed, r is closed
    (ii) p is closed, q is closed, r is open
    (iii) p is closed, q is open, r is closed
    (iv) p is open, q is closed, r is closed
    (v) p is closed, q is open, r is open

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (c)
11. (c)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (c)	**18.** (c)	**19.** (c)	**20.** (a)
21. (a)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (a)	**27.** (d)	**28.** (d)	**29.** (c)	**30.** (b)

Level 2

31. (c)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (d)	**36.** (d)	**37.** (a)	**38.** (c)	**39.** (d)	**40.** (c)
41. (b)	**42.** (a)	**43.** (a)	**44.** (d)	**45.** (a)	**46.** (b)	**47.** (d)	**48.** (d)	**49.** (a)	**50.** (d)

Level 3

51. (b)	**52.** (d)	**53.** (d)	**54.** (b)	**55.** (a)	**56.** (d)	**57.** (b)	**58.** (a)	**59.** (c)	**60.** (b)
61. (a)	**62.** (d)	**63.** (b)	**64.** (a)	**65.** (d)	**66.** (d)	**67.** (a)	**68.** (b)	**69.** (d)	**70.** (c)

CONCEPT APPLICATION

Level 1

2. Recall the definition of statement.
3. Recall the properties.
6. Verify through truth tables.
7. Check from truth tables.
10. Recall the properties.
11. Converse of $p \Rightarrow q$ is $q \Rightarrow p$.
13. Recall the concept of switching network.
14. For only one value of 'x' the equation is true.
15. Check from truth tables.
16. Recall the properties.
17. Verify through truth tables.
18. Recall the concept of switching networking.
19. Check from truth tables.
20. If $p \Rightarrow q$ is false only p is true, q is false.
21. Conjunction is true only when both the statements are true.
22. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
23. Use truth table.
24. Recall bi-conditional truth table.
25. Recall the properties.
26. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
27. $\sim(p \Rightarrow q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$.
28. (i) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

    The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

    (ii) $p \Rightarrow q$ is false only when p is true and q is false.
29. Square of a real number is always non-negative.
30. Conjunction is false if atleast one of the statements is false.

Level 2

31. (i) Check from the options.

    (ii) Go contrary to the given statement by substituting the value of x in it.
32. (i) Use truth table of double implication.

    (ii) Apply the identity, $\sim(p \Leftrightarrow q) \equiv (\sim p \Leftrightarrow q) \equiv (p \Leftrightarrow \sim q)$.
33. $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$.
34. (i) Both p and q are true.

    (ii) $\sim(p \Rightarrow q) \equiv p \wedge \sim q$.
35. A conditional is equivalent to its contrapositive. That is, $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$.
36. Given sentence is not a statement.
37. Contradict the given statement by using different quantifiers.
38. $p \Rightarrow q$ is false only when p is true and q is false.
39. $\sim(p \Rightarrow q) \equiv p \wedge \sim q \equiv \sim(p \vee q)$.
40. Current flows only when p and r are closed or q and s are closed.
41. (i) p is false and q is true.

    (ii) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$. The converse of $p \Rightarrow q$ is $q \Rightarrow p$.

    (iii) $p \Rightarrow q$ is false only when p is true and q is false.
42. Verify through truth table.
43. $\sim(p \Rightarrow q) \equiv p \wedge \sim q \equiv \sim(p \vee q)$.
44. (i) $x \wedge y$ is always false when y is false.

    (ii) Compare with truth table of double implication.
46. We know that, a conditional statement is equivalent to its contrapositive.

Let the given statement be $p \Rightarrow q$.

$\therefore$ Option (a) represents $\sim p \Rightarrow \sim q$

Option (b) represents $\sim q \Rightarrow \sim p$.

$\therefore$ Option (b) follows.

47. Given that p is true and q is true.

$\therefore (p \Rightarrow q) \vee p \equiv t$.

Option (a): $p \vee \sim q \equiv t$.

Option (b): $(p \wedge q) \Rightarrow p \equiv t$.

Option (c): $\sim p \wedge \sim q \equiv f$.

Option (d) follows.

48. Negation of all is all not follows.

49. Let $(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow \sim q) \equiv r$

p	q	$p \Leftrightarrow q$	$\sim p$	$\sim q$	$\sim p \Leftrightarrow \sim q$	r
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

$\therefore$ Given statement is tautology.

50. $x^2 - 6x + 8 \leq 0$

$(x - 2)(x - 4) \leq 0$

$x \in [2, 4]$

$\therefore x = 5 \notin [2, 4]$.

Level 3

51. (i) $\sim(p \Rightarrow q) = \sim p \vee q$.

(ii) $p \vee (q \vee r) = (p \vee q) \vee r$.

52. Current flows irrespective of p, q and r whether closed or open.

53. The contrapositive of $p \Rightarrow q \Rightarrow \sim q \Rightarrow \sim p$.

54. (i) Parallel connection is represented by $x \wedge y$.

(ii) Series connection is represented by $x \vee y$.

55. $\sim(\sim p \vee q) = p \wedge \sim q$.

56. Given $p \vee q$ is true, i.e., any one of p or q is true, so we cannot say the truth value of $p \wedge q$.

57. $(p \vee \sim q) \wedge q$

$\equiv (p \wedge q) \vee (\sim q \wedge q)$ (distributive property)

$\equiv (p \wedge q) \vee f \equiv p \wedge q$.

58. Consider Option (a)

Since $q \vee \sim q$ is true

$p \Rightarrow (q \vee \sim q)$ is always true

$\therefore p \Rightarrow (q \vee \sim q)$ is tautology.

59. $\sim[\sim p \vee (\sim p \Leftrightarrow q)]$

$\equiv [p \wedge \sim(\sim p \Leftrightarrow q)]$

$\equiv p \wedge (p \Leftrightarrow q)$ or $p \wedge (\sim p \Leftrightarrow \sim q)$

From truth tables,

$p \wedge (p \Leftrightarrow q) \equiv p \wedge q$.

60. When q is true, $p \Rightarrow q$ is always true.

$\therefore (p \Rightarrow q) \vee (r \Leftrightarrow s)$ is true. When q is true.

61. $\sim(\sim p \Rightarrow q) \equiv \sim p \wedge \sim q$.

$\therefore \sim p \wedge \sim q = x$ is not a prime number and x^2 is not a composite number.

62. From the truth tables, first three options are not contradictions.

Option (d): $(\sim q) \wedge (p \wedge q)$

$\equiv p \wedge (\sim q \wedge q)$

$\equiv p \wedge (f) \equiv f$.

63. From the options,

$(p \Leftrightarrow q) \Rightarrow (q \Leftrightarrow r)$

$\Rightarrow (r \Leftrightarrow p)$ is false, when p is false, q is true and r is true.

64. Given, p is negation of $q \Rightarrow p \equiv \sim q$

$\therefore (p \Rightarrow q) \vee (q \Rightarrow p)$

$\Rightarrow (\sim q \Rightarrow q) \vee (q \Rightarrow \sim q)$

$\therefore$ If $(\sim q \Rightarrow q)$ is T, then $(q \Rightarrow \sim q)$ is F.

If $(\sim q \Rightarrow q)$ is F, then $(q \Rightarrow \sim q)$ is T.

$\therefore$ T or F $\equiv$ F or T $\equiv$ T.

65. From the truth table

All the options are contingencies.

66. $\sim[\sim(p \wedge q) \vee r] \equiv (p \wedge q) \wedge \sim r$.

67. Let inverse of a conditional statement be $p \Rightarrow q$.

$\therefore$ The conditional statement is $\sim p \Rightarrow \sim q$.

$\therefore$ The contrapositive of the conditional is $q \Rightarrow p$.

That is, if $\angle A + \angle B + \angle C = 180°$, then ABC is a triangle.

68. According to the statement, some prime numbers of are not odd numbers, the quantifier is existential quantifier.

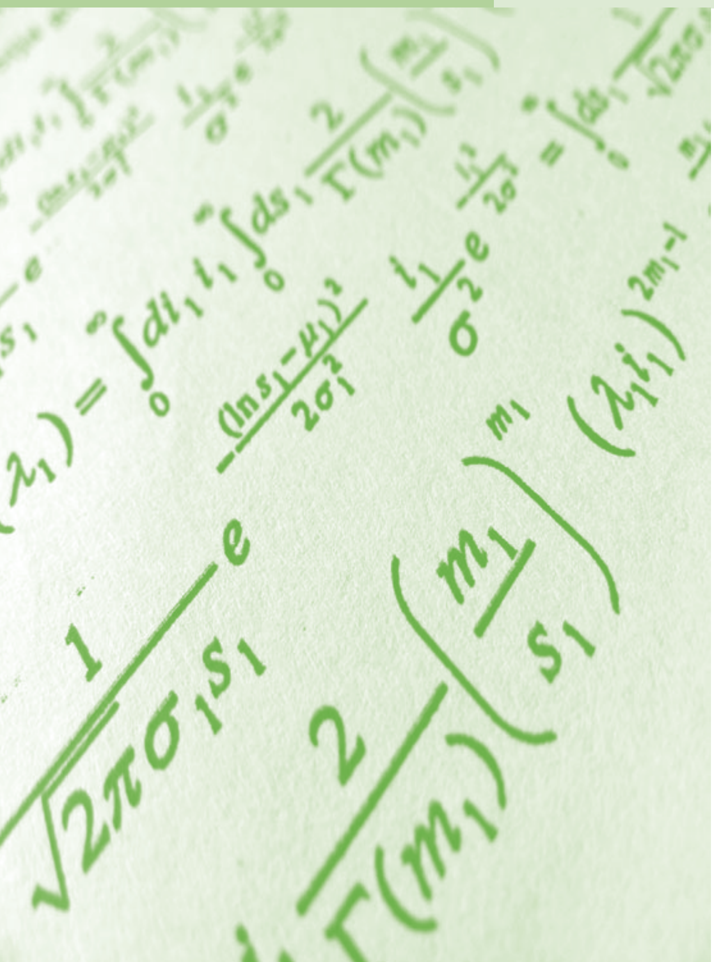
69. For the given network, current flows from M to N when p is open and q is open.

70. For the given network, current flows from A to B when p is closed and q is open.

Chapter

6

Sets, Relations and Functions



REMEMBER

Before beginning this chapter, you should be able to:

- Know the use of sets and some definitions of sets
- Apply fundamental operations on sets
- Recall the term Venn diagram
- Define terms such as relations, functions, etc.

KEY IDEAS

After completing this chapter, you would be able to:

- Learn representation of sets using different methods
- Apply advanced operations on sets such as union, intersection, etc.
- Represent sets using Venn diagrams and apply operations on it
- Define advanced form of terms such as relations, functions, domain, range, etc.
- Present relations and functions by methods such as arrow diagram, tree diagram, graphs, etc.

INTRODUCTION

In everyday life we come across different collections of objects. For example: A herd of sheep, a cluster of stars, a posse of policemen, etc. In Mathematics, we call such collections as sets. The objects are referred as the elements of the sets.

SET

A set is a well-defined collection of objects.

Let us understand the meaning of a well-defined collection of objects.

We say that a collection of objects is well-defined if there is some reasons or rules by which we can say, whether a given object of the universe belongs to or does not belong to the collection.

We usually denote the sets by capital letters A, B, C or X, Y, Z, etc.

To understand the concept of a set, let us look at some examples.

Examples:

1. Let us consider the collection of odd natural numbers less than or equal to 15.

   In this example, we can definitely say what the collection is. The collection comprises the numbers 1, 3, 5, 7, 9, 11, 13 and 15.

2. Let us consider the collection of students in a class who are good at painting. In this example, we cannot say precisely which students of the class belong to our collection. So, this collection is not well-defined.

Hence, the first collection is a set whereas the second collection is not a set. In the first example given above, the set of the odd natural numbers less than or equal to 15 can be represented as set $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$.

Elements of a Set

The objects in a set are called its elements or members.

If a is an element of a set A, then we say that a belongs to A and we write it as, $a \in A$.

If a is not an element of A, then we say that a does not belong to A and we write it as, $a \notin A$.

Some Sets of Numbers and Their Notations

N = Set of all natural numbers = $\{1, 2, 3, 4, 5, \ldots\}$
W = Set of all whole numbers = $\{0, 1, 2, 3, 4, 5, \ldots\}$
Z or I = Set of all integers = $\{0, \pm1, \pm2, \pm3, \ldots\}$

Q = Set of all rational numbers $= \left\{\frac{p}{q} \text{ where } p, q \in Z \text{ and } q \neq 0\right\}$

Cardinal Number of a Set

The number of elements in a set A is called its cardinal number. It is denoted by $n(A)$. A set which has finite number of elements is a finite set and a set which has infinite number of elements is an infinite set.

Examples:

1. Set of English alphabets is a finite set.
2. Set of number of days in a month is a finite set.
3. The set of all even natural numbers is an infinite set.

4. Set of all the lines passing through a point is an infinite set.
5. The cardinal number of the set $X = \{a, c, c, a, b, a\}$ is $n(X) = 3$, as in sets only distinct elements are counted.
6. If $A = \{a, \{a, b\}, b, c, \{c, d\}\}$, then $n(A) = 5$.

Representation of Sets

We represent sets by the following methods:

Roster or List Method

In this method, a set is described by listing out all the elements in the set within curly brackets.

Examples:

1. Let W be the set of all letters in the word JANUARY.
   Then we represent W as, $W = \{A, J, N, R, U, Y\}$.
2. Let M be the set of all multiples of 3 less than 20. Then we represent the set M as, $M = \{3, 6, 9, 12, 15, 18\}$.

Set-builder Method

In this method, a set is described by using a representative and stating the property or properties which the elements of the set satisfy, through the representative.

Examples:

1. Let D be the set of all days in a week. Then we represent D as, $D = \{x/x \text{ is a day in a week}\}$.
2. $N = \{x/10 < x < 20 \text{ and } x \in N\}$.

Types of Sets

Empty Set or Null Set or Void Set

A set with no elements in it is called an empty set (or) void set (or) null set. It is denoted by { } or ϕ. (read as 'phi').

Note $n(\phi) = 0$.

Examples:

1. Set of all positive integers less than 1 is an empty set.
2. Set of all mango trees with apples.

Singleton Set

A set consisting of only one element is called a singleton set.

Examples:

1. The set of all vowels in the word MARCH is a singleton, as A is the only vowel in the word.
2. The set of whole numbers which are not natural numbers is a singleton, as 0 is the only whole number which is not a natural number.
3. The set of all SEVEN WONDERS in India is a singleton, as Tajmahal is the only wonder in the set.

Equivalent Sets

Two sets A and B are said to be equivalent if their cardinal numbers are equal.
We write this symbolically as $A \sim B$ or $A \leftrightarrow B$.

Examples:

1. Sets, $X = \{2, 4, 6, 8\}$ and $Y = \{a, b, c, d\}$ are equivalent as $n(X) = n(Y) = 4$.
2. Sets, $X = \{\text{Dog, Cat, Rat}\}$ $Y = \{\triangle, \bigcirc, \square\}$ are equivalent.
3. Sets, $X = \{-1, -7, -5\}$ and $B = \{\text{Delhi, Hyderabad}\}$ are not equivalent, as $n(X) \neq n(B)$.

Note If the sets A and B are equivalent, we can establish a one-to-one correspondence between the two sets, i.e., we can pair up elements in A and B such that every element of A is paired with a distinct element of set B and every element of set B is paired with a distinct element of set A.

Equal Sets

Two sets A and B are said to be equal if they have the same elements.

Examples:

1. Sets, $A = \{a, e, i, o, u\}$ and $B = \{x/x \text{ is a vowel in the English alphabet}\}$ are equal sets.
2. Sets, $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$ are not equal sets.
3. Sets, $A = \{1, 2, 3, 4, \ldots\}$ and $B = \{x/x \text{ is a natural number}\}$ are equal sets.

Note If A and B are equal sets, then they are equivalent but the converse need not be true.

Disjoint Sets

Two sets A and B are said to be disjoint, if they have no elements in common.

Examples:

1. Sets $X = \{3, 6, 9, 12\}$ and $Y = \{5, 10, 15, 20\}$ are disjoint as they have no element in common.
2. Sets $A = \{a, e, i, o, u\}$ and $B = \{e, i, j\}$ are not disjoint as they have common elements e and i.

Subset and Superset

Let A and B be two sets. If every element of set A is also an element of set B, then A is said to be a subset of B or B is said to be a superset of A. If A is a subset of B, then we write $A \subseteq B$ or $B \supseteq A$.

Examples:

1. Set $A = \{2, 4, 6, 8\}$ is a subset of set $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
2. Set of all primes except 2 is a subset of the set of all odd natural numbers.
3. Set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is a superset of set $B = \{1, 3, 5, 7\}$.

Notes

1. Empty set is a subset of every set.
2. Every set is a subset of itself.
3. If a set A has n elements, then the number of subsets of A is 2^n.
4. If a set A has n elements, then the number of non-empty subsets of A is $2^n - 1$.

Proper Subset

If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B and is denoted by $A \subset B$. When $A \subset B$ then B is called a superset of A and is denoted as $B \supset A$, if $A \subset B$ then $n(A) < n(B)$ and if $B \supset A$ then $n(B) > n(A)$.

Power Set

The set of all subsets of a set A is called its power set. It is denoted by $P(A)$.

Example: Let $A = \{x, y, z\}$. Then the subsets of A are ϕ, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x, y, z\}$.

So, $P(A) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$.

We observe that the cardinality of $P(A)$ is $8 = 2^3$.

Notes

1. If a set A has n elements, then the number of subsets of A is 2^n, i.e., the cardinality of the power set is 2^n.
2. If a set A has n elements, then the number of proper subsets of A is $2^n - 1$.

Universal Set

A set which consists of all the sets under consideration or discussion is called the universal set. It is usually denoted by U or μ.

Example: Let $A = \{a, b, c\}$, $B = \{c, d, e\}$ and $C = \{a, e, f, g, h\}$. Then, the set $\{a, b, c, d, e, f, g, h\}$ can be taken as the universal set here.

$\therefore \mu = \{a, b, c, d, e, f, g, h\}$.

Complement of a Set

Let μ be the universal set and $A \subseteq \mu$. Then, the set of all those elements of μ which are not in set A is called the complement of the set A. It is denoted by A' or A^c. $A' = \{x/x \in \mu \text{ and } x \notin A\}$.

Examples:

1. Let $\mu = \{3, 6, 9, 12, 15, 18, 21, 24\}$ and $A = \{6, 12, 18, 24\}$. Then, $A' = \{3, 9, 15, 21\}$.
2. Let $\mu = \{x/x$ is a student and $x \in$ Class $X\}$

   and $B = \{x/x$ is a boy and $x \in$ Class $X\}$.

   Then, $B' = \{x/x$ is a girl and $x \in$ Class $X\}$.

Notes

1. A and A' are disjoint sets.
2. $\mu' = \phi$ and $\phi' = \mu$.

EXAMPLE 6.1

If $X = \left\{ p : \text{where } p = \dfrac{(n+2)(2n^5 + 3n^4 + 4n^3 + 5n^2 + 6)}{n^2 + 2n} \text{ and } n, p \in Z^+ \right\}$, then find the number of elements in the set X.

(a) 2 **(b)** 3 **(c)** 4 **(d)** 6

HINT

Divide each term by 'n' and find the positive factors of 6.

Operations on Sets

Union of Sets

Let A and B be two sets. Then, the union of A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A and B. That is, $A \cup B = \{x/x \in A \text{ or } x \in B\}$.

Examples:

1. Let $A = \{-1, -3, -5, 0\}$ and $B = \{-1, 0, 3, 5\}$. Then, $A \cup B = \{-5, -3, -1, 0, 3, 5\}$.
2. Let $A = \{x/5 \leq 5x < 25 \text{ and } x \in N\}$ and $B = \{x/5 \leq (10x) \leq 20 \text{ and } x \in N\}$, Then, $A \cup B = \{x/5 \leq 5x \leq 20 \text{ and } x \in N\}$.

Notes

1. If $A \subseteq B$, then $A \cup B = B$.
2. $A \cup \mu = \mu$ and $A \cup \phi = A$.
3. $A \cup A' = \mu$.

Intersection of Sets

Let A and B be two sets. Then the intersection of A and B, denoted by $A \cap B$, is the set of all those elements which are common to both A and B. That is, $A \cap B = \{x/x \in A \text{ and } x \in B\}$.

Examples:

1. Let $A = \{1, 2, 3, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 7\}$. Then, $A \cap B = \{1, 3, 5, 7\}$.
2. Let A be the set of all English alphabet and B be the set of all consonants, then, $A \cap B$ is the set of all consonants in the English alphabet.
3. Let E be the set of all even natural numbers and O be the set of all odd natural numbers. Then $E \cap O = \{ \}$ or ϕ.

Notes

1. If A and B are disjoint sets, then $A \cap B = \phi$.
2. If $A \subseteq B$, then $A \cap B = A$.
3. $A \cap \mu = A$ and $A \cap \phi = \phi$.
4. $A \cap A' = \phi$.

Difference of Sets

Let A and B be two sets. Then the difference $A - B$ is the set of all those elements which are in A but not in B.

That is, $A - B = \{x/x \in A \text{ and } x \notin B\}$.

Example: Let $A = \{3, 6, 9, 12, 15, 18\}$ and $B = \{2, 6, 8, 10, 14, 18\}$. $A - B = \{3, 9, 12, 15\}$ and $B - A = \{2, 8, 10, 14\}$.

Notes

1. $A - B \neq B - A$ unless $A = B$.
2. For any set A, $A' = \mu - A$.

Symmetric Difference of Sets

Let A and B be two sets. Then the symmetric difference of A and B, denoted by $A \Delta B$, is the set of all those elements which are either in A or in B but not in both. That is, $A \Delta B = \{x/x \in A \text{ and } x \notin B \text{ or } x \in B \text{ and } x \notin A\}$.

Note $A \Delta B = (A - B) \cup (B - A)$ (or) $A \Delta B = (A \cup B) - (A \cap B)$.

Example: Let $A = \{1, 2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 12\}$. Then,

$$\begin{aligned} A \Delta B &= (A - B) \cup (B - A) \\ &= \{1, 2, 4, 8, 10\} \cup \{3\} \\ &= \{1, 2, 3, 4, 8, 10\}. \end{aligned}$$

Some Results For any three sets A, B and C, we have the following results.

1. **Commutative Law:**
   (i) $A \cup B = B \cup A$
   (ii) $A \cap B = B \cap A$
   (iii) $A \Delta B = B \Delta A$
2. **Associative Law:**
   (i) $(A \cup B) \cup C = A \cup (B \cup C)$
   (ii) $(A \cap B) \cap C = A \cap (B \cap C)$
   (iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
3. **Distributive Law:**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **De Morgan's Law:**
   (i) $(A \cup B)' = A' \cap B'$
   (ii) $(A \cap B)' = A' \cup B'$
   (iii) $A - (B \cup C) = (A - B) \cap (A - C)$
   (iv) $A - (B \cap C) = (A - B) \cup (A - C)$
5. **Identity Law:**
   (i) $A \cup \phi = \phi \cup A = A$
   (ii) $A \cap \mu = \mu \cap A = A$

6. **Idempotent Law:**
   (i) $A \cup A = A$
   (ii) $A \cap A = A$
7. **Complement Law:**
   (i) $(A')' = A$
   (ii) $A \cup A' = \mu$
   (iii) $A \cap A' = \phi$

Dual of an Identity

An identity obtained by interchanging $\cup$ and $\cap$, and ϕ and μ in the given identity is called the dual of the identity.

Examples:

1. Consider the identity, $A \cup B = B \cup A$.
   Dual of the identity is, $A \cap B = B \cap A$.
2. Consider the identity, $A \cup \mu = \mu$.
   Dual of the identity is, $A \cap \phi = \phi$.

Venn Diagrams

We also represent sets pictorially by means of diagrams called Venn diagrams. In Venn diagrams, the universal set is usually represented by a rectangular region and its subsets by closed regions inside the rectangular region. The elements of the set are written in the closed regions and the elements which belong to the universal set are written in the rectangular region.

Example:
Let $\mu = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 7, 8\}$ and $B = \{2, 3, 4, 5, 9\}$.

We represent these sets in the form of Venn diagram as follows (see Fig. 6.1):

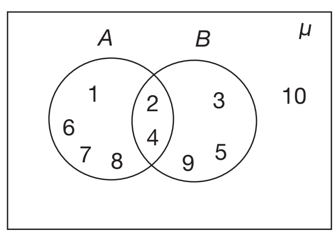


Figure 6.1

We can also represent the sets in Venn diagrams by shaded regions:

Examples:

1. Venn diagram of $A \cup B$, where A and B are two overlapping sets, is as shown in the Fig. 6.2.

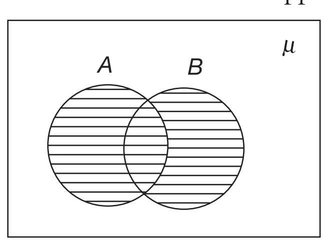


Figure 6.2

2. Let A and B be two overlapping sets. Then, the Venn diagram of $A \cap B$ is as shown in the Fig. 6.3.

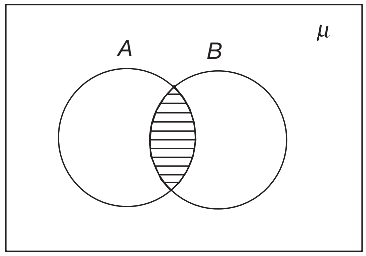


Figure 6.3

3. For a non-empty set A, Venn diagram of A' is as shown in the Fig. 6.4.

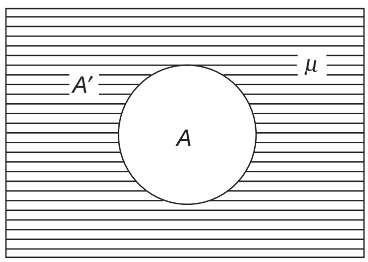


Figure 6.4

4. Let A and B be two overlapping sets. Then, the Venn diagram of $A - B$ is as shown in the Fig. 6.5.

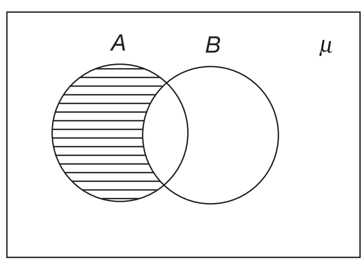


Figure 6.5

5. Let A and B be two sets such $A \subseteq B$. We can represent this relation using Venn diagram as follows (see Fig. 6.6).

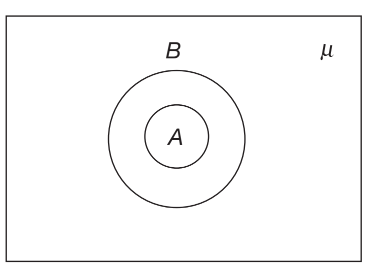


Figure 6.6

Some Formulae on the Cardinality of Sets

Let $A = \{1, 2, 3, 5, 6, 7\}$ and $B = \{3, 4, 5, 8, 10, 11\}$.

Then, $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$ and $A \cap B = \{3, 5\}$.

In terms of the cardinal numbers, $n(A) = 6$, $n(B) = 6$, $n(A \cap B) = 2$ and $n(A \cup B) = 10$.

So, $n(A) + n(B) - n(A \cap B) = 6 + 6 - 2 = 10 = n(A \cup B)$.

We have the following formulae:

For any three sets A, B and C

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
2. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

EXAMPLE 6.2

If $n(A) = 7$, $n(B) = 5$ and $n(A \cup B) =10$, then find $n(A \cap B)$.

SOLUTION

Given, $n(A) = 7$, $n(B) = 5$ and $n(A \cup B) = 10$.

We know that,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

So,

$$10 = 7 + 5 - n(A \cap B)$$

$$\Rightarrow n(A \cap B) = 2.$$

EXAMPLE 6.3

If $n(A) = 8$ and $n(B) = 6$ and the sets A and B are disjoint, then find $n(A \cup B)$.

SOLUTION

Given, $n(A) = 8$, $n(B) = 6$.

A and B are disjoint

$$\Rightarrow A \cap B = \phi \Rightarrow n(A \cap B) = 0$$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 8 + 6 - 0 = 14.$$

Note If A and B are two disjoint sets then, $n(A \cup B) = n(A) + n(B)$. The Venn diagram and the summary of it when three overlapping sets are given, are as follows (see Fig. 6.7):

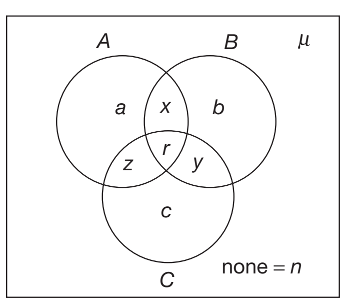


Figure 6.7

$A = a + x + r + z$
Only A (or) exactly $A = a$.
$B = b + x + r + y$
Only B (or) exactly $B = b$.
$C = c + y + r + z$
Only C (or) exactly $C = c$.
$A \cap B = x + r$
$A \cap B \cap C'$ (or) only $A \cap B = x$
$B \cap C = y + r$
$B \cap C \cap A'$ (or) only $B \cap C = y$; $C \cap A = z + r$; $C \cap A \cap B'$ (or) only $C \cap A = z$.
Only two sets (or) exactly two sets $= x + y + z$
$A \cap B \cap C$ (or) all the three $= r$

Atleast one set = Exactly one + Exactly two + Exactly three
$= a + b + c + x + y + z + r$
= Total – None

Atleast two sets = Exactly two + Exactly three
$= x + y + z + r$

Atleast three sets = Exactly three $= r$

Atmost two sets = Exactly two + Exactly one + Exactly zero
$= x + y + z + a + b + c + n$
$= n(\mu) - r$

Atmost three sets = Exactly three + Exactly two + Exactly one + Exactly zero
$= r + x + y + z + a + b + c + n$
$= n(\mu)$.

EXAMPLE 6.4

There are 40 students in a class. Each student speaks at least one of the languages Tamil, English and Hindi. Ten students speak exactly one language. Twenty five students speak atmost two languages. How many students speak atleast two languages?

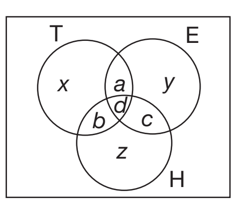
(a) 15 **(b)** 25 **(c)** 30 **(d)** 5

SOLUTION

Number of students who speak at least one of the languages Tamil, English and Hindi = 40
$= (x + y + z + a + b + c + d)$

Number of students who speak only one language = 10
$= (x + y + z)$

$\therefore$ No of students who speak at least two languages $= (a + b + c + d)$
$= 40 - 10 = 30$.



ORDERED PAIR

Let A be a non-empty set and $a, b \in A$. The elements a and b written in the form (a, b) is called an ordered pair. In the ordered pair (a, b), a is called the first coordinate and b is called the second coordinate.

Note Two ordered pairs are said to be equal only when their first as well as the second coordinates are equal, i.e., $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$.

So, $(1, 2) \neq (2, 1)$ and if $(a, 5) = (3, b) \Rightarrow a = 3$ and $b = 5$.

Cartesian Product of Sets

Let A and B be two non-empty sets. The Cartesian product of A and B, denoted by $A \times B$ is the set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$. That is, $A \times B = \{(a, b)/a \in A, b \in B\}$.

Notes

1. $A \times B \neq B \times A$, unless $A = B$.
2. For any two sets A and B, $n(A \times B) = n(B \times A)$.
3. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$.

$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$ and

$B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$.

We observe that, $A \times B \neq B \times A$ and $n(A \times B) = 6 = n(B \times A)$.

Some Results on Cartesian Product

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (or) $(A \cup B) \times (A \cup C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (or) $(A \cap B) \times (A \cap C)$
3. If $A \times B = \phi$, then either,
   (i) $A = \phi$ or
   (ii) $B = \phi$ or
   (iii) both $A = \phi$ and $B = \phi$.

Cartesian product of sets can be represented in following ways:

1. Arrow diagram
2. Tree diagram
3. Graphical representation

Representation of $A \times B$ using Arrow Diagram

EXAMPLE 6.5

If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then find $A \times B$.

SOLUTION

In order to find $A \times B$, represent the elements of A and B as shown in the Fig. 6.8.

Now draw an arrow from each element of A to each element of B, as shown in the figure.

Now, represent all the elements related by arrows in ordered pairs in a set, which is the required $A \times B$.

That is, $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$.

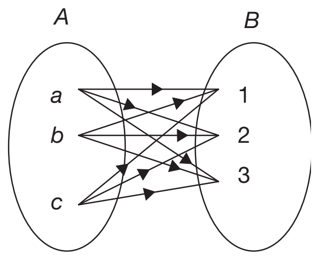


Figure 6.8

Representation of $A \times B$ Using a Tree Diagram

EXAMPLE 6.6

If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then find $A \times B$.

SOLUTION

To represent $A \times B$ using tree diagram, write all the elements of A vertically and then for each element of A, write all the elements of B and draw arrows as shown in the Fig. 6.9.

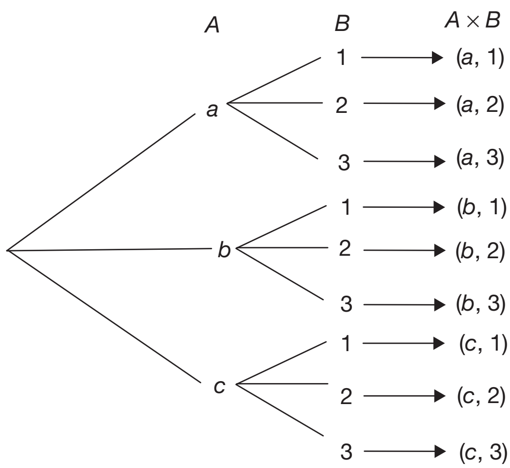


Figure 6.9

$\therefore A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c\ 1), (c, 2), (c, 3)\}$.

Graphical Representation of $A \times B$

EXAMPLE 6.7

If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then find $A \times B$.

SOLUTION

Consider the elements of A on the X-axis and the elements of B on the Y-axis and mark the points (see Fig. 6.10).

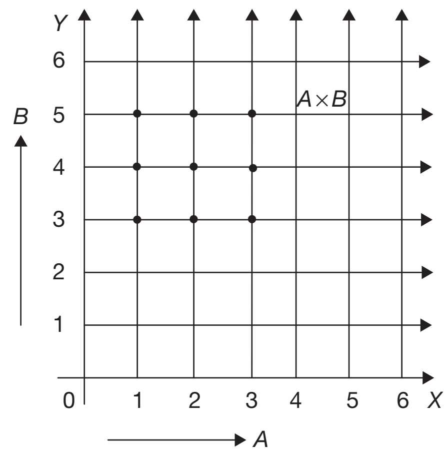


Figure 6.10

$\therefore A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$.

RELATION

We come across certain relations in real life and also in basic geometry, like *is father of*, *is a student of*, *is parallel to*, *is similar to*, etc. We now define the mathematical form of the sum.

Definition

Let A and B be two non-empty sets and $R \subseteq A \times B$. R is called a relation from the set A to B. (Any subset of $A \times B$ is called a relation from A to B).

$\therefore$ A relation contains ordered pairs as elements. Hence, 'A relation is a set of ordered pairs'.

Examples:

1. Let $A = \{1, 2, 4\}$ and $B = \{2, 3\}$

   Then, $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3)\}$

   Let $R_1 = \{(1, 2), (1, 3), (2, 3)\}$

   Clearly, $R_1 \subseteq A \times B$ and we also notice that, for every ordered pair $(a, b) \in R_1$, $a < b$.

   So, R_1 is the relation *is less than* from A to B.

2. In the previous example, let $R_2 = \{(4, 2), (4, 3)\}$

   Clearly, $R_2 \subseteq A \times B$ and we also notice that, for every ordered pair $(a, b) \in R_2$, $a > b$.

   So, R_2 is the relation *is greater than* from A to B.

Notes

1. If $n(A) = p$ and $n(B) = q$, then the number of relations possible from A to B is 2^{pq}.
2. If $(x, y) \in R$, then we write $x\ R\ y$ and read as x is related to y.

Domain and Range of a Relation

Let A and B be two non-empty sets and R be a relation from A to B, we note that:

1. The set of first coordinates of all ordered pairs in R is called the domain of R.
2. The set of second coordinates of all ordered pairs in R is called the range of R.

Example:
Let $A = \{1, 2, 4\}$, $B = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (4, 3)\}$ be a relation from A to B.

Then, domain of $R = \{1, 2, 4\}$ and range of $R = \{1, 2, 3\}$.

Representation of Relations

We represent the relations by the following methods:

Roster Method (or) List Method

In this method we list all the ordered pairs that satisfy the rule or property given in the relation.

Example: Let $A = \{1, 2, 3\}$. If R is a relation on the set A having the property *is less than*, then the roster form of R is, $R = \{(1, 2), (1, 3), (2, 3)\}$.

Set-builder Method

In this method, a relation is described by using a representative and stating the property or properties, which the first and second coordinates of every ordered pair of the relation satisfy, through the representative.

Example: Let $A = \{1, 2, 3\}$. If R is a relation on the set A having the property *is greater than or equal to*, then the set-builder form of R is,

$R = \{(x, y)/x, y \in A \text{ and } x \geq y\}$.

Arrow Diagram

In this method, a relation is described by drawing arrows between the elements which satisfy the property or properties given in the relation.

Example: Let $A = \{1, 2, 4\}$ and $B = \{2, 3\}$. Let R be a relation from A to B with the property *is less than*.
Then, the arrow diagram of R is as shown in the Fig. 6.11.

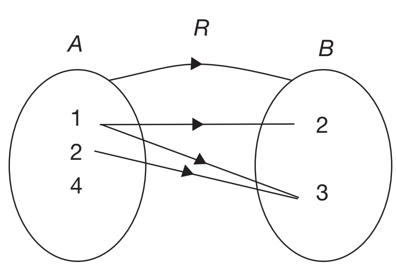


Figure 6.11

Inverse of a Relation

Let R be a relation from A to B. The inverse relation of R, denoted by R^{-1}, is defined as, $R^{-1} = \{(y, x)/(x, y) \in R\}$.

Example: Let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (4, 3)\}$ be a relation from A to B, where $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$. Then, $R^{-1} = \{(1, 1), (1, 2), (2, 1), (3, 2), (3, 4)\}$.

Notes

1. Domain of R^{-1} = Range of R.
2. Range of R^{-1} = Domain of R.
3. If R is a relation from A to B, then R^{-1} is a relation from B to A.
4. If $R \subseteq A \times A$, then R is called a binary relation or simply a relation on the set A.
5. For any relation R, $(R^{-1})^{-1} = R$.

Types of Relations

1. **One-one relation:** A relation $R: A \to B$ is said to be one-one relation if different elements of A are paired with different elements of B. That is, $x \neq y$ in $A \Rightarrow f(x) \neq f(y)$ in B.

   Example: From the given Fig. 6.12, the relation $R = \{(1, 3), (2, 4), (3, 5)\}$.

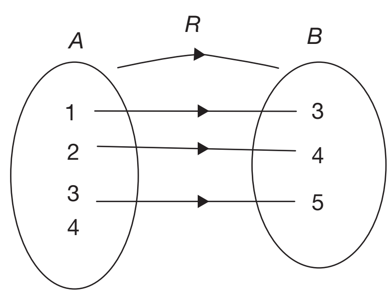


Figure 6.12

2. **One-many relation:** A relation $R: A \to B$ is said to be one-many relation if at least one element of A is paired with two or more elements of B.

   Example: From the given Fig. 6.13, the relation $R = \{(1, 3), (1, 4), (3, 5)\}$.

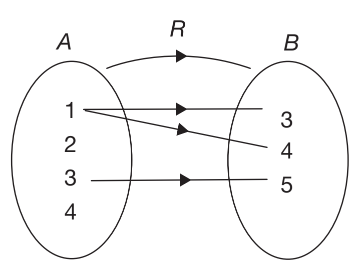


Figure 6.13

3. **Many-one relation:** A relation $R: A \to B$ is said to be many-one relation if two or more elements of A are paired with an element of B.

   Example: From the given Fig. 6.14, the relation $R = \{(1, 2), (2, 4), (3, 2), (5, 3)\}$.

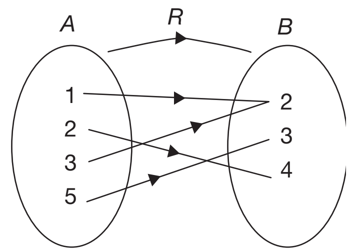


Figure 6.14

4. **Many-many relation:** A relation R: $A \to B$ is said to be many-many relation if two or more elements of A are paired with two or more elements of B.

   Example: From the given Fig. 6.15, the relation $R = \{(1, 3), (1, 4), (2, 3), (2, 5), (3, 4), (3, 5)\}$.

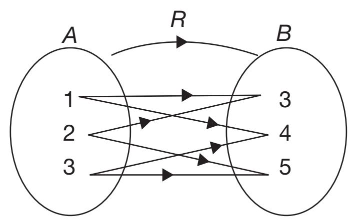


Figure 6.15

Properties of Relations

1. **Reflexive relation:** A relation R on a set A is said to be reflexive if for every $x \in A$, $(x, x) \in R$.

   Examples:

   1. Let $A = \{1, 2, 3\}$ then, $R = \{(1, 1), (1, 2), (2, 2), (3, 3), (2, 3)\}$ is a reflexive on A.
   2. Let $A = \{1, 2, 3\}$ then, $R = \{(1, 1), (2, 3), (1, 2), (1, 3), (2, 2)\}$ is not a reflexive relation as $(3, 3) \notin R$.

   Note Number of reflexive relations defined on set having n elements is 2^{n^2-n}.

2. **Symmetric relation:** A relation R on a set A is said to be symmetric, if for every $(x, y) \in R$, $(y, x) \in R$.

   Examples:

   1. Let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$ is a symmetric relation on A.
   2. Let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (1, 2), (3, 1), (2, 2), (3, 3)\}$ is not a symmetric relation as $(1, 2) \in R$ but $(2, 1) \notin R$.

   Note A relation R on a set A is symmetric, iff $R = R^{-1}$, i.e., R is symmetric, iff $R = R^{-1}$.

3. **Transitive relation:** A relation R on a set A is said to be transitive, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. That is, R is said to be transitive, whenever $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

   Examples:

   1. If $A = \{1, 2, 3\}$, then the relation on set A, $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ is a transitive relation.

2. Let $A = \{1, 2, 3\}$, then the relation on set A, $R = \{(1, 2), (2, 2), (2, 1), (3, 3), (1, 3)\}$ is not a transitive relation as $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.

4. **Anti-symmetric relation:** A relation R on a set A is said to be anti-symmetric, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. That is, R is said to be anti-symmetric if for $x \neq y$, $(x, y) \in R \Rightarrow (y, x) \notin R$.

   Examples:
   1. Let $A = \{1, 2, 3\}$, then $R = \{(1, 1), (1, 2), (3, 3)\}$ is an anti-symmetric relation.
   2. Let $A = \{1, 2, 3\}$, then $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 3)\}$ is not an anti-symmetric relation as $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

5. **Equivalence relation:** A relation R on a set A is said to be an equivalence relation if it is,
   (i) reflexive
   (ii) symmetric and
   (iii) transitive.

   Note For any set A, $A \times A$ is an equivalence relation. In fact it is the largest equivalence relation.

Identity Relation

A relation R on a set A defined as, $R = \{(x, x)/x \in A\}$ is called an identity relation on A. It is denoted by I_A.

Example: Let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

Note Identity relation is the smallest equivalence relation on a set A.

EXAMPLE 6.8

If $A = \{a, b, c, x, y, z\}$, then the maximum number of elements in any relation on A is:

(a) 12 (b) 16 (c) 32 (d) 36

HINT

$A \times A$ has maximum number of elements.

FUNCTION

Let A and B be two non-empty sets. f is a relation from A to B. If f is such that

1. for every $a \in A$, there is $b \in B$ such that $(a, b) \in f$ and
2. no two ordered pairs in f have the same first element, then f is called a function from set A to set B and is denoted as $f: A \rightarrow B$.

Notes

1. If $(a, b) \in f$, then $f(a) = b$ and b is called the f image of a. a is called the pre-image of b.
2. If $f: A \rightarrow B$ is a function, then A is called the domain of f and B is called the co-domain of f.
3. The set $f(A)$ which is all the images of elements of A under the mapping f is called the range of f.

Examples:

1.

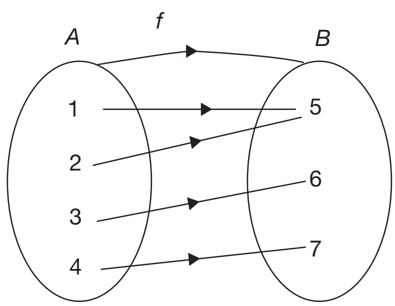


Figure 6.16

2.

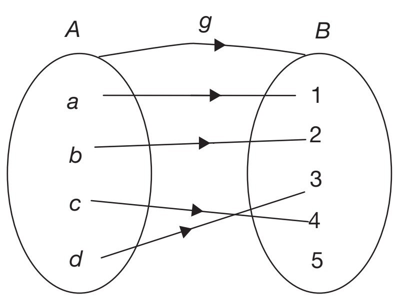


Figure 6.17

Let A and B be two non-empty sets. A relation f from A to B is said to be a function, if every element in A is associated with exactly one element in B. It is denoted by f: $A \to B$ (read as f is mapping from A to B). If $(a, b) \in f$, then b is called the f image of a and is written as $b = f(a)$. a is called the pre-image of b. Also in $f(a) = b$, a is called the independent variable and b is called the dependent variable.

Domain and Co-domain

If f: $A \to B$ is a function, then A is called domain and B is the co-domain of the function.

Range

If f: $A \to B$ is a function, then the set of all images of elements in its domain is called the range of f and is denoted by $f(A)$.

That is, $f(A) = \{f(a)/a \in A\}$.

Note Range of a function is always subset of its co-domain, i.e., $f(A) \subseteq B$.

If f: $A \to B$ is a function, and $n(A) = m$, $n(B) = p$, then the number of functions that can be defined from A to B is p^m.

Examples:

1. $A = \{1, 2, 3, 4\}$; $B = \{2, 3, 4, 5, 6\}$ are two sets. A relation f is defined as $f(x) = x + 2$. The relation f: $A \to B$ is a function and $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$.
2. $A = \{-2, 2, 3, 4\}$, $B = \{4, 9, 16\}$ are two sets. The relation f, defined as $f(x) = x^2$, is a function from A to B, since every element in A is associated with exactly one element in B. The function $f = \{(-2, 4), (2, 4), (3, 9), (4, 16)\}$.

3. $A = \{-1, 1, 2, 5\}$, $B = \{1, 8, 125\}$ are two sets. The relation f defined as $f(x) = x^3$ is not a function from A to B. The relation $f = \{(1, 1), (2, 8), (5, 125)\}$. The number -1 is an element in A but it has no image in B.

4. $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, t, u, v\}$ are two sets. A relation f is defined as follows:

   $f = \{(1, x), (2, y) (3, z), (4, t) (1, u)\}$. Here f is not a function, because the element 1 in A is associated with two elements x, u in B. Therefore, f is not a function.

Arrow Diagram

Functions can be represented by arrow diagrams.

Example: $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 4, 9, 16, 25, 36\}$ are two sets. A relation f is defined as $f(n) = n^2$. The arrow diagram of this function is shown in the Fig. 6.18.

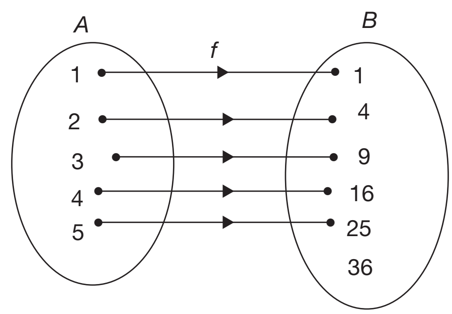


Figure 6.18

Every element in A is associated with exactly one element in B. So, f: $A \to B$ is a function.

Example:

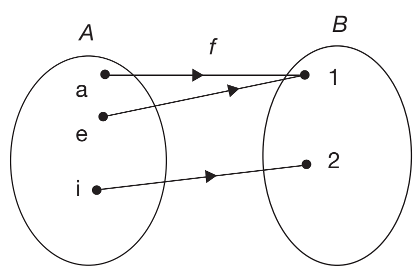


Figure 6.19

Every element in A is associated with exactly one element in B (see Fig. 6.19). So, $f : A \to B$ is a function.

Example:

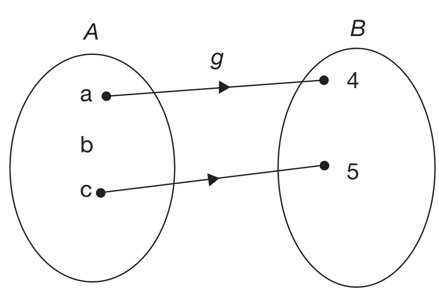


Figure 6.20

b is an element in A and it is not associated with any element in B (see Fig. 6.20). So, g: $A \to B$ is not a function.

Example:

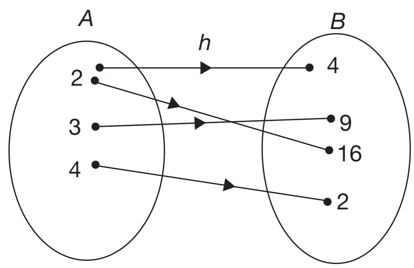


Figure 6.21

2 is an element in A. It is associated with two different elements in B (see Fig. 6.21). That is, 2 has two different images. So, h: $A \to B$ is not a function.

Difference Between Relations and Functions

Every function is a relation but every relation need not be a function. A relation f from A to B is called a function, if

1. Domain $(f) = A$,
2. no two different ordered pairs in f have the same first coordinate.

Example:
Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e\}$.

Some relations f, g, h are defined as follows:

$f = \{(1, a), (2, b), (3, c), (4, d)\}$

$g = \{(1, a), (2, b), (3, c)\}$

$h = \{(1, a), (1, b), (2, c), (3, d), (4, e)\}$.

In the relation f, the domain of f is A and all first coordinates are different. So, f is a function. In the relation g, the domain of g is not A. So g is not a function. In the relation h, the domain of h is A, but two of the first coordinates are equal, i.e., 1 has two different images. So, h is not a function.

EXAMPLE 6.9

Find the domain of the function $f(x) = \dfrac{1}{x} + \dfrac{1}{\log(2-x)}$.

(a) $x > 2$ **(b)** $x \in R - \{2\}$ **(c)** $x < 2, x \neq 0, x \neq 1$ **(d)** $x < 2, x \neq 0$.

HINTS

$\dfrac{1}{x}$ is not defined for $x = 0$ and logarithmic function takes only positive values.

Types of Functions

One-one Function or Injection

Let f: $A \to B$ be a function. If different elements in A are assigned to different elements in B, then the function f: $A \to B$ is called a one-one function or an injection.

That is, a_1, $a_2 \in A$ and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ then f: $A \to B$ is a one-one function (or) if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, then f: $A \to B$ is a one-one function.

Example:

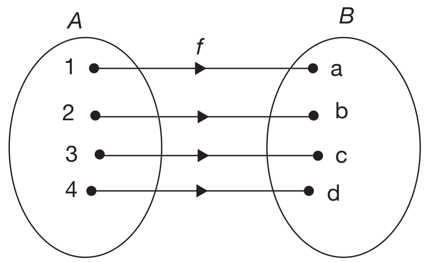


Figure 6.22

Different elements in A are assigned to different elements in B. f: $A \to B$ is a one-one function (see Fig. 6.22).

Example:

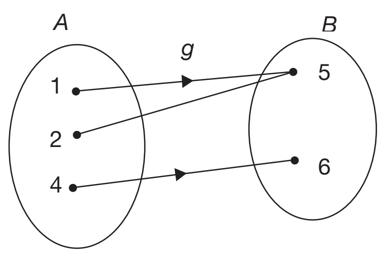


Figure 6.23

1 and 2 are two different elements in A, but they are assigned to the same element in B. So g: $A \to B$ is not a one-one function (see Fig. 6.23).

Note A and B are finite sets and f: $A \to B$ is one-one. Then $n(A) \leq n(B)$.

Many to One Function

If the function f: $A \to B$ is not one-one, then it is called a many to one function; i.e., two or more elements in A are assigned to the same element in B.

Example:

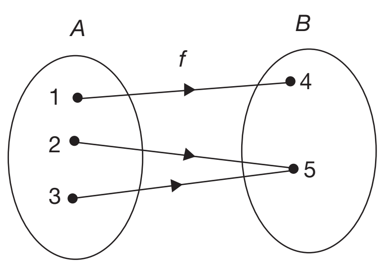


Figure 6.24

2, 3 are different elements in A, and they are assigned to the same element, i.e., 5 in B. So, f: $A \to B$ is a many to one function (see Fig. 6.24).

Onto Function or Surjection

f: $A \to B$ is said to be an onto function, if every element in B is the image of at least one element in A. That is, for every $b \in B$, there exists at least one element $a \in A$, such that $f(a) = b$.

Note If f: $A \to B$ is an onto function, then the co-domain of f must be equal to the range of f, i.e., $f(A) = B$.

Example:

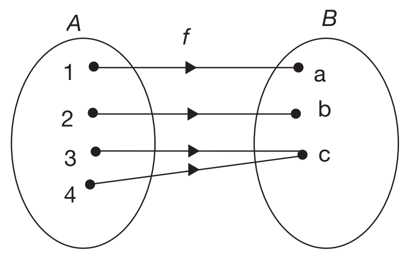


Figure 6.25

Range = {a, b, c}

Co-domain = {a, b, c}

Range = Co-domain

In the Fig. 6.25, every element in B is the image of at least one element in A. Therefore, it is an onto function.

Example: In the Fig. 6.26, d is an element in B, but it is not the image of any element in A. Therefore, it is not an onto function.

Note A and B are finite sets and f: $A \to B$ is onto. Then, $n(B) \leq n(A)$.

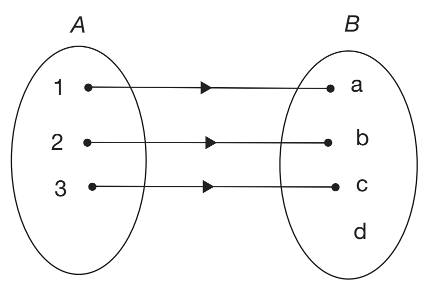


Figure 6.26

Into Function

If a function is not onto, then it is an into function, i.e., at least one element in B is not the image of any element in A, or the range is a subset of the co-domain.

Bijective Function

If the function f: $A \to B$ is both one-one and onto then it is called a bijective function.

Example:

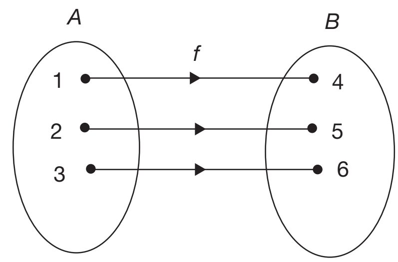


Figure 6.27

In the Fig. 6.27, f: $A \to B$ is both one-one and onto. So f is a bijective function.

Example:

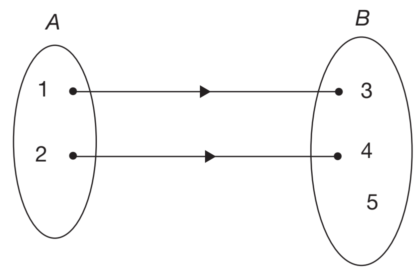


Figure 6.28

In the Fig. 6.28, it is only one-one but not onto, so it is not bijective.

Note A and B are finite sets and f: $A \to B$ is one-one and onto then $n(A) = n(B)$.

Inverse of a Function

If f: $A \to B$ is a function, then the set of ordered pairs obtained by interchanging the first and second coordinates of each ordered pair in f is called the inverse of f and is denoted by f^{-1}. That is, f: $A \to B$ is a function then its inverse is f^{-1}: $B \to A$.

Examples:

1. $f = \{(1, 2), (2, 3), (3, 4)\}$
   $f^{-1} = \{(2, 1), (3, 2), (4, 3)\}$.
2. $g = \{(1, 4), (2, 4), (3, 5), (4, 6)\}$
   $g^{-1} = \{(4, 1), (4, 2), (5, 3), (6, 4)\}$.
3. $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, a, b\}$ are two sets, the function h: $A \to B$ is defined as follows:
   $h = \{(1, a), (2, b), (3, x), (4, z)\}$
   $h^{-1} = \{(a, 1), (b, 2), (x, 3), (z, 4)\}$.

In the above examples only f^{-1} is a function, but g^{-1}, h^{-1} are not functions.

$\therefore$ f: $A \to B$ a function, then f^{-1}: $B \to A$ need not be a function.

Inverse Function If f : $A \to B$ is a bijective function, then f^{-1}: $B \to A$ is also a function. That is, the inverse of a function is also a function, only when the given function is bijective.

Example:
$A = \{1, 2, 3, 4, 5\}$; $B = \{a, e, i, p, u\}$. A function f is defined as follows:
$f = \{(1, a), (2, e), (3, i), (4, u), (5, p)\}$. Clearly, f is a bijective function.
Now $f^{-1} = \{(a, 1), (e, 2), (i, 3), (u, 4), (p, 5)\}$. Clearly f^{-1} is also a function and it is also bijective.

EXAMPLE 6.10

If $f(x) = 2x + 3$ and $g(x) = 3x - 1$, then find $f^{-1}\, o\, g^{-1}$.

(a) $\frac{x+8}{6}$ **(b)** $\frac{x-8}{6}$ **(c)** $\frac{8-x}{6}$ **(d)** $\frac{x-8}{2}$

HINTS

(i) Find $g\, o\, f(x)$.

(ii) We know that, $(g\, o\, f)^{-1} = f^{-1}\, o\, g^{-1}$.

Identity Function

f: $A \to A$ is said to be an identity function on A, if $f(a) = a$ for every $a \in A$, it is denoted by IA.

Example: $A = \{1, 2, 3, 4\}$. The identity function on A is $IA = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

Notes

1. Identity function is always bijective function.
2. The inverse of the identity function is the identify function itself.

Constant Function

A function f: $A \to B$ is a constant function if there is an element $b \in B$, such that $f(a) = b$, for all $a \in A$.

That is, in a constant function the range has only one element.

Example: $A = \{1, 2, 3, 4\}$; $B = \{$a, e, i, u$\}$ are two sets and a function from A to B is defined as follows:

$f = \{(1, \text{a}), (2, \text{a}), (3, \text{a}), (4, \text{a})\}$, therefore f is a constant function.

Note The range of a constant function is a singleton.

Equal Functions

Two functions f and g, defined on the same domain D are said to be equal, if $f(x) = g(x)$ for all $x \in D$.

Example:

Let f: $R - \{2\} \to R$ be defined by $f(x) = x + 2$; and g: $R - \{2\} \to R$ be defined by $g(x) = \frac{x^2 - 4}{x - 2}$.

$\therefore$ f: and g have the same domain $R - \{2\}$.

Given $f(x) = x + 2$;

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

$\therefore$ $f(x) = g(x)$ for all $x \in R - \{2\}$.

$\Rightarrow$ f and g are equal functions.

Composite Function (or) Product Function

Let f and g be two functions such that f: $A \rightarrow B$ and g: $B \rightarrow C$. Let a be an orbitary element in A.

Since f is a function from A to B, there exists an element $b \in B$, such that $f(a) = b$.

Since g is a function from B to C, there exists an element $c \in C$, such that $g(b) = c$.

$\therefore g(f(a)) = c \Rightarrow g \, o \, f(a) = c$.

$\therefore g \, o \, f$ is a function from A to C.

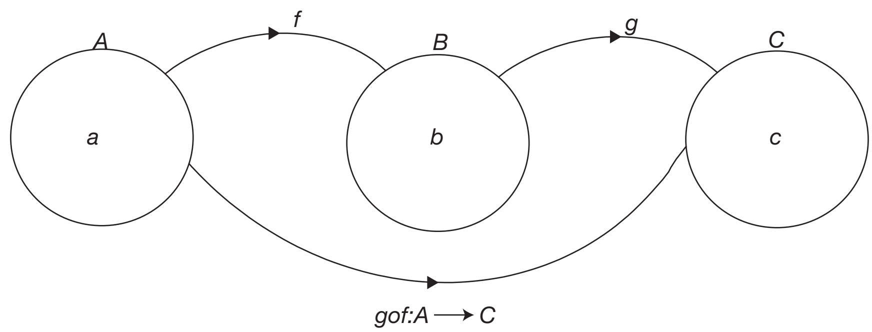


Figure 6.29

If f: $A \rightarrow B$ and $g : B \rightarrow C$ are two functions, then the function $g[f(x)]= g \, o \, f$ from A to C, denoted by $g \, o \, f$ is called the composite function of f and g.

In the composite function $g \, o \, f$,

1. the co-domain of f is the domain of g.
2. the domain of $g \, o \, f$ is the domain of f, the co-domain of $g \, o \, f$ is the co-domain of g.
3. composite function does not satisfy commutative property, i.e., $g \, o \, f \neq f \, o \, g$.
4. if f: $A \rightarrow B$; g: $B \rightarrow C$; h: $C \rightarrow D$ are three functions, $h \, o \, (g \, o \, f) = (h \, o \, g) \, o \, f$, i.e., the composite function satisfies associative property.

Real Function

If f: $A \rightarrow B$ such that $A \subseteq R$, then f is said to be a real variable function.

If f: $A \rightarrow B$ such that $B \subseteq R$, then f is said to be a real valued function.

If f: $A \rightarrow B$, and A and B are both subsets of the set of real numbers (R), then f is called a real function.

Even and Odd Functions

1. If $f(-x) = f(x)$, then the function $f(x)$ is called an even function.

   Example:
   $f(x) = x^2$

   $f(-x) = x^2$

   Here, $f(-x) = f(x)$

   $\therefore f(x) = x^2$ is an even function.

2. If $f(-x) = -f(x)$, then the function $f(x)$ is called an odd function.

Example:
$f(x) = x^3$
$f(-x) = -x^3$
Here, $f(x) = -f(x)$
$f(x) = x^3$ is an odd function.

Notes

1. There are functions which are neither even nor odd.

Example: $2x + 3$, a^x, etc.

2. If $f(x)$ is a real function, then $\frac{f(x)+f(-x)}{2}$ is always even and $\frac{f(x)-f(-x)}{2}$ is always odd.

3. Product of two even functions is even.

4. Product of two odd functions is even.

5. Product of an even function and an odd function is odd.

6. $f(x) = 0$ is both even and odd.

Domain and Range of Some Functions are Listed below

Function	Domain	Range
$\frac{1}{x}$	$R - \{0\}$	R
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\|x\|$	R	$[0, \infty)$
$\log x$	$(0, \infty)$	R
a^x $(a > 0)$	R	$(0, \infty)$

Graphs of Functions

A graph does not represent a function, if there exists a vertical line which meets the graph in two or more points, i.e., a vertical line meets the graph at only one point, then the graph represents a function.

Example:

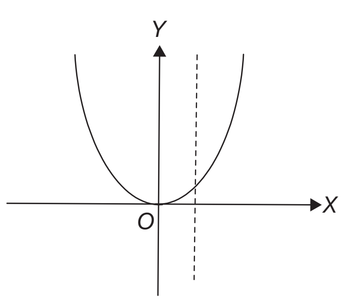


Figure 6.30

In the Fig. 6.30, the vertical dotted line meets the graph at only one point. So, the graph represents a function.

Example:

In the Fig. 6.31, the vertical dotted line meets the graph at two points. So, it is not a function.

Notes

1. The Y-axis does not represent a function.
2. The X-axis represents a many-one function.

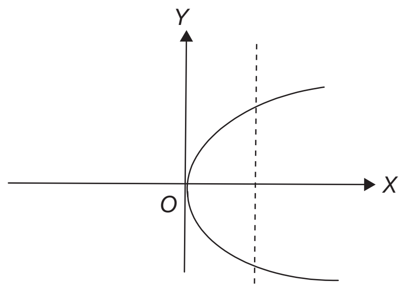


Figure 6.31

Zeroes of a Function

If $f: A \to R$ $(A \subset R)$, then the points $k \in A$, such that $f(k) = 0$ are called the zeroes of the function f.

If k is a zero of $f: A \to R$, then $(k, 0)$ is the corresponding point on the graph of f. k is called the x-intercept of the graph.

Example:

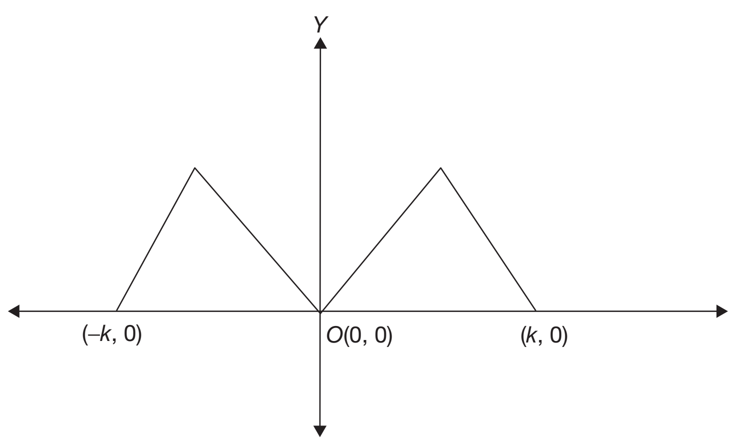


Figure 6.32

Graph of the Fig. 6.32, represents a function, zeroes of the graph are $-k$, 0 and k.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If A is a non-empty set, then $(((A')')')'$ is _______.
2. The number of non-empty proper subsets of a set A is 0, then $n(A)$ = _______.
3. The number of non-empty proper subsets of a set containing 7 elements, is _______.
4. If A and B are disjoint sets, then $A \Delta B$ = _______.
5. $n(A \cup B \cup C)$ = _______.
6. If A and B are disjoint, then $(A \cap B)'$ = _______.
7. In the given figure, if A and B are any two non-empty sets and μ is an universal set, then the shaded region represents ___________.

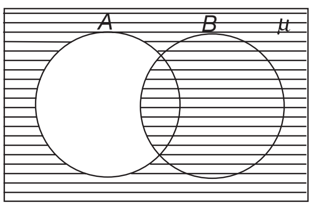


Figure 6.33

8. If any two of the sets $A_1, A_2, \ldots, A_n$ are disjoint, then $A_1 \cap A_2 \cap \ldots \cap A_n$ = _______.
9. If $n(A) = 25$, $n(B) = 10$ and also $B \subset A$, then $n(B - A)$ = _______.
10. If $n(A) = 15$, $n(B) = 13$ and $n(A \cap B) = 10$, then the symmetric difference of A and B is _______.
11. The ordered pair (x, y) is a subset of $\{x, y\}$. (True/False)
12. If $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, then $R = \{(a, 2), (b, 1), (d, 3), (2, c)\}$ is a relation from A to B. (True/False)
13. $A = \{a, b, c\}$ and R is an identity relation on set A. Then the ordered pairs of R are _______.
14. $n(P \times Q) = 200$ and $n(P) = 100$, then $n(Q)$ = _______.
15. Relation $R = \{(x, y) : x > y \text{ and } x + y = 8 \text{ where } x, y \in N\}$, then write R^{-1} in roster form.
16. If $R = R^{-1}$, then the relation R is _______.
17. If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7\}$, then $n(A \times B)$ = _______.
18. If A and B are two equivalent sets and $n(B) = 6$, then $n(A \times B)$ = _______.
19. The domain of the relation $R = \{(x, y): x, y \in N \text{ and } x + y \le 9\}$ is _______.
20. The range of the relation $R = \{(x, y): 3x + 2y = 15 \text{ and } x, y \in N\}$ is _______.
21. $A = \{x, y, z, p\}$; $B = \{7, 8, 9, 10\}$ and a rule f is given by $f(x) = 7$, $f(y) = 7$, $f(z) = 7$, $f(p) = 7$. The relation f is a function. [True/False]
22. Range of the function $|x - 5|$ is _______.
23. If $f(x) = 2x + \frac{3}{2}$ then find $f(3)$ and $f\left(\frac{3}{2}\right)$.
24. If the function $f : A \to \{a, b, c, d\}$ is an onto function, then the minimum number of elements in A must be equal to _______.
25. In a bijection, the number of elements of the domain is equal to the number of elements of the co-domain. [True/False]
26. Domain of the function $\frac{1}{\sqrt{x}}$ is _______.
27. Number of elements of an identity function defined on a set containing four elements is _______.
28. If f is a constant function and $f(100) = 100$, then $f(2007)$ = _______.
29. $f: R \to R$ be defined by $f(x) = 7x + 6$. What is f^{-1}?
30. If $f: A \to B$ and $g: B \to C$ are such that $g \circ f$ is onto then g is necessarily onto. [True/false]

Short Answer Type Questions

31. If $A = \{1, 2, 3, 6, 8, 9\}$, $B = \{3, 4, 5, 6\}$, and $\mu = \{1, 2, 3, \ldots, 10\}$, then find $(A \cup B)'$.
32. Write the following sets in the roster form.

    $X = \{a/30 \le a \le 40 \text{ and } a \text{ is a prime}\}$
33. If $n(P - Q) = x + 37$, $n(Q - P) = 30 + 3x$, $n(P \cup Q) = 120 + 2x$ and $n(P \cap Q) = 35$, then find x.
34. In a club of 70 members 30 play Tennis but not cricket and 55 play Tennis. How many members

PRACTICE QUESTIONS

play cricket but not Tennis? (Each member plays either Tennis or Cricket).

35. Find the value of $n(A \cap B \cap C)$, if $n(A) = 35$, $n(A \cap B \cap C') = 8$, $n(A \cap C \cap B') = 10$ and $n(A \cap B' \cap C') = 6$.

36. $R = \{(a, 2a - b)/a, b \in N$ and $0 < a, b < 3\}$, then find the domain and range of the relation R.

37. If $n(X \cap Y') = 9$, $n(Y \cap X') = 10$ and $n(X \cup Y) = 24$, then find $n(X \times Y)$.

38. In a gathering, two persons are related 'if they have the same bike', then find the properties that are satisfied by the relation.

39. If $A = \{3, 5, 6, 9\}$ and R is a relation in A defined as $R = \{(x, y) \in R$ and $x + y < 18\}$, then write R in roster form.

40. Given $R = \{(a, a), (a, b), (b, c), (a, c), (b, b), (b, a), (c, a), (c, b)\}$ on set $A = \{a, b, c\}$. What are the properties that R satisfies?

41. If $f(x) = \dfrac{2x+3}{4}$, then find $f^{-1}\left(\dfrac{3}{4}\right)$.

42. If $f(x)$ is a polynomial function of 4th degree, $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$ and $f(2) = 17$, then find $f(3)$.

43. If $f(x) = (1 - x^3)^{1/3}$, then find $f\,o\,f(x)$.

44. If $f(x) = \dfrac{x^2 - 1}{3}$ for $x \in \{-2, -1, 0, 1, 2\}$, then find $f^{-1}(x)$.

45. Find the range of the function $f(x) = \dfrac{1}{2x^2 + 1}$.

Essay Type Questions

46. In a colony of 125 members, 70 members watch Telugu channel, 80 members watch Hindi channel and 95 watch English channel, 20 watch only Telugu and Hindi, 35 watch only English and Hindi and 15 watch only Telugu and English. How many members watch all the three channels, if each watches either of the channels?

47. If $f(a) = \log\left(\dfrac{1+a}{1-a}\right)$, then find $f\left(\dfrac{a_1 + a_2}{1 + a_1 a_2}\right)$ in terms of $f(a_1)$, $f(a_2)$.

48. If $f(x) = 2x + 1$ and $g(x) = 3x - 5$, then find $(f\,o\,g)^{-1}(0)$.

49. If $f(x) = x^3 - 1,\ x < 0 = x^2 - 1,\ x \geq 0$ and $g(x) = (x+1)^{1/3},\ x < 1 = (x+1)^{1/2},\ x \geq 1$, then find $g\,o\,f(x)$.

50. If $f = \{(1, 3), (2, 1), (3, 4), (4, 2)\}$ and $g = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$, then find $f\,o\,g$.

CONCEPT APPLICATION

Level 1

1. A, B and C are three non-empty sets. If $A \subset B$ and $B \subset C$, then which of the following is true?
   (a) $B - A = C - B$
   (b) $A \cap B \cap C = B$
   (c) $A \cup B = B \cap C$
   (d) $A \cup B \cup C = A$

2. If S is the set of squares and R is the set of rectangles, then $(S \cup R) - (S \cap R)$ is
   (a) S.
   (b) R.
   (c) set of squares but not rectangles.
   (d) set of rectangles but not squares.

3. If $A = \{1, 2, 3, 4, 5, 6\}$, then how many subsets of A contain the elements 2, 3 and 5?
   (a) 4 (b) 8
   (c) 16 (d) 32

PRACTICE QUESTIONS

4. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, d, e, g\}$, then which of the following is true?
   (a) $C \subset (A \cup B)$
   (b) $C \subset (A \cap B)$
   (c) $A \cup B = A \cup C$
   (d) Both (a) and (c)

5. If $A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_{50}$ and $n(A_x) = x - 1$, then find $n\left[\bigcap_{x=11}^{50} A_x\right]$.
   (a) 49 (b) 50
   (c) 11 (d) 10

6. A group of 30 people take either tea or coffee. If 12 people do not take tea and 15 people take coffee, then how many people take tea?
   (a) 18 (b) 16
   (c) 15 (d) 12

7. If P is the set of parallelograms, and T is the set of trapeziums, then $P \cap T$ is
   (a) P (b) T
   (c) ϕ (d) P T

8. If X, Y and Z are three sets such that $X \supset Y \supset Z$, then $(X \cup Y \cup Z) - (X \cap Y \cap Z) =$ ______.
   (a) $X - Y$ (b) $Y - Z$
   (c) $X - Z$ (d) $Z - X$

9. If $n(A_x) = x + 1$ and $A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_{99}$, then $n\left[\bigcup_{x=1}^{99} A_x\right] =$ ______.
   (a) 99 (b) 98
   (c) 100 (d) 101

10. In a class every student can speak either English or Telugu. The number of students who can speak only English, the number of students who can speak only Telugu and the number of students who can speak both English and Telugu are equal. Then which of the following can represent the number of students of the class?
   (a) 20 (b) 25
   (c) 45 (d) 50

11. If $(2x - y, x + y) = (1, 11)$, then the values of x and y respectively are
   (a) 6, 5. (b) 7, 4.
   (c) 4, 7. (d) 7, 3.

12. A relation between two persons is defined as follows:
   $a\ R\ b$ 'if a and b born in different months', R is
   (a) reflexive. (b) symmetric.
   (c) transitive. (d) equivalence.

13. If A is a non-empty set, then which of the following is false?
   p: There is atleast one reflexive relation on A.
   q: There is atleast one symmetric relation on A.
   (a) p alone (b) q alone
   (c) Both p and q (d) Neither p nor q

14. In a set of teachers of a school, two teachers are said to be related if 'they teach the same subject', then the relation is
   (a) reflexive and symmetric.
   (b) symmetric and transitive.
   (c) reflexive and transitive.
   (d) equivalence.

15. If $A = \{x, y, z\}$, then the relation $R = \{(x, x), (y, y), (z, z), (z, x), (z, y)\}$ is
   (a) symmetric. (b) anti symmetric.
   (c) transitive. (d) Both (b)and (c)

Direction for questions 16 and 17:
In the set of animals, a relation R is defined in each question.

16. $a\ R\ b$ if 'a and b are in different zoological parks', then R is
   (a) only reflexive. (b) only symmetric.
   (c) only transitive. (d) equivalence.

17. On the set of human beings a relation R is defined as follows:
   $a\ R\ b$ if 'a and b have the same brother', then R is
   (a) only reflexive. (b) only symmetric.
   (c) only transitive. (d) equivalence.

18. Consider the following statements:
   p: Every reflexive relation is a symmetric relation.
   q: Every anti-symmetric relation is reflexive.

Which of the following is/are true?

(a) p alone (b) q alone

(c) Both p and q (d) Neither p nor q

19. In a set of ants in a locality, two ants are said to be related if they walk on a same straight line, then the relation is

(a) reflexive and symmetric.

(b) symmetric and transitive.

(c) reflexive and transitive.

(d) equivalence.

20. If $n(A) = 4$ and $n(B) = 4$, then find the number of subsets of $A \times B$.

(a) 65636 (b) 65536

(c) 65532 (d) 65356

21. A function f is constant from set $A = \{1, 2, 3\}$ onto set $B = \{a, b, c\}$ such that $f(1) = a$, then the range of f is

(a) {a, c}. (b) {a}.

(c) {a, b}. (d) {a, b, c}.

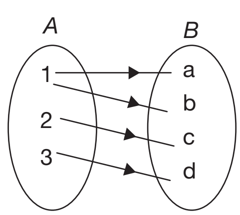
22. Which of the following is an odd function?

(a) $x + x^3$ (b) $x^3 - x^2 - 5$

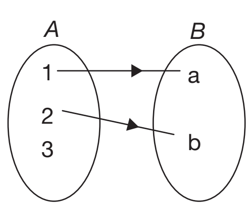
(c) $x^2 + x^4$ (d) $\dfrac{3x^2}{x^2+1}$

23. Which of the following relations is a function?

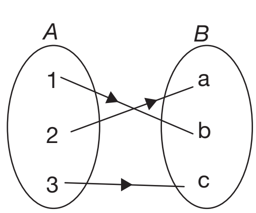
(a)



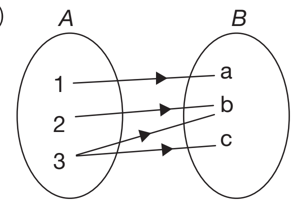
(b)



(c)

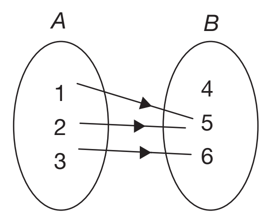


(d)

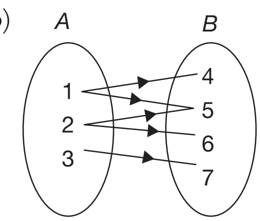


24. Which of the following relation is not a function?

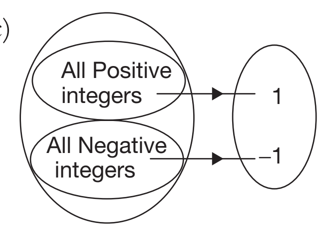
(a)



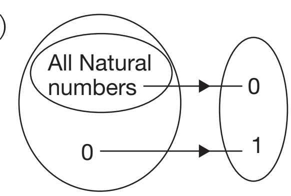
(b)



(c)



(d)



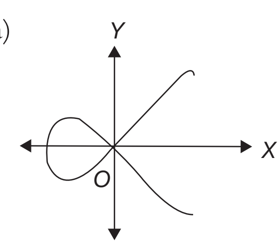
25. If $f : A \to B$ is an onto function defined by $f(x) = 3x - 4$ and $A = \{0, 1, 2, 3\}$, then the co-domain of f is

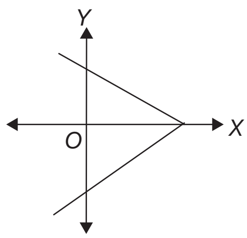
(a) $\{-4, 0, 2, 5\}$ (b) $\{-1, 2, 5, 6\}$

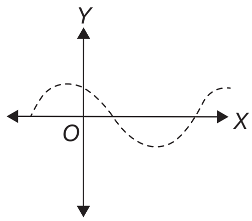
(c) $\{-4, -1, 2, 5\}$ (d) $\{-4, 1, 2, 5\}$

26. Which of the following graphs represents a function?

(a)



(b) 

(c) 

(d) None of these

27. If $f(x) = 2x - 3x^2 - 5$ and $g(x) = \dfrac{f(x) + f(-x)}{2}$, then $g(x)$ is

(a) odd.

(b) even.

(c) even as well as odd.

(d) Neither (a) nor (b)

28. Domain of the function $f(x) = \dfrac{5-x}{|3-x|}$ is

(a) $x \in R$ (b) $x \in Z$

(c) $R - \{3\}$ (d) $R - \{5\}$

29. $A = \{-1, 0, 1, 2\}$, $B = \{0, 1, 2\}$ and $f: A \to B$ defined by $f(x) = x^2$, then f is

(a) only one-one function.

(b) only onto function.

(c) bijective.

(d) not a function.

30. If two sets A and B have p and q number of elements respectively and $f: A \to B$ is one-one, then the relation between p and q is

(a) $p \geq q$ (b) $p > q$

(c) $p \leq q$ (d) $p = q$

Level 2

31. If $n(A \cap B) = 10$, $n(B \cap C) = 20$ and $n(A \cap C) = 30$, then find the greatest possible value of $n(A \cap B \cap C)$.

(a) 15 (b) 20

(c) 10 (d) 4

32. If X, Y and Z are any three non-empty sets such that any two of them are disjoint, then $(X \cup Y \cup Z) \cap (X \cap Y \cap Z)$ is

(a) X. (b) Y.

(c) Z. (d) ϕ.

33. A and B are any two non-empty sets and A is proper subset of B. If $n(A) = 5$, then find the minimum possible value of $n\ (A\ \Delta\ B)$.

(a) 1

(b) 5

(c) Cannot be determined

(d) None of these

34. If $n(A \cap B) = 5$, $n(A \cap C) = 7$ and $n(A \cap B \cap C) = 3$, then the minimum possible value of $n(B \cap C)$ is _______.

(a) 0 (b) 1

(c) 3 (d) 2

35. If a set contains n elements, then which of the following cannot be the number of reflexive relations in the set?

(a) 2^n (b) 2^{n-1}

(c) 2^{n^2-1} (d) 2^{n+1}

36. If $A = \{4, 6, 10, 12\}$ and R is a relation defined on A as 'two elements are related if they have exactly one common factor other than 1'. Then the relation R is

(a) anti-symmetric.

(b) only transitive.

(c) only symmetric.

(d) equivalence.

37. X is the set of all members in a colony and R is a relation defined on X as 'two persons are related if they speak same language'. The relation R is
(a) only symmetric.
(b) only reflexive.
(c) both symmetric and reflexive but not transitive.
(d) equivalence.

38. The relation 'is a factor of' on the set of natural numbers is not
(a) reflexive. (b) symmetric.
(c) anti-symmetric. (d) transitive.

39. If $f(x) = \log x$, then $\dfrac{f(xy)+f\left(\dfrac{x}{y}\right)}{f(x)f(y)} =$ ______.
(a) $\dfrac{2}{\log x}$ (b) $2\log y$
(c) $2\log x$ (d) $\dfrac{2}{\log y}$

40. If $f(x)=\dfrac{x-1}{x+1}, x \neq -1$; then find $f\left(\dfrac{x-1}{x+1}\right)$.
(a) x (b) $-\dfrac{1}{x}$
(c) $f(x)$ (d) $f\left(\dfrac{1}{x}\right)$

41. If $f: R \to R$ defined by $f(x) = 3x - 5$, then $f^{-1}(\{-1, -2, 1, 2\}) =$ ______.
(a) $\left\{1, \dfrac{4}{3}, \dfrac{7}{3}\right\}$ (b) $\left\{-1, 2, \dfrac{-4}{3}\right\}$
(c) $\left\{1, 2, \dfrac{4}{3}, \dfrac{7}{3}\right\}$ (d) $\{1, 2, -1, -2\}$

42. If $f: R \to R$ is a function defined as $f(\alpha - f(\alpha)) = 5f(\alpha)$ and $f(1) = 7$, then find $f(-6)$.
(a) 37 (b) 35
(c) 7 (d) 21

Level 3

43. All the students of a class like Horlicks, Maltova or Viva. Number of students who like only Horlicks and Maltova, only Maltova and Viva and only Horlicks and Viva are all equal to twice the number of students who like all the three foods. Number of students who like only Horlicks, only Maltova and only Viva are all equal to thrice the number of students who like all the three foods. If four students like all the three, then find the number of students in the class.
(a) 64 (b) 48
(c) 68 (d) 52

44. The inverse of the function $f(x)=(x^3-1)^{1/4}-12$ is
(a) $[1+(x+12)^3]^{1/4}$
(b) $[1-(x+12)^4]^{1/4}$
(c) $[(x+12)^3-1]^{1/4}$
(d) $[1+(x+12)^4]^{1/4}$

45. The relation, $R = \{(1, 3), (3, 5)\}$ is defined on the set with minimum number of elements of natural numbers. The minimum number of elements to be included in R so that R is equivalence, is
(a) 5. (b) 6.
(c) 7. (d) 8.

46. If $f(2x + 3) = 4x^2 + 12x + 15$, then the value of $f(3x + 2)$ is
(a) $9x^2 - 12x + 36$
(b) $9x^2 + 12x + 10$
(c) $9x^2 - 12x + 24$
(d) $9x^2 - 12x - 5$

47. If a relation $f: A \to B$ is defined by $f(x) = x + 2$, where $A = \{-1, 0, 1\}$ and $B = \{1, 2, 3\}$, then f is
(a) only one-one function.
(b) only onto function.
(c) bijective.
(d) None of these

48. The domain of the function, $f(x)=\dfrac{|x|-2}{|x|-3}$ is ______.
(a) R (b) $R - \{2, 3\}$
(c) $R - \{2, -2\}$ (d) $R - \{-3, 3\}$

PRACTICE QUESTIONS

49. If $f(x + y) = f(xy)$ and $f(1) = 5$, then find the value of $\sum_{k=0}^{6} f(k)$.

(a) 25 (b) 35

(c) 36 (d) 24

50. If $f(x) = (x + 1)$ and $g(x) = (x - 1)$, then find $(f \text{ o } g)((g \text{ o } f)(2))$.

(a) 1 (b) 2

(c) 3 (d) 4

51. If $f(x) = x$, $g(x) = x^2$ and $h(x) = x^3$, then find $[(h \text{ o } g) \text{ o } f](x)$.

(a) x (b) x^2

(c) x^3 (d) x^6

52. If $f = \{(2, 4), (3, 6), (4, 8)\}$ and $g = \{(4, 3), (6, 4), (8, 2)\}$, then find $f \text{ o } g$.

(a) $\{(2, 3), (3, 4), (4, 2)\}$

(b) $\{(4, 6), (6, 8), (8, 4)\}$

(c) $\{(3, 2), (4, 3), (2, 4)\}$

(d) $\{(6, 4), (8, 6), (4, 8)\}$

53. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{2x^2 + 5x + 2}}$.

(a) R

(b) $\left(-2, \dfrac{-1}{2}\right)$

(c) $(-\infty, -2] \cup \left[\dfrac{-1}{2}, \infty\right)$

(d) $(-\infty, -2) \cup \left(\dfrac{-2}{2}, \infty\right)$

54. Find the domain of function, $\sum_{p=1}^{10} \dfrac{1}{|2x - p|}$.

(a) R

(b) $R - \left\{\dfrac{1}{2}, 1, \dfrac{3}{2}, 2, \ldots, 10\right\}$

(c) $R - \left\{\dfrac{1}{2}, 1, \dfrac{3}{2}, 2, \ldots, 5\right\}$

(d) $R - \{1, 2, \ldots, 10\}$

55. Find the inverse function of $f(x) = 2x - 3$.

(a) $3x + 2$ (b) $3x - 2$

(c) $(x + 3)/2$ (d) $(x - 3)/2$

56. If $f(x) = x + 1$ and $g(x) = x - 2$, then find $(f^{-1} \text{o } g^{-1})(x)$.

(a) $x - 1$ (b) $x + 2$

(c) $g(x)$ (d) $f(x)$

57. If $f(x) + f(1 - x) = 10$, then the value of $f\left(\dfrac{1}{10}\right) + f\left(\dfrac{2}{10}\right) + \cdots + f\left(\dfrac{9}{10}\right)$

(a) is 45.

(b) is 50.

(c) is 90.

(d) is 40.

58. There are 60 students in a class. The number of students who passed in Mathematics is 45 and the number of students who passed in Physics is 40. The number of students who failed in both the subjects is 5. Find the number of students who passed in exactly one of the subjects.

(a) 35

(b) 25

(c) 15

(d) 40

59. If $X = \{2, 3, 5, 7, 11\}$ and $Y = \{4, 6, 8, 9, 10\}$, then find the number of one-one functions from X to Y.

(a) 720 (b) 120

(c) 24 (d) 12

Directions for questions 60 and 61:

These questions are based on the following data.

For any two sets A and B, $n(A) = 15$, $n(B) = 12$, $A \cap B \neq \phi$ and $B \not\subset A$.

60. Find the maximum possible value of $n(A \Delta B)$.

(a) 27 (b) 26

(c) 24 (d) 25

61. Find the minimum possible value of $n(A \Delta B)$.

(a) 3 (b) 4

(c) 5 (d) 6

Directions for questions 62 and 63:
These questions are based on the following data.

A and B are two finite sets. The difference of the number of elements of the power sets is 96. (Assume $n(A) > n(B)$)

62. Find $n(A) + n(B)$.

(a) 11 (b) 12
(c) 13 (d) 14

63. Find $n(A) - n(B)$.

(a) 2 (b) 3
(c) 4 (d) 5

Directions for questions 64 and 65:
These questions are based on the following data.

The relation R is defined on a set $P = \{a, b, c, d, e\}$ and R is a reflexive relation.

64. Which of the following is true about the number of elements of R?

(a) $1 \leq n(R) \leq 5$ (b) $1 \leq n(R) \leq 2^5$
(c) $5 \leq n(R) < 2^5$ (d) $5 \leq n(R) \leq 25$

65. How many reflexive relations are possible on P?

(a) 2^5 (b) 2^{25}
(c) 2^{20} (d) 2^{18}

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. A
2. 1
3. 126
4. $A \cup B$
5. $n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
6. μ
7. A'
8. ϕ
9. 0
10. 8
11. False
12. False
13. $R = \{(a, a), (b, b), (c, c)\}$
14. 2
15. $R^{-1} = \{(1, 7), (2, 6), (3, 5)\}$
16. symmetric
17. 24
18. $n(A \times B) = 36$
19. $\{1, 2, 3, 4, 5, 6, 7, 8\}$
20. $\{3, 6\}$
21. True
22. $(0, \infty)$
23. $\frac{15}{2}, \frac{9}{2}$
24. 4
25. True
26. R^+
27. 4
28. 100
29. $f^{-1}(y) = \frac{y-6}{7}$
30. True

Short Answer Type Questions

31. $\{7, 10\}$
32. $X = \{31, 37\}$
33. 9
34. 15
35. 11
36. Domain = $\{1, 2\}$, Range = $\{1, 2, 0, 3\}$
37. 210.
38. It is reflexive, symmetric, and transitive.
39. $R = \{(3, 3), (3, 5), (3, 6), (3, 9), (5, 3), (5, 5), (5, 6), (5, 9), (6, 3), (6, 5), (6, 6), (6, 9), (9, 3), (9, 5), (9, 6)\}$
40. R is only symmetric.
41. 0
42. 82
43. x
44. Inverse does not exist
45. $(0, 1)$

Essay Type Questions

46. 25
47. $f(a_1) + f(a_2)$
48. $\frac{3}{2}$
49. When $x < 0$ or $x \geq 1$, $g \circ f(x) = x$ and when $0 \leq x < 1$, $g \circ f(x) = x^{2/3}$
50. $\{(1, 1), (2, 4), (3, 2), (4, 3)\}$

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (a)	**7.** (a)	**8.** (c)	**9.** (c)	**10.** (c)
11. (c)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (d)	**16.** (b)	**17.** (d)	**18.** (d)	**19.** (d)	**20.** (b)
21. (b)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (c)	**29.** (d)	**30.** (c)

Level 2

31. (c)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (d)	**36.** (c)	**37.** (d)	**38.** (b)	**39.** (d)	**40.** (b)
41. (c)	**42.** (b)								

Level 3

43. (a)	**44.** (d)	**45.** (c)	**46.** (c)	**47.** (c)	**48.** (d)	**49.** (b)	**50.** (b)	**51.** (d)	**52.** (b)
53. (d)	**54.** (c)	**55.** (c)	**56.** (d)	**57.** (a)	**58.** (b)	**59.** (b)	**60.** (d)	**61.** (c)	**62.** (b)
63. (a)	**64.** (d)	**65.** (c)							

CONCEPT APPLICATION

Level 1

1. Using the concept of subset, verify the options.
2. Every square is a rectangle.
4. Find $A \cup B$, $A \cap B$ and $A \cup C$.
5. If $A_1 \subset A_2 \subset A_3$, ..., then $n(A_1 \cap A_2 \cap A_3 \ldots) = n(A_1)$.
6. Use Venn diagram concept.
7. Recall the properties of parallelogram and trapezium.
8. Recall the concept of superset, union of sets and intersection.
9. Recall the concepts of subset and union of sets.
10. $n(A \cup B) = n(\text{only } A) + n(\text{only } B) + n(A \cap B)$.
11. Equate the corresponding coordinates.
12. Use the definitions of reflexive, symmetric, anti-symmetric and transitive.
13. Recall the different types of relations.
14. Use the definitions of reflexive, symmetric, anti-symmetric and equivalence.
15. Recall the properties of relations.
16. Recall the properties of relations.
17. Use the definitions of reflexive, symmetric, anti-symmetric, transitive and equivalence.
18. Recall the definitions of reflexive, symmetric and anti-symmetric relations.
19. Recall the properties of relations.
20. Number of subsets of $A \times B$ is $2^{n(A)\cdot n(B)}$.
21. Constant function contains only one element in the range.
22. If $f(x)$ is an odd function, then $f(-x) = -f(x)$.
23. Recall the definition of function.
24. Recall the definition of function.
25. Substitute domain values in $f(x)$.
26. Use the definition of a function.
27. Find $\dfrac{f(x)+f(-x)}{2}$ and verify even or odd.
28. Function is defined when denominator is not equal to zero.
29. Substitute the domain values in $f(x)$ and find range.
30. Recall the definition of one-one function.

Level 2

31. The greatest possible value of $n(A \cap B \cap C)$ is the least value amongst the values of $n(A \cap B)$, $(B \cap C)$ and $n(A \cap C)$.
32. A and B are disjoint $\Rightarrow A \cap B = \phi$ and $A \cap \phi = \phi$.
33. (i) A is proper subset of B, $A - B = \phi$, i.e., $n(A - B) = 0$.
    (ii) Given $n(A) = 5$, the minimum number of elements in B is 6.
    (iii) The minimum possible value of $n(A \,\Delta\, B)$ is $n(B) - n(A)$.
34. Number of elements in $(A \cap B \cap C)$ becomes the minimum number of elements in $B \cap C$.
35. Number of reflexive relations in a set containing 'n' elements is 2^{n^2-n}.
36. (i) Write the elements in R.
    (ii) Apply definition of symmetric relation.
37. Recall the properties of relations.
38. Use the definitions of reflexive, symmetric, anti-symmetric and equivalence.
39. (i) Substitute $f(xy)$, $f(x/y)$ in the given equation.
    (ii) $\log ab = \log a + \log b$.
    (iii) $\log\left(\dfrac{a}{b}\right) = \log a - \log b$.

(iv) Replace x by xy and also by $\frac{x}{y}$ in $f(x)$ and simplify.

40. (i) Replace x by $\frac{x-1}{x+1}$ in $f(x)$.

(ii) Then simplify.

41. (i) Find $f^{-}(x)$.

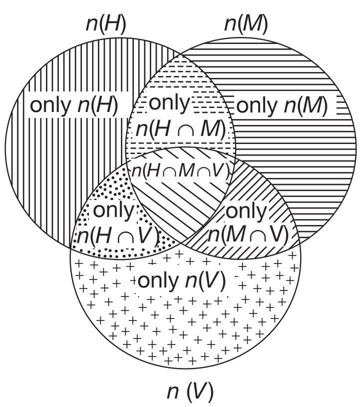
(ii) Put $x = -1, -2, 1, 2$ in $f^{-1}(x)$ and find their values.

42. Put $\alpha = 1$ in the given equation and simplify.

Level 3

43. (i) Let the number of students who like all the three be 'x.

(ii)



(iii) Substitute the given values in the above figure.

44. (i) Let $f(x) = y$.

(ii) Write the value of x in terms of y.

45. Recall the properties of equivalence relation.

46. (i) Put $2x + 3 = t$.

(ii) Replace x by t.

(iii) Again put $t = 3x + 2$.

47. Given, $f: A \to B$ $f(x) = x + 2$ A

$= \{-1, 0, 2\}$ and $B = \{1, 2, 3\}$ $f(A) = B$ and $\forall$ $x \in A$, $f(x_1) \neq f(x_2)$.

$\therefore$ f is bijective.

48. $F(x) = \dfrac{|x|-2}{|x|-3}$. Here, x cannot be ± 3.

(f is not defined when $x = \pm 3$)

$\therefore$ Domain $= R - \{-3, 3\}$

49. Given $f(x + y) = f(xy)$ (1)

and $f(1) = 5$

Put $x = 1$ and $y = 0$

$f(1 + 0) = f(0) \Rightarrow f(0) = 5$

Put $x = 1$, $y = 1$ in Eq. (1)

$f(1 + 1) = f(1)$

$\therefore$ $f(2) = 5$

Put $x = 2$, $y = 1$ in Eq. (1)

$f(2 + 1) = f(2 \times 1)$

$\therefore$ $f(3) = 5$

Similarly, $f(4) = f(5) = f(6) = 5$

$$\therefore \sum_{k=0}^{6} f(k) = f(0) + f(1) + \cdots + (6) = 7 \times 5 = 35.$$

50. Given $f(x) = x + 1$, $g(x) = x - 1$

$(g \text{ o } f)(x) = g[f(x)] = g[x + 1] = (x + 1) - 1 = x$

$\therefore$ $(g \text{ o } f)(2) = 2$ (1)

$(g \text{ o } f)(x) = f[g(x)]$

$= f(x - 1) = (x - 1) + 1$

$= x$ (2)

$\therefore$ $(g \text{ o } f)[(g \text{ o } f)(2)]$

$= (g \text{ o } f)(2)$ (from Eq. (1))

$= 2.$

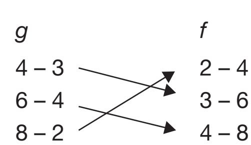
51. Given $f(x) = x$, $g(x) = x^2$ and $h(x)$

$= x^3$ $[(h \text{ o } g) \text{ o } f](x) = (h \text{ o } g)[f(x)]$

$= (h \text{ o } g)(x) = h[g(x)] = h(x^2)$

$= (x^2)^3 = x^6.$

52. For $f \text{ o } g$:



$\therefore$ $f \text{ o } g = \{(4, 6), (6, 8), (8, 4)\}$.

53. $f(x)=\dfrac{1}{\sqrt{2x^2+5x+2}}$

$\therefore\ 2x^2+5x+2>0$

$\Rightarrow (2x+1)\ (x+2)>0$

$\Rightarrow 2x+1>0$ and $x+2>0$

(Or)

$\Rightarrow 2x+1<0$ and $x+2<0$

$\Rightarrow x<-2$ or $x>\dfrac{-1}{2}$

$\Rightarrow x\in(-\infty,-2)\cup\left(\dfrac{-1}{2},\infty\right).$

54. $\displaystyle\sum_{p=1}^{10}\frac{1}{|2x-p|}$

$=\dfrac{1}{|2x-1|}+\dfrac{1}{|2x-2|}+\cdots+\dfrac{1}{|2x-10|}$

Here, for $x=\dfrac{1}{2},1,\dfrac{3}{2},\ldots,5,$ it is not defined.

$\therefore$ Domain $=R-\left\{\dfrac{1}{2},1,\dfrac{3}{2},2\ldots5\right\}.$

55. Let $y=f(x)=2x-3$

$\Rightarrow y=2x-3\Rightarrow x=\dfrac{y+3}{2}$

$y=f(x)\Rightarrow x=f^{-1}(y)$

$f^{-1}(y)=\dfrac{y+3}{2}$

Replace y by x.

$f^{-1}(x)=\dfrac{x+3}{2}.$

56. $f(x)=x+1$ $\qquad g(x)=x-2$

$\Rightarrow x=yx-1$ $\qquad z=x-2$

$x=z+2$

$\therefore f^{-1}(y)=y-1$ $\qquad g^{-1}(z)=z+2$

$\therefore f^{-1}(x)=x-1$ $\qquad g^{-1}(x)=x+2$

Now, $(f^{-1}o\ g^{-1})\ (x)=f^{-1}\ (g^{-1}\ (x)\}$

$=f^{-1}\ (x+2)=(x+2)-1$

$=x+1=f(x).$

57. Given $f(x)+f(1-x)=10$ $\qquad$ (1)

$f\left(\dfrac{1}{10}\right)+f\left(\dfrac{2}{10}\right)+\cdots+f\left(\dfrac{9}{10}\right)$

$=f\left(\dfrac{1}{10}\right)+f\left(\dfrac{9}{10}\right)+f\left(\dfrac{2}{10}\right)$

$+f\left(\dfrac{8}{10}\right)+\cdots+f\left(\dfrac{5}{10}\right)$

$=\left[f\left(\dfrac{1}{10}\right)+f\left(1-\dfrac{1}{10}\right)\right]+\left[f\left(\dfrac{2}{10}\right)+f\left(1-\dfrac{2}{10}\right)\right]$

$+\cdots+f\left(\dfrac{5}{10}\right)$

$=4(10)+f\left(\dfrac{1}{2}\right)$

Put $x=\dfrac{1}{2}$ in Eq. (1)

$\therefore f\left(\dfrac{1}{2}\right)+f\left(1-\dfrac{1}{2}\right)=10$

$\Rightarrow 2\left[f\left(\dfrac{1}{2}\right)\right]=10\Rightarrow f\left(\dfrac{1}{2}\right)=5$

$\therefore f\left(\dfrac{1}{10}\right)+f\left(\dfrac{1}{10}\right)+\cdots+f\left(\dfrac{9}{10}\right)40+5=45.$

58. Number of students $=60$

Number of students who passed in Mathematics $=45$

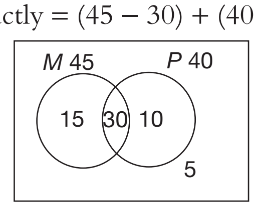
Number of students who passed in Physics $=40$

Number of students who failed in both the subjects $=5$

$\therefore$ Number of students who passed in one of the subjects $=60-5=55$

$\therefore$ Number of students who passed in both the subjects $=45+40-55=30$

$\therefore$ Number of students who passed in one of the subjects exactly $=(45-30)+(40-30)=25.$



59. Given $X=\{2,3,5,7,11\}$ $Y=\{4,6,8,9,10\}$.

$\therefore\ n(x)=5$ and $n(y)=5$.

Number of one to one functions from X to $Y=5!$ $=120$.

Solutions 60 and 61:

$A \cap B \neq \phi$, $B \not\subset A$

$n(A) = 15$ and $n(B) = 12$

As $A \cap B \neq \phi$, $n(A \cap B) \neq 0$.

60. For the maximum value of $n(A \Delta B)$, $n(A \cap B)$ should be minimum.

The minimum value of $n(A \cap B) = 1$, ($\because$ $n(A \cap B) \neq 0$).

Now, $n(A \Delta B) = n(A \cup B) - n(A \cap B)$

$= n(A) + n(B) - 2[\, n(A \cap B)\,]$

$= 15 + 12 - 2 = 25$.

61. For the minimum value of $n(A \Delta B)$, $n(A \cap B)$ should be maximum.

The maximum value of $n(A \cap B) = 11$, ($\therefore$ $B \not\subset A$ and $n(B) = 12$).

Now $n(A \Delta B) = n(A \cup B) - n(A \cap B)$

$= n(A) + n(B) - 2\,[n(A \cap B)]$

$= 15 + 12 - 2(11) = 5$.

Solutions 62 and 63:

Let $n(A) = x$ and $n(B) = y$.

Given, $n[p(A)] - n[p(B)] = 96$.

$\therefore$ $2^x - 2^y = 96$

$\Rightarrow 2^x - 2^y = 2^5\ (3)$

$\Rightarrow 2^{x-5} - 2^{y-5} = 3$

$\Rightarrow 2^{x-5} - 2^{y-5} = 2^2 - 2^\circ$

$\Rightarrow x - 5 = 2$ and $y - 5 = 0$

$\Rightarrow x = 7$ and $y = 5$.

62. $n(A) + n(B) = x + y = 7 + 5 = 12$.

63. $n(A) - n(B) = x - y = 7 - 5 = 2$.

Solutions 64 and 65:

64. R is reflexive relation defined on $P = \{a, b, c, d, e\}$.

And also $R \subseteq P \times P$.

$\therefore$ $5 \leq n(R) \leq 25$.

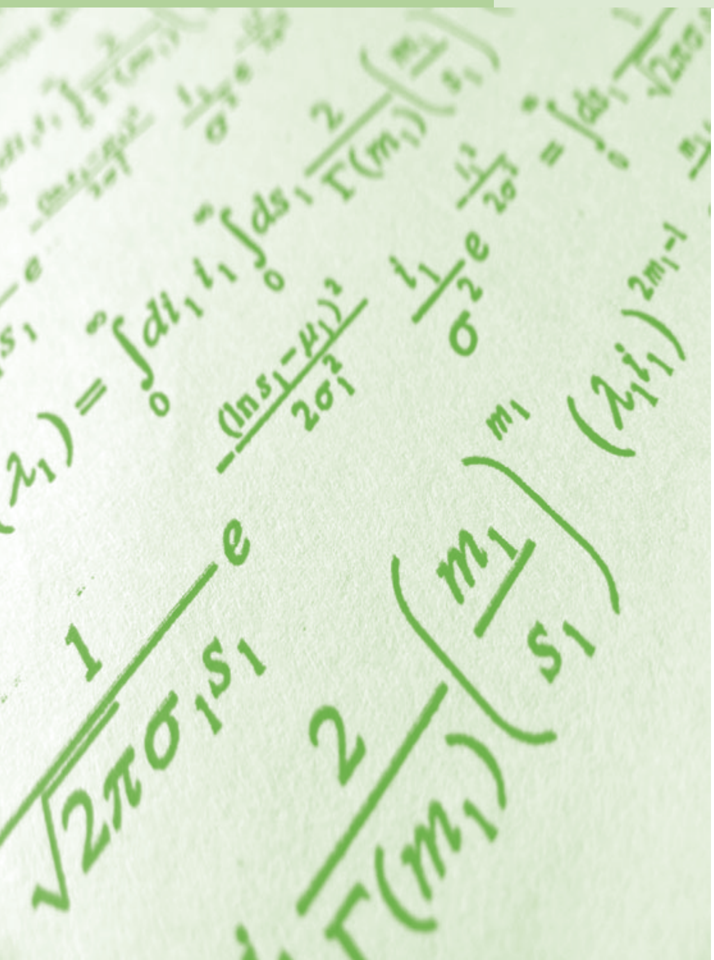
65. Number of possible reflexive relations

$= 2^{5^2-5} = 2^{20}$.

HINTS AND EXPLANATION

Chapter 7

Progressions



REMEMBER

Before beginning this chapter, you should be able to:

- Recall different types of numbers
- Apply basic operations on numbers

KEY IDEAS

After completing this chapter, you would be able to:

- Understand terms such as sequences, series and progressions
- Study arithmetic progression (AP) and also obtain arithmetic mean (AM)
- Formulate geometric progression (GP)
- Insert harmonic mean (HM) between two numbers
- Study relation between AM, GM and HM

INTRODUCTION

Let us observe the following pattern of numbers:

1. 5, 11, 17, 23, …
2. 6, 12, 24, 48, …
3. 4, 2, 0, –2, –4, …
4. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

In example 1, every number (except 5) is formed by adding 6 to the previous number. Hence a specific pattern is followed in the arrangement of these numbers. Similarly, in example 2, every number is obtained by multiplying the previous number by 2. Similar cases are followed in examples 3 and 4.

SEQUENCE

A systematic arrangement of numbers according to a given rule is called a sequence.

The numbers in a sequence are called its terms. We refer the first term of a sequence as T_1, second term as T_2 and so on. The nth term of a sequence is denoted by T_n, which may also be referred to as the general term of the sequence.

Finite and Infinite Sequences

1. A sequence which consists of a finite number of terms is called a finite sequence.

   Example: 2, 5, 8, 11, 14, 17, 20, 23 is a finite sequence of 8 terms.

2. A sequence which consists of an infinite number of terms is called an infinite sequence.

   Example: 3, 10, 17, 24, 31, … is an infinite sequence.

Note If a sequence is given, then we can find its nth term and if the nth term of a sequence is given we can find the terms of the sequence.

Examples:

Find the first four terms of the sequences whose nth terms are given as follows.

1. $T_n = 3n + 1$

   Substituting $n = 1$,

   $T_1 = 3(1) + 1 = 4$

   Similarly, $T_2 = 3(2) + 1 = 7$

   $T_3 = 3(3) + 1 = 10$

   $T_4 = 3(4) + 1 = 13$

   ∴ The first four terms of the sequence are 4, 7, 10, 13.

2. $T_n = 2n^2 - 3$

   Substituting $n = 1$,

   $T_1 = 2(1)^2 - 3 = -1$

Similarly, $T_2 = 2(2)^2 - 3 = 5$

$T_3 = 2(3)^2 - 3 = 15$

$T_4 = 2(4)^2 - 3 = 29$

$\therefore$ The first four terms of the sequence are $-1, 5, 15, 29$.

Series

The sum of the terms of a sequence is called the series of the corresponding sequence.

Example:

$1 + 2 + 3 + \cdots + n$ is a finite series of first n natural numbers.

The sum of first n terms of series is denoted by S_n.

Here, $S_n = T_1 + T_2 + \cdots + T_n$.

Here, $S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

…

…

$S_n = T_1 + T_2 + T_3 + \cdots + T_n$

We have,

$S_2 - S_1 = T_2$

$S_3 - S_2 = T_3$

Similarly,

$S_n - S_{n-1} = T_n$.

EXAMPLE 7.1

In the series, $T_n = 2n + 5$, find S_4.

SOLUTION

$T_n = 2n + 5$

$T_1 = 2(1) + 5 = 7$

$T_2 = 2(2) + 5 = 9$

$T_3 = 2(3) + 5 = 11$

$T_4 = 2(4) + 5 = 13$

$S_4 = T_1 + T_2 + T_3 + T_4 = 7 + 9 + 11 + 13 = 40$.

Sequences of numbers which follow specific patterns are called progressions. Depending on the pattern, the progressions are classified as follows:

1. Arithmetic Progression
2. Geometric Progression
3. Harmonic Progression.

Arithmetic Progression (AP)

Numbers (or terms) are said to be in arithmetic progression when each one, except the first, is obtained by adding a constant to the previous number (or term).

An arithmetic progression can be represented by a, $a + d$, $a + 2d$, …, $[a + (n - 1)d]$. Here, d is added to any term to get the next term of the progression. The term a is the first term of the progression, n is the number of terms in the progression and d is the common difference.

1. The nth term (general term) of an arithmetic progression is, $T_n = a + (n - 1)d$.
2. Sum to n terms of an AP $= S_n = \frac{n}{2}[2a + (n - 1)d]$.

The sum to n terms of an AP can also be written in a different manner. That is, sum of n terms $= \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + \{a + (n - 1)d\}]$.

But, when there are n terms in an AP, a is the first term and $\{a + (n - 1)d\}$ is the last term. Hence, $S_n = \left(\frac{n}{2}\right)$[first term + last term].

Arithmetic Mean (AM)

The average of all the terms in an AP is called the arithmetic mean (AM) of the AP.

The average of a certain terms $= \dfrac{\text{Sum of all the terms}}{\text{Number of terms}}$

$\therefore$ AM of n terms in an AP $= \dfrac{S_n}{n} = \dfrac{1}{n} \times \dfrac{n}{2}[\text{first term + last term}] = \dfrac{(\text{first term + last term})}{2}$,

i.e., the AM of an AP is the average of the first and the last terms of the AP.

The AM of an AP can also be obtained by considering any two terms which are EQUIDISTANT from the two ends of the AP and taking their average, i.e.,

1. the average of the second term from the beginning and the second term from the end is equal to the AM of the AP
2. the average of the third term from the beginning and the third term from the end is also equal to the AM of the AP and so on.

In general, the average of the kth term from the beginning and the kth term from the end is equal to the AM of the AP.

If the AM of an AP is known, the sum to n terms of the series (S_n) can be expressed as $S_n = n$(AM)

In particular, if three numbers are in arithmetic progression, then the middle number is the AM, i.e., if a, b and c are in AP, then b is the AM of the three terms and $b = \dfrac{a+c}{2}$.

If a and b are any two numbers, then their AM $= \dfrac{a+b}{2}$.

Inserting arithmetic mean between two numbers:

When n arithmetic means a_1, a_2, …, a_3 are inserted between a and b, then a, a_1, a_2, …, a_n, b are in AP.

$\Rightarrow t_1 = a$ and $t_{n+2} = b$ of AP

The common difference of the AP can be obtained as follows:

Given that, n arithmetic means are there between a and b.

$\therefore a = t_1$ and $b = t_{n+2}$

Let d be the common difference.

$\Rightarrow b = t_1 + (n + 1)d$

$\Rightarrow b = a + (n + 1)d$

$\Rightarrow d = \dfrac{(b-a)}{(n+1)}$.

Notes

1. If three numbers are in AP, we can take the three terms to be $(a - d)$, a and $(a + d)$.
2. If four numbers are in AP, we can take the four terms to be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. The common difference in this case is $2d$ and not d.
3. If five numbers are in AP, we can take the five terms to be $(a - 2d)$, $(a - d)$, a, $(a + d)$ and $(a + 2d)$.

Some Important Results

The sum of first n terms of the following series are quite useful and hence should be remembered by students.

1. Sum of first n natural numbers $= \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. Sum of the squares of first n natural numbers $= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
3. Sum of the cubes of first n natural numbers $= \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \left[\sum_{1}^{n} i\right]^2$

EXAMPLE 7.2

Find the 14th term of an AP whose first term is 3 and the common difference is 2.

SOLUTION

The nth term of an AP is given by $t_n = a + (n - 1)d$, where a is the first term and d is the common difference.

$\therefore\ t_{14} = 3 + (14 - 1)\,2 = 29.$

EXAMPLE 7.3

Find the first term and the common difference of an AP, if the 3rd term is 6 and the 17th term is 34.

SOLUTION

If a is the first term and d is the common difference, then we have

$$a + 2d = 6 \tag{1}$$

$$a + 16d = 34 \tag{2}$$

On subtracting Eq. (1) from Eq. (2), we get

$$14d = 28 \Rightarrow d = 2$$

Substituting the value of d in Eq. (1), we get $a = 2$

$\therefore\ a = 2$ and $d = 2$.

EXAMPLE 7.4

Find the sum of the first 22 terms of an AP whose first term is 4 and the common difference is $\frac{4}{3}$.

SOLUTION

Given that, $a = 4$ and $d = \frac{4}{3}$.

We have, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{22} = \left(\frac{22}{2}\right)\left[(2)(4) + (22-1)\left(\frac{4}{3}\right)\right] = (11)(8+28) = 369.$$

EXAMPLE 7.5

Divide 124 into four parts in such a way that they are in AP and the product of the first and the 4th part is 128 less than the product of the 2nd and the 3rd parts.

SOLUTION

Let the four parts be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. The sum of these four parts is 124, i.e., $4a = 124 \Rightarrow a = 31$

$(a - 3d)\ (a + 3d) = (a - d)\ (a + d) - 128$

$\Rightarrow a^2 - 9d^2 = a^2 - d^2 - 128$

$\Rightarrow 8d^2 = 128 \Rightarrow d = \pm 4.$

As $a = 31$, taking $d = 4$, the four parts are 19, 27, 35 and 43.

Note If d is taken as -4, then the same four numbers are obtained, but in decreasing order.

EXAMPLE 7.6

Find the three terms in AP, whose sum is 36 and product is 960.

SOLUTION

Let the three terms of an AP be $(a - d)$, a and $(a + d)$.

Sum of these terms is $3a$.

$3a = 36 \Rightarrow a = 12$

Product of these three terms is

$(a + d)\ a\ (a - d) = 960$

$\Rightarrow (12 + d)\ (12 - d) = 80$

$\Rightarrow 144 - d^2 = 80 \Rightarrow d = \pm 8.$

Taking $d = 8$, we get the terms as 4, 12 and 20.

Note If d is taken as -8, then the same numbers are obtained, but in decreasing order.

EXAMPLE 7.7

Find the sum of all natural numbers and lying between 100 and 200 which leave a remainder of 2 when divided by 5 in each case.

(a) 2990 **(b)** 2847 **(c)** 2936 **(d)** None of these

SOLUTION

The natural number which leaves remainder 2 greater than 100 is 102, and less than 200 is 197.

$\therefore$ The series is 102, 107, 112, 117, 122, …, 197.

Number of terms

$$= \frac{200}{5} - \frac{100}{5} = 40 - 20 = 20.$$

$$S_n = \frac{n}{2}[a+l] = \frac{20}{2}[102+197]$$

$$= \frac{20}{2}[299] = 2990.$$

EXAMPLE 7.8

Find the sum of 100 terms of the series 1(3) + 3(5) + 5(7) + …

(a) 1353300 **(b)** 1353400 **(c)** 1353200 **(d)** 1353100

SOLUTION

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots$

$$t_n = t_{n1} \times t_{n2}$$
$$= 1 + (n-1)2 + 3 + (n-1)2$$
$$= (2n-1)(2n+1)$$
$$t_n = 4n^2 - 1.$$
$$\sum t_n = \sum(4n^2 - 1)$$
$$= 4\sum n^2 - \sum 1$$
$$= 4 \times \frac{n(n+1)(2n+1)}{6} - n$$
$$= \frac{2n(n+1)(2n+1) - 3n}{3}$$
$$= \frac{n}{3}[2(n+1)(2n+1) - 3].$$
$$\sum t_{100} = \frac{100}{3}[2 \times 101 \times 201 - 3]$$
$$= \frac{100}{3}[202 \times 201 - 3]$$
$$= 100[202 \times 67 - 1]$$
$$= 100[13534 - 1] = 100 \times 13533$$
$$= 1353300.$$

GEOMETRIC PROGRESSION (GP)

Numbers are said to be in geometric progression when the ratio of any quantity to the number that follows it is the same. In other words, any term of a GP (except the first one) can be obtained by multiplying the previous term by the same constant.

The constant is called the common ratio and is normally represented by r. The first term of a GP is generally denoted by a.

A geometric progression can be represented by a, ar, ar^2, … where a is the first term and r is the common ratio of the GP. The nth term of the GP is ar^{n-1}, i.e., $t_n = ar^{n-1}$.

Sum of first n terms $= S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1} = \dfrac{r(ar^{n-1})-a}{r-1}$

$\therefore$ The sum of first n terms of a geometric progression can also be written as

$$S_n = \frac{r(\text{Last term}) - \text{First term}}{r-1}.$$

Notes

1. If n terms viz., $a_1, a_2, a_3, \ldots, a_n$ are in GP, then the geometric mean (GM) of these n terms is given by $= \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$.
2. If three terms are in geometric progression, then the middle term is the geometric mean of the GP, i.e., if a, b and c are in GP, then b is the geometric mean of the three terms.
3. If there are two terms say a and b, then their geometric mean is given by $\text{GM} = \sqrt{ab}$.
4. When n geometric means are there between a and b, the common ratio of the GP can be derived as follows. Given that, n geometric means are there between a and b.

   $\therefore \quad a = t_1$ and $b = t_{n+2}$.

   Let 'r' be the common ratio

   $\Rightarrow b = (t_1)(r^{n+1}) \Rightarrow b = ar^{n+1}$

   $\Rightarrow r^{n+1} = \dfrac{b}{a} \Rightarrow r = \sqrt[(n+1)]{\dfrac{b}{a}}$
5. For any two positive numbers a and b, their arithmetic mean is always greater than or equal to their geometric mean, i.e., for any two positive numbers a and b, $\dfrac{a+b}{2} \geq \sqrt{ab}$. The equality holds if and only if $a = b$.
6. When there are three terms in geometric progression, we can take the three terms to be a/r, a and ar.

Infinite Geometric Progression

If $-1 < r < 1$ (or $|r| < 1$), then the sum of a geometric progression does not increase infinitely but 'converges' to a particular value, no matter how many terms of the GP we take. The sum of an infinite geometric progression is represented by S_∞ and is given by the formula,

$$S_\infty = \frac{a}{1-r}, \text{ if } |r| < 1.$$

EXAMPLE 7.9

Find the 7th term of the GP whose first term is 6 and common ratio is $\dfrac{2}{3}$.

SOLUTION

Given that, $t_1 = 6$ and $r = \dfrac{2}{3}$.

We have, $t_n = a \cdot r^{n-1}$

$$t_7 = (6)\left(\frac{2}{3}\right)^6 = \frac{(6)(64)}{729} = \frac{128}{243}.$$

EXAMPLE 7.10

Find the common ratio of the GP whose first and last terms are 25 and $\frac{1}{625}$ respectively and the sum of the GP is $\frac{19531}{625}$.

SOLUTION

We know that the sum of a GP is $\frac{\text{first term} - r(\text{last term})}{1-r}$

or $\frac{19531}{625} = \frac{25 - \left(\frac{r}{625}\right)}{1-r}$

$\therefore r = \frac{1}{5}$.

EXAMPLE 7.11

Find three numbers of a GP whose sum is 26 and product is 216.

SOLUTION

Let the three numbers be a/r, a and ar.

Given that,

$a/r \cdot a \cdot ar = 216;$

$\Rightarrow a^3 = 216; a = 6.$

$a/r + a + ar = 26$

or, $6 + 6r + 6r^2 = 26r$

or, $6r^2 - 20r + 6 = 0$

or, $6r^2 - 18r - 2r + 6 = 0$

or, $6r(r-3) - 2(r-3) = 0$

or, $r = 1/3$ (or) $r = 3$.

Hence, the three numbers are 2, 6 and 18 (or) 18, 6 and 2.

EXAMPLE 7.12

If $|x| < 1$, then find the sum of the series $2 + 4x + 6x^2 + 8x^3 + \cdots$.

SOLUTION

Let $S = 2 + 4x + 6x^2 + 8x^3 + \cdots$ (1)

$xS = 2x + 4x^2 + 6x^3 + \cdots$ (2)

Eq. (1) – Eq. (2) gives

$S(1-x) = 2 + 2x + 2x^2 + 2x^3 + \cdots$

$= 2(1 + x + x^2 + \cdots)$

$1 + x + x^2 + \cdots$ is an infinite GP with $a = 1$, $r = x$ and $|r| = |x| < 1$

$\therefore$ Sum of the series $= \frac{1}{1-x}$

$\therefore S(1-x) = \frac{2}{(1-x)}$

$\therefore S = \frac{2}{(1-x)^2}$.

EXAMPLE 7.13

Find the sum of the series $1, \frac{2}{5}, \frac{4}{25}, \frac{8}{125}, \ldots \propto$.

SOLUTION

Given that, $a = 1, r = \frac{2}{5}$ and $|r| = \left|\frac{2}{5}\right| < 1$

$$\therefore S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}.$$

EXAMPLE 7.14

S_{10} is the sum of the first 10 terms of a GP and S_5 is the sum of the first 5 terms of the same GP.

If $\frac{S_{10}}{S_5} = 244$, then find the common ratio.

(a) 3 **(b)** 4 **(c)** 5 **(d)** 2

SOLUTION

$$\frac{S_{10}}{S_5} = \frac{\frac{a\cdot(r^{10}-1)}{(r-1)}}{\frac{a(r^5-1)}{r-1}}$$

or, $$\frac{S_{10}}{S_5} = \frac{r^{10}-1}{r^5-1} = r^5+1$$

$$\frac{S_{10}}{S_5} = \frac{(r^5-1)^2+2r^5-2}{r^5-1}$$

$$244 = \frac{(r^5-1)[r^5-1+2]}{(r^5-1)} \Rightarrow 244 = r^5+1$$

or, $r^5 = 243 \Rightarrow r = 3.$

EXAMPLE 7.15

The difference between two hundred-digit numbers consisting of all 1's and a hundred-digit number consisting of all 2's is equal to

(a) $\underbrace{99\ldots9}_{100 \text{ times}}$

(b) $\left(\frac{333\ldots3}{80 \text{ times}}\right)^2$

(c) $\left(\frac{333\ldots3}{100 \text{ times}}\right)^2$

(d) $\underbrace{99\ldots9}_{200 \text{ times}}$

HINT

Apply the principle $11 = 1 + 10^{2-1}$, 111 = and proceed.

HARMONIC PROGRESSION (HP)

A progression is said to be a harmonic progression if the reciprocal of the terms in the progression form an arithmetic progression.

For example, consider the series $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$

The progression formed by taking reciprocals of terms of the above series is 2, 5, 8, 11, ... Clearly, these terms form an AP whose common difference is 3.

Hence, the given progression is a harmonic progression.

*n*th term of an HP:

We know that if a, $a + d$, $a + 2d$, ... are in AP, then the nth term of this AP is $a + (n - 1)d$. Its reciprocal is $\frac{1}{a+(n-1)d}$.

So, nth term of an HP whose first two terms are and is $\frac{1}{a}$ and $\frac{1}{a+d}$ is $\frac{1}{a+(n-1)d}$.

Note There is no concise general formula for the sum to n terms of an HP.

EXAMPLE 7.16

Find the 10th term of the HP $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \ldots$

SOLUTION

The given HP is $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \ldots$

The corresponding AP is $\frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots$

Here $a = \frac{2}{3}; d = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ T_{10} of the corresponding AP is $a + (10 - 1)d = \frac{2}{3} + (9)\frac{1}{3} = \frac{11}{3}$

Hence, required term in HP is $\frac{3}{11}$.

Harmonic Mean (HM)

If three terms are in HP, then the middle term is the HM of other two terms.

The harmonic mean of two terms a and b is given by $\text{HM} = \frac{2ab}{a+b}$.

Inserting *n* Harmonic Means Between Two Numbers

To insert n harmonic means between two numbers, we first take the corresponding arithmetic series and insert n arithmetic means, and next, we find the corresponding harmonic series.

This is illustrated by the example below:

EXAMPLE 7.17

Insert three harmonic means between $\frac{1}{12}$ and $\frac{1}{20}$.

SOLUTION

After inserting the harmonic means, let the harmonic progression be,

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \frac{1}{a+4d}$$

Given, $\frac{1}{a} = \frac{1}{12}$ and $\frac{1}{a+4d} = \frac{1}{20} \Rightarrow a = 12$ and $d = 2$

$\therefore$ The required harmonic means are $\frac{1}{14}, \frac{1}{16}$ and $\frac{1}{18}$.

Relation between AM, HM and GM of Two Numbers

Let x and y be two numbers.

$\therefore$ AM $= \frac{x+y}{2}$, GM $= \sqrt{xy}$ and HM $= \frac{2xy}{x+y}$

$\Rightarrow$ (AM) (HM) = $(\text{GM})^2$.

EXAMPLE 7.18

The ratio of geometric and arithmetic mean of two real numbers is 3 : 5. Then find the ratio of their harmonic mean and geometric mean.

(a) 3 : 5 **(b)** 9 : 25 **(c)** 9 : 5 **(d)** 5 : 9

SOLUTION

$G^2 = AH \Rightarrow GG = AH$

$\frac{G}{A} = \frac{H}{G}$

It is given, $G : A = 3 : 5$

$\therefore H : G = 3 : 5.$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Third term of the sequence whose nth term is $2n + 5$ is ______.
2. If a is the first term and d is the common difference of an AP, then the $(n + 1)$th term of the AP is ______.
3. If the sum of three consecutive terms of an AP is 9, then the middle term is ______.
4. General term of the sequence 5, 25, 125, 625, … is ______.
5. The arithmetic mean of 7 and 8 is ______.
6. The arrangement of numbers $\frac{1}{2}, \frac{-3}{4}, \frac{-5}{6}, \frac{-7}{8}, \ldots$ is an example of sequence. [True/False]
7. If $\frac{a}{2}$ is the first term and d is the common difference of an AP, then the sum of n terms of the AP is ______.
8. In a sequence, if S_n is the sum of n terms and S_{n-1} is the sum of $(n - 1)$ terms, then the nth term is ________.
9. If $T_n = 3n + 8$, then T_{n-1} = ______.
10. The sum of the first $(n + 1)$ natural numbers is ______.
11. For a series in geometric progression, the first term is a and the second term is $3a$. The common ratio of the series is ______.
12. In a series, starting from the second term, if each term is twice its previous term, then the series is in ______ progression.
13. All the multiples of 3 form a geometric progression. [True/False]
14. If a, b and c are in geometric progression then, a^2, b^2 and c^2 are in ______ progression.
15. If every term of a series in geometric progression is multiplied by a real number, then the resulting series also will be in geometric progression. [True/False]
16. Geometric mean of 5, 10 and 20 is ______ .
17. Sum of the infinite terms of the GP, −3, −6, −12, … is 3. [True/False]
18. The reciprocals of all the terms of a series in geometric progression form a ______ progression.
19. The nth term of the sequence $\frac{1}{100}, \frac{1}{10000}, \frac{1}{1000000}, \ldots$ is ______.
20. In a series, $T_n = x^{2n-2}$ $(x \neq 0)$, then write the infinite series.
21. The harmonic mean of 1, 2 and 3 is $\frac{3}{2}$. [True/False]
22. If a, b, c and d are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ are in ______ progression.
23. If the AM of two numbers is 9 and their HM is 4, then their GM is 6. [True/False]
24. If a, b and c are the arithmetic mean, geometric mean and harmonic mean of two distinct terms respectively, then b^2 is equal to ______.
25. If the sum of first n terms which are in GP is $a(r+1)$, then the number of terms is ______. (Where a is the first term and r is the common ratio)
26. Write the first three terms of the sequence whose nth term is $T_n = 8 - 5n$.
27. Write the first three terms of the sequence whose nth term is $T_n = 5^{n+1}$
28. If three arithmetic means are inserted between 4 and 5, then the common difference is ______.
29. If the 7th and the 9th terms of a GP are x and y respectively, then the common ratio of the GP is ______.
30. In a series, $T_n = 3 - n$, then S_5 = ______.

Short Answer Type Questions

31. If the 5th term and the 14th term of an AP are 35 and 8 respectively, then find the 20th term of the AP.
32. Which term of the series 21, 15, 9, … is −39?
33. If the seventh term of an AP is 25 and the common difference is 4, then find the 15th term of AP.

34. Find the general term of AP whose sum of n terms is given by $4n^2 + 3n$.

35. Find the sum of all three-digit numbers which leave a remainder 2, when divided by 6.

36. If the ratio of the sum of first three terms of a GP to the sum of first six terms is 448 : 455, then find the common ratio.

37. If in a GP, 5th term and the 12th term are 9 and $\frac{1}{243}$ respectively, find the 9th term of GP.

38. A person opens an account with ₹50 and starts depositing every day double the amount he has deposited on the previous day. Then find the amount he has deposited on the 10th day from the beginning.

39. Find the sum of 5 geometric means between $\frac{1}{3}$ and 243, by taking common ratio positive.

40. Using progressions express the recurring decimal $2\cdot\overline{123}$ in the form of $\frac{p}{q}$, where p and q are integers.

41. A ball is dropped from a height of 64 m and it rebounces $\frac{3}{4}$ of the distance every time it touches the ground. Find the total distance it travels before it comes to rest.

42. Find the sum to n terms of the series $5 + 55 + 555 + \ldots$.

43. In an HP, if the 3rd term and the 12th term are 12 and 3 respectively, then find the 15th term of the HP.

44. If lth, mth and nth terms of an HP are x, y and z respectively, then find the value of $yz(m - n) + xz(n - l) + xy(l - m)$.

45. The AM of two numbers is 40 more than GM and 64 more than HM. Find the numbers.

Essay Type Questions

46. Find the sum to n terms of the series $1 \cdot 2 \cdot 3 + 2 \cdot 4 \cdot 6 + 3 \cdot 6 \cdot 9 + \ldots$.

47. One side of an equilateral triangle is 36 cm. The mid-points of its sides are joined to form another triangle. Again another triangle is formed by joining the mid-points of the sides of this triangle and the process is continued indefinitely. Determine the sum of areas of all such triangles including the given triangle.

48. Three numbers form a GP. If the third term is decreased by 128 then, the three numbers, thus obtained, will form an AP. If the second term of this AP is decreased by 16, a GP will be formed again. Determine the numbers.

49. If A, G and H are AM, GM and HM of any two positive numbers, then prove that $A \geq G \geq H$.

50. The product of three numbers of a GP is $\frac{64}{27}$. If the sum of their products when taken in pairs is $\frac{148}{27}$, then find the numbers.

CONCEPT APPLICATION

Level 1

1. Find t_5 and t_6 of the arithmetic progression $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$.
   (a) $1, \frac{5}{4}$ (b) $\frac{5}{4}, 1$
   (c) $1, \frac{7}{4}$ (d) $\frac{7}{4}, 1$

2. If $t_n = 6n + 5$, then $t_{n+1} =$ ______.
   (a) $6n - 1$ (b) $6n + 11$
   (c) $6n + 6$ (d) $6n - 5$

3. Which term of the arithmetic progression 21, 42, 63, 84, … is 420?
   (a) 19 (b) 20
   (c) 21 (d) 22

4. Find the 15th term of the arithmetic progression 10, 4, −2, ...
   (a) −72 (b) −74
   (c) −76 (d) −78

5. If the kth term of the arithmetic progression 25, 50, 75, 100, … is 1000, then k is ______.

PRACTICE QUESTIONS

(a) 20 (b) 30
(c) 40 (d) 50

6. The sum of the first 20 terms of an arithmetic progression whose first term is 5 and common difference is 4, is ______.

(a) 820 (b) 830
(c) 850 (d) 860

7. Two arithmetic progressions have equal common differences. The first term of one of these is 3 and that of the other is 8, then the difference between their 100th terms is ______.

(a) 4 (b) 5
(c) 6 (d) 3

8. If a, b and c are in arithmetic progression, then $b + c$, $c + a$ and $a + b$ are in

(a) arithmetic progression.
(b) geometric progression.
(c) harmonic progression.
(d) None of these

9. The sum of the first 51 terms of the arithmetic progression whose 2nd term is 2 and 4th term is 8, is ______.

(a) 3774 (b) 3477
(c) 7548 (d) 7458

10. Three alternate terms of an arithmetic progression are $x + y$, $x - y$ and $2x + 3y$, then $x =$ ______.

(a) $-y$ (b) $-2y$
(c) $-4y$ (d) $-6y$

11. Find the 15th term of the series 243, 81, 27, ….

(a) $\frac{1}{3^{14}}$ (b) $\frac{1}{3^{8}}$
(c) $\left(\frac{1}{3}\right)^9$ (d) $\left(\frac{1}{3}\right)^{10}$

12. If t_8 and t_3 of a geometric progression are $\frac{4}{9}$ and $\frac{27}{8}$ respectively, then find t_{12} of the geometric progression.

(a) $\frac{64}{729}$ (b) $\frac{32}{243}$
(c) $\frac{729}{64}$ (d) $\frac{243}{32}$

13. If $t_n = 3^{n-1}$, then $S_6 - S_5 =$ ______.

(a) 243 (b) 81
(c) 77 (d) 27

14. Find the sum of the first 10 terms of geometric progression 18, 9, 4.5, ….

(a) $9\frac{(2^{10}-1)}{2^8}$ (b) $9\frac{(2^{10}-1)}{2^{10}}$
(c) $36\left(\frac{2^{10}-1}{2^8}\right)$ (d) $8\frac{(2^{10}-1)}{2^8}$

15. If the 3rd, 7th and 11th terms of a geometric progression are p, q and r respectively, then the relation among p, q and r is ______.

(a) $p^2 = qr$ (b) $r^2 = qp$
(c) $q^2 = p^2r^2$ (d) $q^2 = pr$

16. Evaluate $\Sigma(3 + 2^r)$, where $r = 1, 2, 3, \ldots, 10$.

(a) 2051 (b) 2049
(c) 2076 (d) 1052

17. Find the sum of the series $\frac{27}{8}+\frac{9}{4}+\frac{3}{2}+\ldots\infty$.

(a) $\frac{81}{8}$ (b) $\frac{27}{8}$
(c) $\frac{81}{16}$ (d) $\frac{9}{8}$

18. If $3x - 4$, $x + 4$ and $5x + 8$ are the three positive consecutive terms of a geometric progression, then find the terms.

(a) 2, 8, 32 (b) 2, 10, 50
(c) 2, 6, 18 (d) 12, 6, 3

19. Find the geometric mean of the first twenty five powers of twenty five.

(a) 5^{13} (b) 5^{19}
(c) 5^{24} (d) 5^{26}

20. Find the sum of 3 geometric means between $\frac{1}{3}$ and $\frac{1}{48}$ $(r > 0)$.

(a) $\frac{1}{4}$ (b) $\frac{5}{24}$
(c) $\frac{7}{24}$ (d) $\frac{1}{3}$

21. If the second and the seventh terms of a Harmonic Progression are $\frac{1}{5}$ and $\frac{1}{25}$, then find the series.

(a) $1, \frac{1}{5}, \frac{1}{9}, \ldots$ (b) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$

(c) $\frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \ldots$ (d) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

22. The 10th term of harmonic progression $\frac{1}{5}, \frac{4}{19}, \frac{2}{9}, \frac{4}{17}, \ldots$ is ______.

(a) $\frac{11}{4}$ (b) $\frac{13}{4}$

(c) $\frac{4}{13}$ (d) $\frac{4}{11}$

23. If the ratio of the arithmetic mean and the geometric mean of two positive numbers is 3 : 2, then find the ratio of the geometric mean and the harmonic mean of the numbers.

(a) 2 : 3 (b) 9 : 4

(c) 3 : 2 (d) 4 : 9

24. If A, G and H are AM, GM and HM of any two given positive numbers, then find the relation between A, G and H.

(a) $A^2 = GH$ (b) $G^2 = AH$

(c) $H^2 = AG$ (d) $G^3 = A^2H$

25. Find the least value of n for which the sum $1 + 2 + 2^2 + \ldots$ to n terms is greater than 3000.

(a) 8 (b) 10

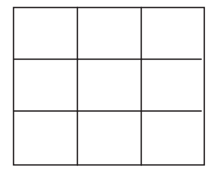
(c) 12 (d) 15

26. Find the HM of $\frac{1}{7}$ and $\frac{1}{12}$.

(a) $\frac{1}{19}$ (b) $\frac{2}{19}$

(c) $\frac{3}{19}$ (d) $\frac{4}{19}$

27. Number of rectangles in the following figure is ______.



(a) 9 (b) 10

(c) 24 (d) 36

28. In a series, if $t_n = \frac{n^2 - 1}{n + 1}$, then $S_6 - S_3 =$ ______.

(a) 3 (b) 12

(c) 22 (d) 25

29. Find the number of terms to be added in the series 27, 9, 3, … so that the sum is $\frac{1093}{27}$.

(a) 6 (b) 7

(c) 8 (d) 9

30. Find the value of p $(p > 0)$ if $\frac{15}{4} + p$, $\frac{5}{2} + 2p$ and $2 + p$ are the three consecutive terms of a geometric progression.

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$

(c) $\frac{5}{3}$ (d) $\frac{1}{2}$

Level 2

31. If $\frac{1}{b+c}, \frac{1}{c+a}$ and $\frac{1}{a+b}$ are in AP, then a^2, b^2 and c^2 are in

(a) geometric progression.

(b) arithmetic progression.

(c) harmonic progression.

(d) None of these

32. Among the following, which term belongs to the arithmetic progression –5, 2, 9, …?

(a) 342 (b) 343

(c) 344 (d) 345

33. Five distinct positive integers are in arithmetic progression with a positive common difference. If their sum is 10020, then find the smallest possible value of the last term.

(a) 2002 (b) 2004

(c) 2006 (d) 2008

34. In a right triangle, the lengths of the sides are in arithmetic progression. If the lengths of the sides of the triangle are integers, which of the following could be the length of the shortest side?

(a) 2125 (b) 1700

(c) 1275 (d) 1150

35. If $S_1 = 3, 7, 11, 15, \ldots$ upto 125 terms and $S_2 = 4, 7, 10, 13, 16, \ldots$ upto 125 terms, then how many terms are there in S_1 that are there in S_2?

(a) 29 (b) 30 (c) 31 (d) 32

36. The first term and the mth term of a geometric progression are a and n respectively and its nth term is m. Then its $(m + 1 - n)$th term is ______.

(a) $\frac{ma}{n}$ (b) $\frac{na}{m}$

(c) mna (d) $\frac{mn}{a}$

37. The sum of the terms of an infinite geometric progression is 3 and the sum of the squares of the terms is 81. Find the first term of the series.

(a) 5 (b) $\frac{27}{5}$

(c) $\frac{31}{6}$ (d) $\frac{19}{3}$

38. If $\log_{\sqrt{2}} x + \log_{\sqrt{\sqrt{2}}} x + \log_{\sqrt{\sqrt{\sqrt{2}}}} x + \ldots$ upto 7 terms = 1016, the find the value of x.

(a) 4 (b) 16 (c) 64 (d) 2

39. For which of the following values of x is $8^{1+\sin x+\sin^2 x+\sin^3 x+\cdots+\infty} = 64$?

(a) 60° (b) 135° (c) 45° (d) 30°

40. Find the sum of all the multiples of 6 between 200 and 1100.

(a) 96750 (b) 95760

(c) 97560 (d) 97650

41. If the kth term of a HP is λp and the λth term is kp and $k \neq \lambda$, then the pth term is ______.

(a) $k^2\lambda$ (b) k^2p

(c) p^2k (d) λk

42. If six harmonic means are inserted between 3 and $\frac{6}{23}$, then the fourth harmonic mean is

(a) $\frac{6}{11}$ (b) $\frac{6}{17}$ (c) $\frac{3}{7}$ (d) $\frac{3}{10}$

43. If a, b and c are positive numbers in arithmetic progression. and a^2, b^2 and c^2 are in geometric progression, then a^3, b^3 and c^3 are in

(A) arithmetic progression.

(B) geometric progression.

(C) harmonic progression.

(a) (A) and (B) only

(b) only (C)

(c) (A), (B) and (C)

(d) only (B)

44. The arithmetic mean A of two positive numbers is 8. The harmonic mean H and the geometric mean G of the numbers satisfy the relation $4H + G^2 = 90$. Then one of the two numbers is ______.

(a) 6 (b) 8

(c) 12 (d) 14

45. The infinite sum $\sum_{n=1}^{\infty}\left(\frac{5^n + 3^n}{5^n}\right)$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{5}$

(c) $\frac{2}{3}$ (d) None of these

46. (i) If $x = 3 + \frac{3}{y} + \frac{3}{y^2} + \frac{3}{y^3} + \cdots + \infty$, then, show that $y = \frac{x}{x-3}$. (Where $|y| < 1$). The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) $xy - 3y = x$ (B) $x = 3\left(\frac{1}{1-\frac{1}{y}}\right)$

(C) $y(x - 3) = x$ (D) $x = 3\left(\frac{y}{y-1}\right)$

(a) BDCA (b) BDAC

(c) CABD (d) ACBD

47. Find the harmonic mean of 5 and 3.

The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) $\text{HM} = \frac{2 \times 5 \times 3}{5 + 3}$

(B) We know that the harmonic mean of a, b is $\frac{2ab}{a+b}$.

(C) Here, $a = 5$ and $b = 3$

(D) $\text{HM} = \frac{30}{8} = \frac{15}{4}$

(a) BCDA (b) BCAD

(c) ABCD (d) BADC

Level 3

48. The numbers $h_1, h_2, h_3, h_4, \ldots, h_{10}$ are in harmonic progression and $a_1, a_2, \ldots, a_{10}$ are in arithmetic progression. If $a_1 = h_1 = 3$ and $a_7 = h_7 = 39$, then the value of $a_4 \times h_4$ is ______.

(a) $\frac{13}{49}$ (b) $\frac{182}{3}$

(c) $\frac{7}{13}$ (d) 117

49. Find the value of

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{16}\right)\left(1+\frac{1}{256}\right)\ldots\infty.$$

(a) 1 (b) 2

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$

50. The ratio of the sum of n terms of two arithmetic progressions is given by $(2n+3):(5n-7)$. Find the ratio of their nth terms.

(a) $(4n+5):(10n+2)$

(b) $(4n+1):(10n-12)$

(c) $(4n-1):(10n+8)$

(d) $(4n-5):(10n-2)$

51. There are n arithmetic means (were $n \in N$) between 11 and 53 such that each of them is an integer. How many distinct arithmetic progressions are possible from the above data?

(a) 7 (b) 8

(c) 14 (d) 16

52. If $x = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \cdots + \infty$, then find the value of $x + \frac{1}{x}$.

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $3\sqrt{2}$ (d) $4\sqrt{2}$

53. In a GP of 6 terms, the first and last terms are $\frac{x^3}{y^2}$ and $\frac{y^3}{x^2}$ respectively. Find the ratio of 3rd and 4th terms of that GP.

(a) $x^2 : 1$ (b) $y^2 : x$

(c) $y : x$ (d) $x : y$

54. If $x = 3 + \frac{3}{y} + \frac{3}{y^2} + \frac{3}{y^3} + \cdots + \infty$, then $y =$ ______ (where $|y| > 1$).

(a) $\frac{x}{3}$ (b) $\frac{x}{x-3}$

(c) $\frac{1-x}{3}$ (d) $1-\frac{3}{x}$

55. Find the sum of $\frac{0.3}{0.5} + \frac{0.33}{0.55} + \frac{0.333}{0.555} + \ldots$ to 15 terms.

(a) 10 (b) 9

(c) 3 (d) 5

56. In a GP, if the fourth term is the square of the second term, then the relation between the first term and common ratio is ______.

(a) $a = r$ (b) $a = 2r$

(c) $2a = r$ (d) $r^2 = a$

57. For which of the following values of x is $(0° < x < 90°)$ $16^{1+\cos x+\cos^2 x+\cos^3 x+\ldots\infty} = 256$?

(a) 30° (b) 45°

(c) 60° (d) 15°

58. If t_2 and t_3 of a GP are p and q, respectively, then t_5 = ______.

(a) $p\left(\frac{q}{p}\right)^3$ (b) $p\left(\frac{q}{p}\right)^2$

(c) $\frac{p^2}{q^3}$ (d) p^2q^2

59. If a, b, c, d are in GP, then $(b+c)^2 =$ ______.

(a) $(b+d)(a+d)$ (b) $(a+d)(c+d)$

(c) $(a+b)(c+d)$ (d) $(a+c)(b+d)$

60. a, b, c are in GP. If a is the first term and c is the common ratio, then $b =$ ______.

(a) 1 (b) $\frac{1}{a}$

(c) $\frac{1}{c}$ (d) None of these

61. In a GP of 7 terms, the last term is $\frac{64}{81}$ and the common ratio is $\frac{2}{3}$. Find the 3rd term.

(a) 4 (b) 9

(c) 8 (d) 12

PRACTICE QUESTIONS

62. An AP starts with a positive fraction and every alternate term is an integer. If the sum of the first 11 terms is 33, then find the fourth term.

(a) 2 (b) 3

(c) 5 (d) 6

63. If the sum of 16 terms of an AP is 1624 and the first term is 500 times the common difference, then find the common difference.

(a) 5 (b) $\frac{1}{2}$

(c) $\frac{1}{5}$ (d) 2

64. Find the sum of the series $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \cdots + (1 + 2 + 3 + \cdots + 20)$.

(a) 1470

(b) 1540

(c) 1610

(d) 1370

65. Evaluate $\Sigma 2^i$, where $i = 2, 3, 4, \ldots, 10$.

(a) 2044

(b) 2048

(c) 1024

(d) 1022

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 11
2. $a + nd$
3. 3
4. $5n$
5. 7.5
6. False
7. $\frac{n}{2}[a+(n-1)d]$
8. $S_n - S_{n-1}$
9. $3n + 5$
10. $\frac{(n+1)(n+2)}{2}$
11. 3
12. geometric
13. False
14. geometric
15. True
16. 10
17. False
18. geometric
19. $\frac{1}{100^n}$
20. $1 + x^2 + x^4 + x^6 + \ldots$
21. False
22. arithmetic
23. True
24. ac
25. 2
26. 3, –2, –7.
27. 25, 125, 625.
28. $\frac{1}{4}$
29. $\pm\sqrt{\frac{y}{x}}$
30. 0

Shot Answer Type Questions

31. −10
32. −39
33. 57
34. $8n - 1$
35. 82650
36. $r = \frac{1}{4}$
37. $\frac{1}{9}$
38. ₹25600
39. 121
40. $\frac{707}{333}$
41. 448 m
42. $\frac{50}{81}(10^n - 1) - \frac{5n}{9}$
43. $\frac{12}{5}$
44. 0
45. 180 and 20

Essay Type Questions

46. $\frac{3}{2}n^2(n+1)^2$
47. $432\sqrt{3}$ cm^2
48. $\frac{8}{9}, \frac{104}{9}$ and $\frac{1352}{9}$
50. $1, \frac{4}{3}, \frac{16}{9}$

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (d)	**7.** (b)	**8.** (a)	**9.** (a)	**10.** (d)
11. (c)	**12.** (a)	**13.** (a)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (d)	**20.** (c)
21. (a)	**22.** (d)	**23.** (c)	**24.** (b)	**25.** (c)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (b)

Level 2

31. (b)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (b)	**39.** (d)	**40.** (d)
41. (d)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (d)	**46.** (b)	**47.** (b)			

Level 3

48. (d)	**49.** (b)	**50.** (b)	**51.** (c)	**52.** (b)	**53.** (d)	**54.** (b)	**55.** (b)	**56.** (a)	**57.** (c)
58. (a)	**59.** (c)	**60.** (a)	**61.** (a)	**62.** (a)	**63.** (c)	**64.** (b)	**65.** (a)		

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. The nth term in AP = $t_n = a + (n - 1)\ d$.
2. Substitute $n = n + 1$ in t_n.
3. Use formula to find nth term of an AP.
4. Use the formula to find the nth term of an AP.
5. Use the formula of nth term of an AP.
6. Use the formula to find S_n of an AP.
7. The difference of nth term of two AP's having same common difference is the difference of their first terms.
8. If a, b and c are in AP, then $2b = a + c$.
9. Find a and d using the given data and use the formula to find S_n of AP.
10. The difference between t_3 and t_1 is same as t_5 and t_3.
11. Use the formula to find the nth term of a GP.
12. Find a and r using the given data.
13. $S_6 - S_5 = t_6$.
14. Use the formula of S_n of a GP.
15. $p = ar_1^2, q = ar_1^6$ and $r = ar_1^{10}$.
16. $\Sigma\ (3 + 2r) = 3\Sigma 1 + \sum_{r=1}^{10} 2^r$.
17. Use the formula to find S_∞ of a GP.
18. If a, b and c are in GP, then $b^2 = ac$.
19. Form the series and use the formula to find S_n.
20. The 3 geometric means are t_2, t_3 and t_4 terms.
21. Use the formula of nth term of a HP.
22. Use the formula of the nth term of a HP.
23. Use the relation between AM, GM and HM.
24. (i) Consider two numbers as a and b.
    (ii) Then find AM, GM and HM.
25. (i) $S_n = \dfrac{a(r^n - 1)}{(r - 1)}$
    (ii) $a = 1$ and $r = 2$
    (iii) Given that $S_n > 3000$
    (iv) Then find least possible value of n.
26. HM of a and b is $\dfrac{2ab}{a + b}$.
28. $S_6 - S_3 = T_4 + T_5 + T_6$.
29. Use the formula of S_n of a GP.
30. If a, b and c are in GP, then $b^2 = ac$.

Level 2

31. $t_2 - t_1 = t_3 - t_2$.
32. Each term is in the form of $7n + 2$, where $n = -1, 0, 1, 2, 3, \ldots$.
33. Let the five integers be $a - 2d$, $a - d$, a, $a + d$ and $a + 2d$.
34. (i) The sides must be in the ratio 3 : 4 : 5.
    (ii) Shortest side should be a multiple of 3.
35. (i) Common terms in S_1 and S_2 are 7, 19, 31, ….
    (ii) The first term of S_1 is 3. The 125th term is 499.
    The first term of S_2 is 4. The 125th term is 376.
    (iii) The last common term is 367 (and not 499).
    (iv) $7 = 12(1) - 5$, $19 = 12(2) - 5$, …, $367 = 12(31) - 5$.
36. (i) Use the formula to find nth term of a GP.
    (ii) $ar^{m-1} = n$, $ar^{n-1} = m$.
    (iii) $(m + 1 - n)$th term $= ar^{m-n}$.
37. (i) Given $S_\infty = \dfrac{a}{1 - r} = 3$.
    (ii) $a^2 + a^2r^2 + a^2r^4 + \cdots = 81$.
38. (i) Use $\log_{b^n} a = \dfrac{1}{n}\log_b a$.
    (ii) $S_n = \dfrac{a(r^n - 1)}{(r - 1)}$
    (iii) If $\log_a x = b$, then $a^b = x$.
39. (i) Use the formula to find S_∞ of a GP.
    (ii) Equate the powers on either sides by making equal bases.

40. Form the series and find the value of n and use the formula $S_n = \frac{n}{2}(2a+(n-1)d)$.

41. $t_n = \frac{1}{a+(n-1)d}$ in HP.

42. (i) $\frac{1}{a} = 3, \frac{1}{a+7d} = \frac{6}{23}$.

(ii) Find a and d by using above relation find t_5 using $\frac{1}{a+4d}$.

43. Use $b = \frac{a+c}{2}$ and $(b^2)^2 = a^2c^2$.

44. Use $G^2 = AH$.

45. (i) $\sum_{n=1}^{\infty}\left(1+\left(\frac{3}{5}\right)^n\right) = 1+\frac{3}{5}+\left(\frac{3}{5}\right)^2+\ldots\infty$

(ii) $S_\infty = \frac{a}{1-r}$, where a is the first term and r is the common ratio.

46. BDAC is the sequential order of steps.

47. BCAD is the sequential order of steps.

Level 3

48. (i) In AP, $t_n = a + (n-1)d$ and in HP, $t_n = \frac{1}{a+(n-1)d}$.

(ii) Find the values of a and d by using the data.

49. (i) Let $p = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\ldots$

(ii) Multiply with $\left(1-\frac{1}{2}\right)$ on both sides.

(iii) $(a+b)(a-b) = a^2 - b^2\ a,\ n - \infty\ \frac{1}{2^n} = 0$.

50. (i) Let a_1 and d_1 be the first term and common difference of the first AP and a_2, d_2 be the corresponding values for the second AP.

(ii) $S_n : S'_n$ has to be converted in the form of $t_n : t'_n$.

51. (i) n arithmetic means are in between a and b; $d = \frac{b-a}{n+1}$.

(ii) Evaluate for how many values of $(n+1)$, d is an integer.

52. $x = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \ldots\infty$

$$x = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}$$

$$x = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \Rightarrow \frac{1}{x} = \sqrt{2}-1$$

$$\therefore x + \frac{1}{x} = \sqrt{2}+1+\sqrt{2}-1 = 2\sqrt{2}.$$

53. $a = \frac{x^3}{y^2} = t_1; b = \frac{y^3}{x^2} = t_6$

There are 4 GM's between a and b.

$$\therefore r = \left(\frac{b}{a}\right)^{1/n+1}$$

$$r = \left[\frac{\left(\frac{y^3}{x^2}\right)}{\left(\frac{x^3}{y^2}\right)}\right]^{1/4+1} \Rightarrow r = \left[\left(\frac{y}{x}\right)^5\right]^{1/5}$$

$$\Rightarrow r = \frac{y}{x}$$

But the ratio of 3rd and 4th term is $\frac{1}{r}$.

$$\Rightarrow \frac{1}{r} = \frac{x}{y}.$$

Therefore, the required ratio is $x : y$.

54. $x = 3 + \frac{3}{y} + \frac{3}{y^2} + \frac{3}{y^3} + \cdots + \infty$

$$x = 3\left[\frac{1}{1-\frac{1}{y}}\right]$$

$$x = 3\left[\frac{y}{y-1}\right] \Rightarrow xy - x = 3y$$

$$xy - 3y = x$$

$$y(x-3) = x \Rightarrow y = \frac{x}{x-3}.$$

HINTS AND EXPLANATION

55. $\dfrac{0.3}{0.5}+\dfrac{0.33}{0.55}+\dfrac{0.333}{0.555}+\cdots 15 \text{ terms}$

$S=\dfrac{3}{5}+\dfrac{33}{55}+\dfrac{333}{555}+\cdots 15 \text{ terms}$

$S=\dfrac{3}{5}\left[1+\dfrac{11}{11}+\dfrac{111}{111}+\cdots 15 \text{ terms}\right]$

$S=\dfrac{3}{5}\times 15 \Rightarrow S=9.$

56. $t_4 = t_2{}^2$

$\Rightarrow ar^3 = (ar)^2 \Rightarrow ar^3 = a^2r^2 \Rightarrow r = a.$

57. $16^{1+\cos x+\cos^2 x+\cos^3 x+\ldots\infty} = 256$

$16^{\frac{1}{1-\cos x}} = 256 \Rightarrow 16^{\frac{1}{1-\cos x}} = 16^2$

$\Rightarrow \dfrac{1}{1-\cos x} = 2 \Rightarrow 1-\cos x = \dfrac{1}{2}$

$1-\dfrac{1}{2} = \cos x.$

$\dfrac{1}{2} = \cos x \Rightarrow x = 60°.$

58. $t_2 = ar = p$

$t_3 = ar^2 = q$

$\Rightarrow \dfrac{t_3}{t_2} = \dfrac{q}{p} = \dfrac{ar^2}{ar} \Rightarrow r = \dfrac{q}{p}$

$ar = p$

$\Rightarrow a\cdot\dfrac{q}{p} = p \Rightarrow a = \dfrac{p^2}{q}$

$t_5 = a\cdot r^4$

$=\dfrac{p^2}{q}\left(\dfrac{q}{p}\right)^4$

$=\dfrac{p^2}{q}\times\dfrac{q^4}{p^4} = \dfrac{q^3}{p^2} = p\left(\dfrac{q}{p}\right)^3.$

59. a, b, c, d are in GP

$a = a,\ b = ar,\ c = ar^2,\ d = ar^3$

$b + c = ar + ar^2 = ar(1 + r)$

$(b + c)^2 = a^2r^2\ (1 + r^2 + 2r)$

$= a^2r^2 + a^2r^4 + 2a^2r^3$

$= a\cdot ar^2 + ar\cdot ar^3 + a^{2r3} + a^{2r3}$

$= a\cdot c + bd + bc + ad = a(c + d) + b\ (c + d)$

$= (a + b)(c + d).$

60. a, b, c are in GP

$r = \dfrac{b}{a} = \dfrac{c}{b}$

but given that $r = c$

$\Rightarrow \dfrac{c}{b} = c \Rightarrow b = 1.$

61. Let a be the first term and r the common ratio of the GP.

$T_y = ar^6 = \dfrac{64}{81} \Rightarrow a = 9$

Therefore the 3rd term, $T_3 = ar^2$

$T_3 = 9\left(\dfrac{2}{3}\right)^2 = 4.$

62. Given, $\dfrac{11}{2}\,[2a + 10d] = 33$

$\Rightarrow a + 5d = 3$

$\Rightarrow a + 3d = 2$

Since, alternate terms are integers and the given sum is positive.

63. $S_n = \dfrac{n}{2}[2a + (n-1)d]$

$\dfrac{16}{2}[2\times 500d + (16-1)d] = 1624$

$8[1000d + 15d] = 1624$

$1015d = \dfrac{1624}{8}$

$1015d = 203$

$d = \dfrac{203}{1015} \Rightarrow d = \dfrac{1}{5}.$

64. 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + … 20 terms

= 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91 + 105 + 120 + 136 + 153 + 171 + 190 + 210

= 1540.

Alternative method:

$1 + (1 + 2) + (1 + 2 + 3) + \ldots$

nth term of the series is

$1 + 2 + 3 + 4 + \ldots n = \frac{n(n+1)}{2}$

Sum of n terms of the series

$= \sum t_n$

$= \sum \frac{n(n+1)}{2} \Rightarrow = \frac{1}{2}\sum(n^2 + n)$

$= \frac{1}{2}\left(\sum n^2 + \sum n\right)$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{n(n+1)}{4}\left(\frac{2n+1}{3} + 1\right)$

$\Rightarrow \frac{n(n+1)(2n+4)}{12} = \frac{n(n+1)(n+2)}{6}.$

If $n = 20$

$\sum t_{20} = \frac{20(21)(22)}{6}$

$\Rightarrow \frac{10 \times 21 \times 22}{3} = 1540.$

65. $\sum 2^i, i = 2, 3, \ldots 10$

$= (2^2 + 2^3 + \cdots + 2^{10})$

$= 2^2 \times \frac{(2^9 - 1)}{2 - 1}$

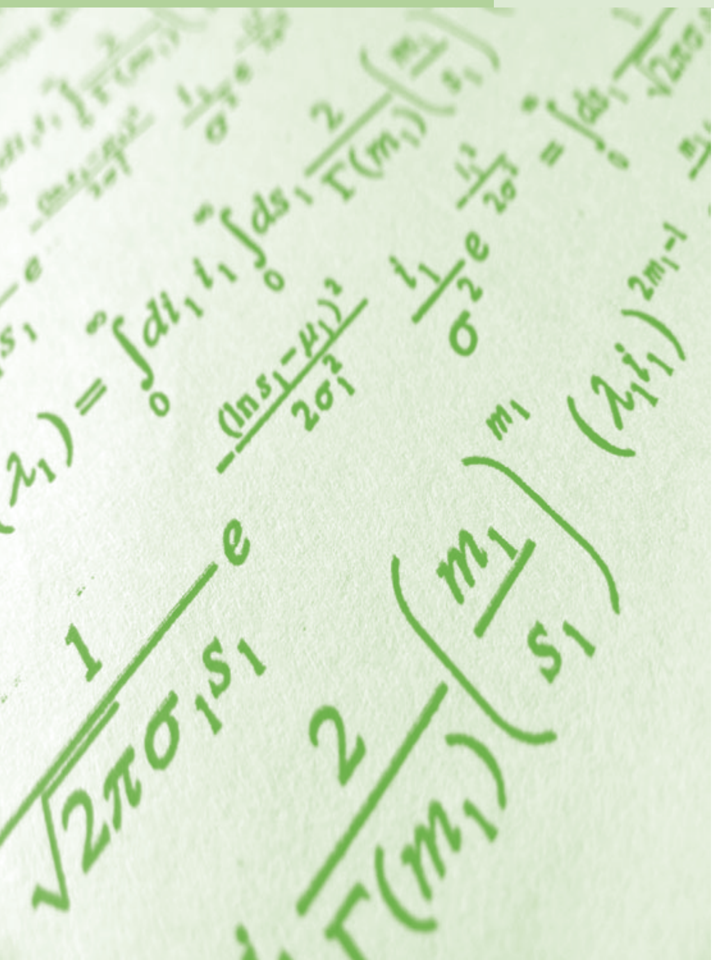
$= \frac{2^{11} - 2^2}{1}$

$= 2044.$

Chapter

8

Trigonometry



REMEMBER

Before beginning this chapter, you should be able to:

- Basic trigonometric identities
- Know systems of measurement of angle

KEY IDEAS

After completing this chapter, you would be able to:

- Convert angles using three systems of measurements of angles
- Study trigonometric identities and solve numerical problems using them
- Understand trigonometric ratios of specific and compound angles
- Learn signs of trigonometric ratios in different quadrants and how to use trigonometric table
- Calculate heights of poles, longer distances, etc., using trigonometric identities

INTRODUCTION

The word trigonometry is originated from the Greek word 'tri' means three, 'gonia' means angle and 'metron' means measure. Hence, the word trigonometry means three angle measure, i.e., it is the study of geometrical figures which have three angles, i.e., triangles.

The great Greek mathematician Hipparchus of 140 BCE gave relation between the angles and sides of a triangle. Further trigonometry is developed by Indian (Hindu) mathematicians. This was migrated to Europe via Arabs.

Trigonometry plays an important role in the study of Astronomy, Surveying, Navigation and Engineering. Nowadays it is used to predict stock market trends.

ANGLE

A measure formed between two rays having a common initial point is called an angle. The two rays are called the arms or sides of the angle and the common initial point is called the vertex of the angle.

In the Fig. 8.1, OA is said to be the initial side and the other ray OB is said to be the terminal side of the angle.

The angle is taken positive when measured in anti-clockwise direction and is taken negative when measured in clockwise direction (see Fig. 8.2).

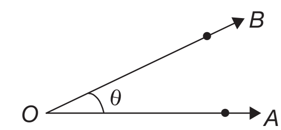


Figure 8.1

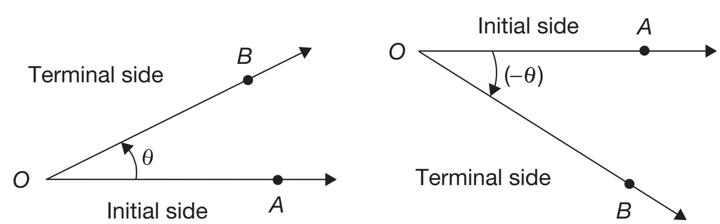


Figure 8.2

Systems of Measurement of Angle

We have the following systems of the measurement of angle.

Sexagesimal System

In this system, the angle is measured in degrees(°).

Degree When the initial ray is rotated through $\left(\frac{1}{360}\right)$ of one revolution, we say that an angle of one degree (1°) is formed at the initial point. A degree is divided into 60 equal parts and each part is called one minute (1′). Further, a minute is divided into 60 equal parts called seconds (″).

So, 1 right angle = 90°

$1° = 60'$ (minutes)

$1' = 60''$ (seconds).

Note This system is also called as the British system.

Centesimal System

In this system, the angle is measured in grades.

Grade When the initial ray is rotated through $\left(\frac{1}{400}\right)$ of one revolution, an angle of one grade is said to be formed at the initial point. It is written as 1^g.

Further, one grade is divided into 100 equal parts called minutes and one minute is further divided into 100 equal parts called seconds.

So, 1 right angle $= 100^g$

$1^g = 100'$ (minutes)

$1' = 100''$ (seconds).

Note This system is also called as the French system.

Circular System

In this system, the angle is measured in radians.

Radian The angle subtended by an arc of length equal to the radius of a circle at its centre is said to have a measure of one radian. It is written as 1^c (see Fig. 8.3).

Note This measure is also known as radian measure.

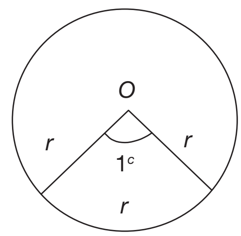


Figure 8.3

Relation Between the Units of the Three Systems

When a rotating ray completes one revolution, the measure of angle formed about the vertex is $360°$ or 400^g or $2\pi^c$.

So,

$$360° = 400^g = 2\pi^c$$

(or)

$$90° = 100^g = \frac{\pi^c}{2}.$$

For convenience, the above relation can be written as, $\frac{D}{90} = \frac{G}{100} = \frac{R}{\frac{\pi}{2}}$.

Where D denotes degrees, G grades and R radians.

Remember

1. $1° = \frac{\pi}{180}$ radians $= 0.0175$ radians (approximately)
2. $1^c = \frac{180}{\pi}$ degrees $= 57°17'\,44''$ (approximately)

Note

1. The measure of an angle is a real number.
2. If no unit of measurement is indicated for any angle, it is considered as radian measure.

EXAMPLE 8.1

Convert 45° into circular measure.

SOLUTION

Given, $D = 45°$

We have, $\frac{D}{90} = \frac{R}{\frac{\pi}{2}}$

So, $\frac{45}{90} = \frac{R}{\frac{\pi}{2}}$

$$\text{or, } \frac{1}{2} \times \frac{\pi}{2} = R$$

$$\text{or, } R = \frac{\pi}{4}.$$

Hence, circular measure of 45° is $\frac{\pi^c}{4}$.

EXAMPLE 8.2

Convert 150^g into sexagesimal measure.

SOLUTION

Given, $G = 150^g$

We have, $\frac{D}{90} = \frac{G}{100}$

So, $\frac{D}{90} = \frac{150}{100}$

$$\text{or, } D = \frac{3}{2} \times 90 = 135.$$

Hence, sexagesimal measure of 150^g is 135°.

EXAMPLE 8.3

What is the sexagesimal measure of angle measuring $\frac{\pi^c}{3}$?

SOLUTION

Given, $R = \frac{\pi^c}{3}$

We have, $\frac{D}{90} = \frac{R}{\frac{\pi}{2}}$

So, $\dfrac{D}{90} = \dfrac{\frac{\pi}{3}}{\frac{\pi}{2}}$

$$\Rightarrow D = \frac{2}{3} \times 90 = 60°.$$

Hence, the sexagesimal measure of $\dfrac{\pi^c}{3}$ is 60°.

TRIGONOMETRIC RATIOS

Let AOB be a right triangle with $\angle AOB$ as 90°. Let $\angle OAB$ be θ. Notice that $0° < \theta < 90°$, i.e., θ is an acute angle (see Fig. 8.4).

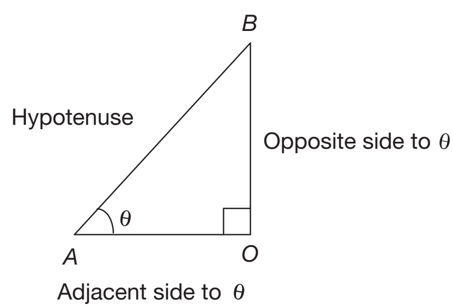


Figure 8.4

We can define six possible ratios among the three sides of the triangle AOB, known as trigonometric ratios. They are defined as follows.

1. Sine of the angle θ (or) simply $\sin\theta$:

$$\sin\theta = \frac{\text{Side opposite to angle } \theta}{\text{Hypotenuse}} = \frac{OB}{AB}.$$

2. Cosine of the angle θ or simply $\cos\theta$:

$$\cos\theta = \frac{\text{Side adjacent to angle } \theta}{\text{Hypotenuse}} = \frac{OA}{AB}.$$

3. Tangent of the angle θ or simply $\tan\theta$:

$$\tan\theta = \frac{\text{Side opposite to } \theta}{\text{Side adjacent to } \theta} = \frac{OB}{OB}.$$

4. Cotangent of the angle θ or simply $\cot\theta$:

$$\cot\theta = \frac{\text{Side adjacent to } \theta}{\text{Side opposite to } \theta} = \frac{OA}{OB}.$$

5. Cosecant of the angle θ or simply $\operatorname{cosec}\theta$:

$$\operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Side opposite to } \theta} = \frac{AB}{OB}.$$

6. Secant of the angle θ or simply sec θ:

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Side adjacent to } \theta} = \frac{AB}{OA}.$$

Observe that,

1. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$.

2. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$.

EXAMPLE 8.4

If $\sin\theta = \dfrac{3}{5}$, then find the values of $\tan\theta$ and $\sec\theta$.

SOLUTION

Given, $\sin\theta = \dfrac{3}{5}$.

Let AOB be the right triangle such that, $\angle OAB = \theta$.

Assume that $OB = 3$ and $AB = 5$ (see Fig. 8.5).

Then, $OA = \sqrt{AB^2 - OB^2} = \sqrt{25 - 9} = 4$.

So, $\tan\theta = \dfrac{\text{Side opposite to } \theta}{\text{Side adjacent to } \theta} = \dfrac{OB}{OA} = \dfrac{3}{4}$ and

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Side adjacent to } \theta} = \frac{AB}{OA} = \frac{5}{4}.$$

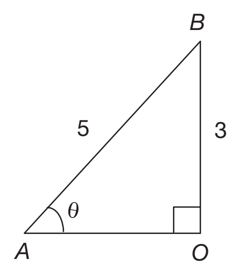


Figure 8.5

Some Pythagorean Triplets

1. 3, 4, 5
2. 5, 12, 13
3. 8, 15, 17
4. 7, 24, 25
5. 9, 40, 41

Trigonometric Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\sec^2\theta - \tan^2\theta = 1$
3. $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Table of Values of Trigonometric Ratios for Specific Angles

Trigonometric Ratios	Angles 0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

From the above table, we observe that

1. $\sin\theta = \cos\theta$, $\tan\theta = \cot\theta$ and $\sec\theta = \operatorname{cosec}\theta$, if $\theta = 45°$.
2. $\sin\theta$ and $\tan\theta$ are increasing functions, when $0° \leq \theta \leq 90°$.
3. $\cos\theta$ is a decreasing function, when $0° \leq \theta \leq 90°$.

EXAMPLE 8.5

Find the value of $\tan 45° + 2\cos 60° - \sec 60°$.

SOLUTION

$$\tan 45° + 2\cos 60° - \sec 60° = 1 + 2\left(\frac{1}{2}\right) - 2 = 1 + 1 - 2 = 0$$

$\therefore$ $\tan 45° + 2\cos 60° - \sec 60° = 0$.

EXAMPLE 8.6

Using the trigonometric table, evaluate

(a) $\sin^2 30° + \cos^2 30°$

(b) $\sec^2 60° - \tan^2 60°$.

SOLUTION

(a) $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$

Hence, $\sin^2 30° + \cos^2 30° = 1$.

(b) $\sec^2 60° - \tan^2 60° = (2)^2 - \left(\sqrt{3}\right)^2 = 4 - 3 = 1$

Hence, $\sec^2 60° - \tan^2 60° = 1$.

EXAMPLE 8.7

Find the values of $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$ and $\tan 30°$. What do you observe?

SOLUTION

$$\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} = \frac{\sqrt{3} - \dfrac{1}{\sqrt{3}}}{1 + \sqrt{3}.\dfrac{1}{\sqrt{3}}} = \frac{\dfrac{(3-1)}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}} \text{ and } \tan 30° = \frac{1}{\sqrt{3}}.$$

Hence, $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} = \tan 30°$.

Trigonometric Ratios of Compound Angles

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\sin(A - B) = \sin A \cos B - \cos A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$ and $\cos(A - B) = \cos A \cos B + \sin A \sin B$
3. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Also, by taking $A = B$ in the above relations, we get,

1. $\sin 2A = 2\sin A\cos A$
2. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
3. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

EXAMPLE 8.8

Find the value of $\sin 75°$.

SOLUTION

We have, $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30° = \dfrac{1}{\sqrt{2}} \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

$\therefore \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

EXAMPLE 8.9

Find the value of tan 15°.

SOLUTION

We have, $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$$= \frac{(\sqrt{3} - 1)}{(\sqrt{3} + 1)} \times \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)}$$

$$= \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{3 + 1 - 2\sqrt{3}}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

$\therefore \tan 15° = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$ or $2 - \sqrt{3}$.

EXAMPLE 8.10

Eliminate θ from the equations $x = p \sin \theta$ and $y = q \cos \theta$.

SOLUTION

We know that trigonometric ratios are meaningful when they are associated with some θ, i.e., we cannot imagine any trigonometric ratio without θ. Eliminate θ means, eliminating the trigonometric ratio itself by suitable identities.

Given, $x = p \sin \theta$ and $y = q \cos \theta$

$$\Rightarrow \frac{x}{p} = \sin \theta \text{ and } \frac{y}{q} = \cos \theta$$

We know that, $\sin^2\theta + \cos^2\theta = 1$.

So,
$$\left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2 = 1$$

Hence, the required equation is $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} = 1$.

EXAMPLE 8.11

Find the relation obtained by eliminating θ from the equations $x = r \cos \theta + s \sin \theta$ and $y = r \sin \theta - s \cos \theta$.

SOLUTION

Given,
$$x = r \cos \theta + s \sin \theta$$
$$\Rightarrow x^2 = (r \cos \theta + s \sin \theta)^2$$
$$= r^2 \cos^2\theta + 2rs \cos \theta \cdot \sin \theta + s^2 \sin^2\theta.$$

Also, $y = r\sin\theta - s\cos\theta$.

$$\text{or } y^2 = r^2\sin^2\theta + s^2\cos^2\theta - 2rs\sin\theta\cdot\cos\theta$$

$$\text{or, } x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) + s^2(\sin^2\theta + \cos^2\theta)$$

$$\text{or, } \quad = r^2(1) + s^2(1) \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= r^2 + s^2.$$

Hence, the required relation is $x^2 + y^2 = r^2 + s^2$.

EXAMPLE 8.12

Eliminate θ from the equations

$x = \operatorname{cosec}\theta + \cot\theta$

$y = \operatorname{cosec}\theta - \cot\theta$.

SOLUTION

Given, $x = \operatorname{cosec}\theta + \cot\theta$ and $y = \operatorname{cosec}\theta - \cot\theta$.

Multiplying these equations, we get

$$xy = (\operatorname{cosec}\theta + \cot\theta)(\operatorname{cosec}\theta - \cot\theta)$$

$$= \operatorname{cosec}^2\theta - \cot^2\theta = 1.$$

Hence, the required relation is $xy = 1$.

EXAMPLE 8.13

Eliminate θ from the equations $m = \tan\theta + \cot\theta$ and $n = \tan\theta - \cot\theta$.

SOLUTION

Given,

$m = \tan\theta + \cot\theta$ (1)

$n = \tan\theta - \cot\theta$ (2)

Adding Eqs. (1) and (2), we get

$$m + n = 2\tan\theta$$

$$\Rightarrow \tan\theta = \frac{m+n}{2}$$

Subtracting Eq. (2) from Eq. (1), we get

$$m - n = 2\cot\theta$$

$$\Rightarrow \cos\theta = \frac{m-n}{2}$$

$$\therefore \tan\theta\cdot\cot\theta = \left(\frac{m+n}{2}\right)\cdot\left(\frac{m-n}{2}\right)$$

$$\Rightarrow \tan\theta\cdot\frac{1}{\tan\theta} = \frac{m^2-n^2}{4}$$

$$\Rightarrow 1 = \frac{m^2-n^2}{4} \quad \text{or} \quad m^2 - n^2 = 4.$$

Hence, by eliminating 'θ', we obtain the relation $m^2 - n^2 = 4$.

EXAMPLE 8.14

If $\cos(A+B)=\frac{1}{2}$ and $B=\sqrt{2}$, then find A and B.

SOLUTION

Given,

$$\cos(A+B)=\frac{1}{2}$$

$\cos(A+B) = \cos 60°$

$A + B = 60°$ (1)

$\sec B=\sqrt{2}=\sec 45°$

$B = 45°$ (2)

From Eqs. (1) and (2), we have

$\therefore$ $A = 15°$ and $B = 45°$.

EXAMPLE 8.15

Find the length of the chord which subtends an angle of 120° at the centre 'O' and which is at a distance of 5 cm from the centre.

SOLUTION

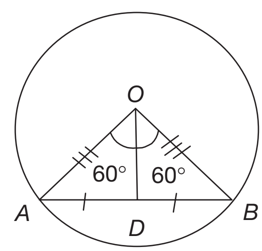


Figure 8.6

Let the chord be AB and OD be the distance of chord from the centre of circle.

Given $\angle AOB = 120°$ and $OD = 5$ cm, clearly, $\Delta OAD \cong \Delta OBD$, by SSS axiom

$$\angle AOD=\angle BOD=\frac{1}{2}(\angle AOB)=60° \text{ in } \Delta AOD$$

$$\tan 60°=\frac{AD}{OD}$$

$$\Rightarrow \sqrt{3}=\frac{AD}{5}\Rightarrow AD=5\sqrt{3}.$$

$\therefore$ The length of the chord, $AB=2AD=10\sqrt{3}$ cm.

EXAMPLE 8.16

Evaluate: $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$.

SOLUTION

Given, $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

Rationalize the denominator, i.e.,

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}}$$

$$=\sqrt{\frac{(1-\sin\theta)^2}{(1)^2-\sin^2\theta}}=\sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}=\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}\qquad(\because\ \sin^2\theta+\cos^2\theta=1)$$

$$=\sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2}=\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}=\sec\theta-\tan\theta\qquad\left(\because\ \sec\theta=\frac{1}{\cos\theta},\ \tan\theta=\frac{\sin\theta}{\cos\theta}\right)$$

$$\therefore\ \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta.$$

Standard Position of the Angle

The angle is said to be in its standard position if its initial side coincides with the positive X-axis (see Fig. 8.7).

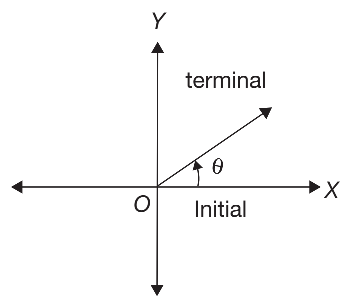


Figure 8.7

Notes

1. Rotation of terminal side in anti-clockwise direction we consider the angle formed is positive and rotation in clockwise direction the angle formed is negative.
2. Depending upon the position of terminal side we decide the angle in different quadrants.

Co-terminal Angles

The angles that differ by either 360° or the integral multiples of 360° are called co-terminal angles (see Fig. 8.8).

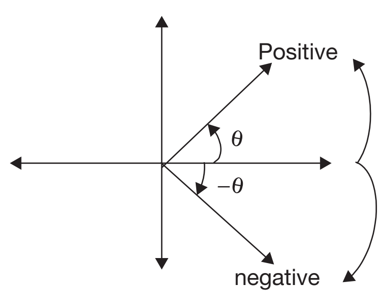


Figure 8.8

Example: 60°, 360° + 60° = 420°, 2 · 360° + 60° = 780° are co-terminal angles.

Notes

1. If θ is an angle then its co-terminal angle is in the form of $(n \cdot 360° + \theta)$.
2. The terminal side of co-terminal angles in their standard position coincides (see Fig. 8.9).

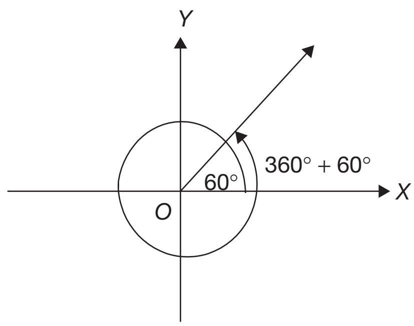


Figure 8.9

Signs of Trigonometric Ratios

1. If θ lies in the first quadrant, i.e., $0 < \theta < \dfrac{\pi}{2}$, then all the trigonometric ratios are taken positive.
2. If θ lies in the second quadrant, i.e., $\dfrac{\pi}{2} < \theta < \pi$, then only $\sin\theta$ and $\text{cosec}\,\theta$ are taken positive and all the other trigonometric ratios are taken negative.
3. If θ lies in the third quadrant, i.e., $\pi < \theta < \dfrac{3\pi}{2}$, then only $\tan\theta$ and $\cot\theta$ are taken positive and all the other trigonometric ratios are taken negative.
4. If θ lies in the fourth quadrant, i.e., $\dfrac{3\pi}{2} < \theta < 2\pi$, then only $\cos\theta$ and $\sec\theta$ are taken positive and all the other trigonometric ratios are taken negative.

Trigonometric Ratios of (90° – θ)

$\sin(90° - \theta) = \cos\theta$ $\qquad$ $\cos(90° - \theta) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$ $\qquad$ $\cot(90° - \theta) = \tan\theta$
$\text{cosec}(90° - \theta) = \sec\theta$ $\qquad$ $\sec(90° - \theta) = \text{cosec}\,\theta$

Trigonometric Ratios of (90° + θ)

$\sin(90° + \theta) = \cos\theta$ $\qquad$ $\cos(90° + \theta) = -\sin\theta$
$\tan(90° + \theta) = -\cot\theta$ $\qquad$ $\cot(90° + \theta) = -\tan\theta$
$\text{cosec}(90° + \theta) = \sec\theta$ $\qquad$ $\sec(90° + \theta) = -\text{cosec}\,\theta$

Trigonometric Ratios of (180° – θ)

$\sin(180° - \theta) = \sin\theta$ $\qquad$ $\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$ $\qquad$ $\cot(180° - \theta) = -\cot\theta$
$\text{cosec}(180° - \theta) = \text{cosec}\,\theta$ $\qquad$ $\sec(180° - \theta) = -\sec\theta$

Trigonometric Ratios of (180° + θ)

$\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$ $\cot(180° + \theta) = \cot\theta$
$\text{cosec}(180° + \theta) = -\text{cosec}\,\theta$ $\sec(180° + \theta) = -\sec\theta$

Similarly, the trigonometric ratios of $270° \pm \theta$ and $360° \pm \theta$ can be written.

Note The trigonometric ratios of $(-\theta)$ are the same as the trigonometric ratios of $(360° - \theta)$. So, $\sin(-\theta) = \sin(360° - \theta) = -\sin\theta$ and so on.

EXAMPLE 8.17

What is the value of tan 315°?

SOLUTION

$\tan 315° = \tan(360° - 45°) = -\tan 45° = -1$

$\therefore \tan 315° = -1$.

EXAMPLE 8.18

Find the value of $\sin^2 135° + \sec^2 135°$.

SOLUTION

$$\sin 135° = \sin(180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}.$$

$$\sec 135° = \sec(180° - 45°) = -\sec 45° = -\sqrt{2}.$$

$$\therefore \sin^2 135° + \sec^2 135° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\sqrt{2}\right)^2 = \frac{1}{2} + 2 = \frac{5}{2}.$$

EXAMPLE 8.19

If $\cos A = \frac{5}{13}$ and A is not in first quadrant, then find the value of $\frac{\sin A - \cos A}{\tan A + 1}$.

SOLUTION

Given that $\cos A = \frac{5}{13}$ and A is not in first quadrant

$\Rightarrow A$ is in fourth quadrant.

$\sin A = \frac{-12}{13}$ and $\tan A = \frac{-12}{5}$

Now, $$\frac{\sin A - \cos A}{\tan A + 1} = \frac{\frac{-12}{13} - \frac{5}{13}}{\frac{-12}{5} + 1} = \frac{-17}{13} \times \frac{-5}{7} = \frac{85}{91}.$$

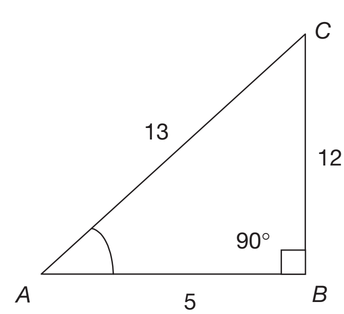


Figure 8.10

EXAMPLE 8.20

If $ABCD$ is a cyclic quadrilateral, then find the value of $\cos A \cos B - \cos C \cos D$.

SOLUTION

Given, $ABCD$ is a cyclic quadrilateral

$A + C = 180°$ and $B + D = 180°$

Now, $\cos A \cos B - \cos C \cos D$

$= \cos A \cos B - \cos(180° - A) \cos(180° - B) = \cos A \cos B - (-\cos A)(-\cos B)$

$= \cos A \cos B - \cos A \cos B = 0.$

EXAMPLE 8.21

If $\cot 15° = m$, then find $\dfrac{\cot 195° + \cot 345°}{\tan 15° - \cot 105°}$.

SOLUTION

Given, $\cot 15° = m \Rightarrow \tan 15° = \dfrac{1}{m}$ and $\tan 75° = m$ $\qquad (\because \tan(90° - \theta) = \cot\theta)$

$$\frac{\cot 195° + \cot 345°}{\tan 15° - \cot 105°} = \frac{\cot(180° + 15°) + \cot(360° - 15°)}{\tan 15° - \cot(90° + 15°)} = \frac{\cot 15° - \cot 15°}{\tan 15° - (-\tan 15°)} = 0.$$

EXAMPLE 8.22

If $\sin\theta$ and $\cos\theta$ are the roots of the equation $mx^2 + nx + 1 = 0$, then find the relation between m and n.

SOLUTION

The given equation is $mx^2 + nx + 1 = 0$

Here, $a = m$, $b = n$ and $c = 1$

$$\sin\theta + \cos\theta = \frac{-b}{a} = \frac{-n}{m}$$

$$\sin\theta + \cos\theta = \frac{c}{a} = \frac{1}{m}$$

Consider,

$$\sin\theta + \cos\theta = \frac{-n}{m}$$

or $\quad (\sin\theta + \cos\theta)^2 = \left(\dfrac{-n}{m}\right)^2$

or, $\quad \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \dfrac{n^2}{m^2}$

$$\Rightarrow 1 + 2\sin\theta\cos\theta = \frac{n^2}{m^2} \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$1 + 2\left(\frac{1}{m}\right) = \frac{n^2}{m^2} \qquad \left(\because \sin\theta\cos\theta = \frac{1}{m}\right)$$

$$1 + \frac{2}{m} = \frac{n^2}{m^2}$$

$$\Rightarrow n^2 - m^2 = 2m.$$

EXAMPLE 8.23

If $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, then find $\sin(\alpha + \beta)$.

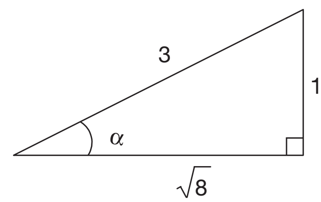


Figure 8.11

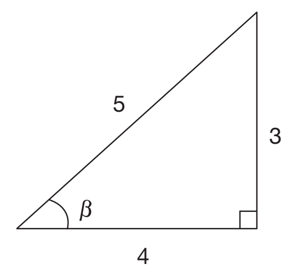


Figure 8.12

SOLUTION

Given, $\sin\alpha = \frac{1}{3}$

$$\cos\alpha = \frac{\sqrt{8}}{3}$$

$$\cos\beta = \frac{4}{5}$$

$$\sin\beta = \frac{3}{5}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{1}{3}\cdot\frac{4}{5} + \frac{\sqrt{8}}{3}\cdot\frac{3}{5} = \frac{4 + 3\sqrt{8}}{15}.$$

$$\sin(\alpha + \beta) = \frac{4 + 3\sqrt{8}}{15}.$$

EXAMPLE 8.24

Express the following as a single trigonometric ratio:

(a) $\sqrt{3}\cos\theta - \sin\theta$

(b) $\sin\theta - \cos\theta.$

SOLUTION

(a) Given, $\sqrt{3}\cos\theta - \sin\theta = 2\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$

$$= 2(\cos\theta \cdot \cos 30° - \sin\theta \cdot \sin 30°) = 2(\cos(\theta + 30°))$$

$$\Rightarrow \sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + 30°).$$

(b) $\sin\theta - \cos\theta = \sqrt{2}\left(\dfrac{\sin\theta - \cos\theta}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right) = \sqrt{2}\left[\sin\theta\cos\left(\frac{\pi}{4}\right) - \cos\theta\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) \qquad [\because \sin(A-B) = \sin A\cos B - \cos A\sin B]$$

EXAMPLE 8.25

If $A + B = 90°$, then prove that

(a) $\sin^2 A + \sin^2 B = 1$

(b) $\tan^2 A - \cot^2 B = 0$.

SOLUTION

Given, $A + B = 90°$

$$\Rightarrow B = 90° - A.$$

(a) $\sin^2 A + \sin^2 B = \sin^2 A + \sin^2 (90° - A) = \sin^2 A + \cos^2 A \qquad (\therefore \sin(90° - \theta) = \cos\theta)$

$$= 1 \ \sin^2 A + \sin^2 B = 1.$$

(b) $\tan^2 A - \cot^2 B = \tan^2 A - \cot^2 (90° - A) = \tan^2 A - \tan^2 A$ $(\because \ \tan\theta = \cot(90° - \theta) = 0$
$\tan^2 A - \cot^2 B = 0.$

EXAMPLE 8.26

Simplify the following:

(a) $\begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix}$

(b) $\log(\cot 1°) + \log\cot 2° + \log(\cot 3°) + \cdots + \log(\cot 89°)$

(c) $\sin 1° \cdot \sin 2° \cdots \sin 181°$.

SOLUTION

(a) $\begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix} = \cos A(-\cos A) - \sin A\sin A = -\cos^2 A - \sin^2 A$

$$= -(\sin^2 A + \cos^2 A) = -1$$

$\therefore \quad \begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix} = -1.$

(b) $\log(\cot 1°) + \log\cot 2° + \cdots + \log\cot 89°$

$= \log(\cot 1° \cdot \cot 2° \cdots \cot 89°). \qquad [\because \log a + \log b + \cdots + \log n = \log(abc \ldots n)]$

We know that,

$\cot 1° = \tan 89°$, $\cot 2° = \tan 88°$ and so on

$\cot 1° \cdot \cot 89° = \cot 1° \cdot \tan 1° = 1$ and so on

$\cot 1° \cdot \cot 2° \ldots \cot 89° = 1$

$\Rightarrow \log(\cot 1° \cot 2° \cdots \cot 89°) = \log(1) = 0.$

(c) $\sin 1° \cdot \sin 2° \cdot \sin 3° \cdots \sin 181°.$

We know that, $\sin 180° = 0$

$\sin 1° \cdot \sin 2° \cdots \sin 180° \cdot \sin 181° = 0.$

Trigonometric Tables

The values of trigonometric ratios of different angles can be found by using natural sines, natural cosines, natural tangents, etc. These tabular values are approximate and tally upto three decimal places.

The following example gives an idea how to find any value of trigonometric ratios.

EXAMPLE 8.27

Find the value of $\sin 65°28'$.

SOLUTION

Step 1: We have to refer the table of natural sines in order to find the value of $\sin 65°28'$.

Step 2: We have to slide our view from left to right in the row containing 65° until we reach the intersection with 28′ and directly minutes table, so as to locate the nearest approximation in the minutes, i.e., in this problem 24′ and note down the value.

Step 3: We locate the difference of minutes (i.e., here 4′) in the mean difference and add it to the value in step 2.

To get the value of $\sin 65°28'$, consider the following procedure.

0′	6′	12′	18′	24′	…	54′	Mean Differences				
							1	2	3	4	5
0.9063	9070	9078	9085	9092	…	9128	1	2	4	5	6

Now,

$$\begin{aligned}\sin 65°\,28' &= \sin 65° \cdot 24' + 4' \\ &= 0.9092 + 0.0005 \\ &= 0.9097.\end{aligned}$$

Hence, $\sin 65°28' = 0.9097$.

EXAMPLE 8.28

Find the area of the right angle triangle with one of the acute angles being 65° and hypotenuse 6 cm.

SOLUTION

Let the right triangle be ABC, $\angle B = 90°$, $\angle A = 65°$ and $AC = 6$ cm.

From ΔABC,

$$\cos 65° = \frac{AB}{AC}$$

$$0.4226 = \frac{AB}{6}$$

$$AB = 2.5356$$

$$\sin 65° = \frac{BC}{AC} \Rightarrow 0.9063 = \frac{BC}{6}$$

$$BC = 5.4378$$

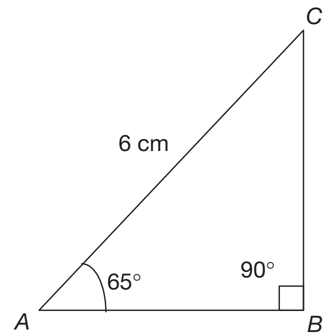


Figure 8.13

$$\text{Area of } \Delta ABC = \frac{1}{2} AB \times BC = \frac{1}{2}(2.5356)(5.4378)$$

$$= 6.8940 \text{ cm}^2 \text{ (approximately)}$$

Hence, the area of the ΔABC is 6.8940 cm^2.

EXAMPLE 8.29

Find the length of the chord which subtends an angle of 110° at the centre of the circle of radius 7 cm.

SOLUTION

Let the chord be AB. O be the centre of the circle and OD is shortest distance of the chord from the centre of circle.

Given $OA = OB = 7$ cm, $\angle AOB = 110°$.

Clearly, $\Delta AOD \cong \Delta BOD$ by SSS axiom

$$\angle AOD = \angle BOD = \frac{1}{2}\angle AOB = 55°.$$

In ΔAOD,

$$\sin 55° = \frac{AD}{AO}$$

$$0.8192 = \frac{AD}{7}$$

$$AD = 5.7344.$$

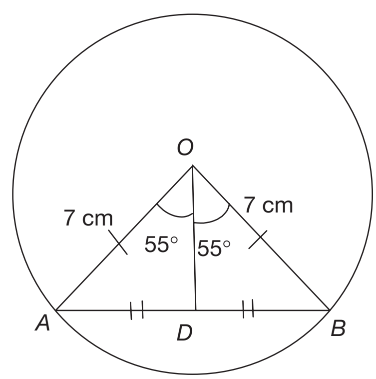


Figure 8.14

The length of the chord $= 2AD$

$$= 2(5.7344)$$

$$= 11.4688 \text{ cm.}$$

HEIGHTS AND DISTANCES

1. Let AB be a vertical line and PA and $P'B$ be two horizontal lines as shown in the Fig. 8.15.
2. Let $\angle APB = \alpha$ and $\angle PBP' = \beta$. Then,
   - (i) α is called the angle of elevation of the point B as seen from the point P and
   - (ii) β is called the angle of depression of the point P as seen from the point B.

Note Angle of elevation is always equal to the angle of depression.

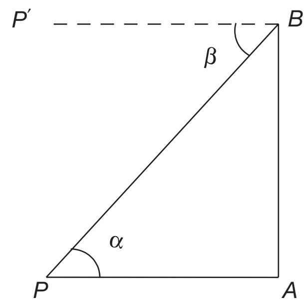


Figure 8.15

EXAMPLE 8.30

From a point on the ground which is at a distance of 50 m from the foot of the tower, the angle of elevation of the top of the tower is observed to be 30°. Find the height of the tower.

SOLUTION

Let the height of the tower be h m.

From ΔPAB, $\tan 30° = \dfrac{AB}{PA}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{50} \Rightarrow h = \frac{50}{\sqrt{3}} \text{ (or) } \frac{50\sqrt{3}}{3}.$$

Hence, the height of the tower is $\dfrac{50\sqrt{3}}{3}$ m.

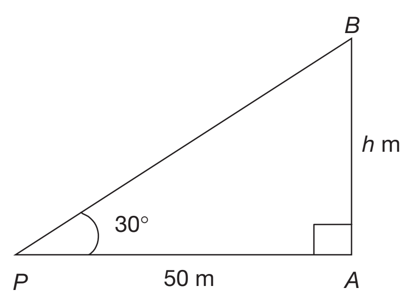


Figure 8.16

EXAMPLE 8.31

The angle of elevation of the top of a tower is 45°. On walking 20 m towards the tower along the line joining the foot of the observer and foot of the tower, the angle of elevation is found to be 60°. Find the height of the tower.

SOLUTION

Let the height of the tower be h m.

Let $QA = x$ m

In ΔPAB,

$$\tan 45° = \frac{AB}{PA} \Rightarrow 1 = \frac{h}{20+x} \Rightarrow 20 + x = h$$

$$\Rightarrow x = h - 20 \quad (1)$$

From $\Delta\ QAB$,

$$\tan 60° = \frac{AB}{QA}$$

$$\Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}\,x \Rightarrow h = \sqrt{3}\,(h-20), \quad \text{(using Eq. (1))}$$

$$\Rightarrow (\sqrt{3}-1)\,h = 20\sqrt{3} \Rightarrow h = \frac{20\sqrt{3}}{\sqrt{3}-1} = \frac{20\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)\,(\sqrt{3}+1)} = \frac{20(3+\sqrt{3})}{3-1} = 10(3+\sqrt{3})$$

Hence, the height of the tower is $10(3+\sqrt{3})$ m.

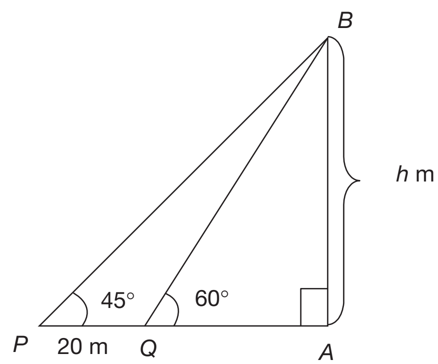


Figure 8.17

EXAMPLE 8.32

From the top of a building 100 m high, the angles of depression of the bottom and the top of an another building just opposite to it are observed to be 60° and 45° respectively. Find the height of the building.

SOLUTION

Let the height of the building be h m

Let $AC = BD = d$ m

From ΔBDE,

$$\tan 45° = \frac{ED}{BD} \Rightarrow 1 = \frac{100-h}{d}$$

$$\Rightarrow d = 100 - h \quad (1)$$

From $\Delta\ ACE$,

$$\tan 60° = \frac{CE}{AC}$$

$$\Rightarrow \sqrt{3} = \frac{100}{d} \Rightarrow \sqrt{3}d = 100 \Rightarrow d = \frac{100}{\sqrt{3}}$$

$$\Rightarrow 100 - h = \frac{100}{\sqrt{3}} \qquad \text{(using Eq. (1))}$$

$$\Rightarrow h = 100 - \frac{100}{\sqrt{3}}$$

$$= 100\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) = \frac{100(3-\sqrt{3})}{3}.$$

Hence, the height of the tower is $\dfrac{100(3-\sqrt{3})}{3}$ m.

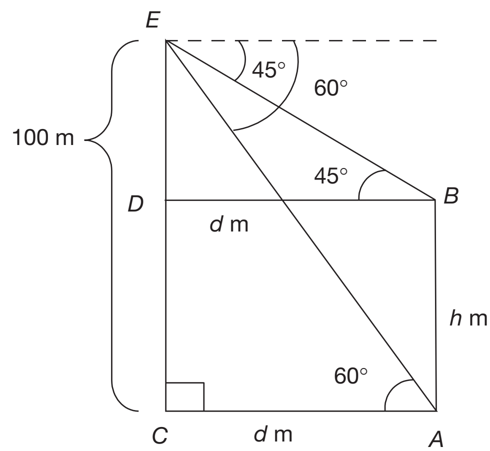


Figure 8.18

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If $\sin\theta = \frac{1}{2}$ where $0° \le \theta \le 180°$, then the possible values of θ are ______.
2. $\cot\theta$ in terms of $\sin\theta =$ ______ $(0 \le \theta \le 90°)$.
3. If A and B are two complementary angles, then $\sin A \cdot \cos B + \cos A \cdot \sin B =$ ______.
4. If the angle of a sector is 45° and the radius of the sector is 28 cm then the length of the arc is ______.
5. If $ABCD$ is a cyclic quadrilateral, then $\tan A + \tan C =$ ______.
6. $\frac{1-\cos 2\theta}{2} =$ ______ (in terms of $\sin\theta$).
7. $\cos 1° \cdot \cos 2° \cdot \cos 3° \cdots \cos 120° =$ ______.
8. The $\frac{3\pi}{2}$ is equivalent to ______ in centesimal system.
9. If $A + B = 360°$, then $\frac{\tan A + \tan B}{1 - \tan A \tan B} =$ ______.
10. If $\tan\theta + \cot\theta = 2$, then $\tan^{10}\theta + \cot^{10}\theta =$ ______ (where $0 < \theta < 90°$).
11. Write an equation eliminating θ from the equations $a = d\sin\theta$ and $c = d\cos\theta$.
12. Convert 250^g into other two measures.
13. $\sin(180+\theta) + \cos(270+\theta) + \cos(90+\theta) + \sin(360+\theta) =$ ______.
14. If $\sin\theta + \cos\theta = 1$ and $0° \le \theta \le 90°$, then the possible values of θ are ______.
15. Evaluate $\sin^2 45° + \cos^2 60° + \mathrm{cosec}^2 30°$.
16. If $ABCD$ is a cyclic quadrilateral, then find the value of $\cos A + \cos B + \cos C + \cos D$.
17. $\mathrm{cosec}(7\pi + \theta) \cdot \sin(8\pi + \theta) =$ ______.
18. If $\theta_1 = \frac{7}{25}$ and $\theta_2 = \frac{24}{25}$, then find the relation between θ_1 and θ_2.
19. Find the value of tan 1140°.
20. If $\sin(A+B) = \cos(A-B) = \frac{\sqrt{3}}{2}$, then $\cot 2A =$ ______.
21. If ΔABC is an isosceles triangle and right angled at B, then $\frac{\tan A + \tan C}{\cot A + \cot C} =$ ______.
22. $[\sin(x-\pi) + \cos(x-\pi/2)] \cdot \cos(x-2\pi) =$ ______.
23. $\tan(A+B)\tan(A-B) =$ ______.
24. The angles of a quadrilateral are in the ratio 1 : 2 : 3 : 4. Then the smallest angle in the centesimal system is ______.
25. If $\alpha + \beta = 90°$ and $\alpha = \frac{\beta}{2}$ then $\tan\alpha \cdot \tan\beta =$ ______.
26. $[\sin\alpha + \sin(180-\alpha) + \sin(180+\alpha)]\,\mathrm{cosec}\,\alpha =$ ______.
27. Express $\frac{\tan\theta + 1}{\tan\theta - 1}$ as a single trigonometric ratio.
28. If $\mathrm{cosec}\,\theta + \cot\theta = 3$, then find $\cos\theta$.
29. The top of a building from a fixed point is observed at an angle of elevation 60° and the distance from the foot of the building to the point is 100 m, then the height of the building is ______.
30. If $\cot\theta = \frac{4}{3}$ where $180 < \theta < 270$, then $\sin\theta + \cos\theta =$ ______.

Short Answer Type Questions

31. If the tip of the pendulum of a clock travels 13.2 cm in one oscillation and the length of the pendulum is 6.3 cm, then the angle made by the pendulum in half oscillation in radian system is ______.
32. If $\mathrm{cosec}\,\theta$, $\sec\theta$ and $\cot\theta$ are in HP, then $\frac{\sin\theta + \tan\theta}{\cos\theta} =$ ______.
33. $\cot\frac{\pi}{18} \cdot \cot\frac{\pi}{9} \cdot \cot\frac{\pi}{4} \cdot \cot\frac{4\pi}{18} \cdot \cot\frac{7\pi}{18} =$ ______.
34. If $\cot\theta = \frac{3}{4}$ and θ is acute, then find the value of $\frac{\tan\theta + \cot\theta}{\sec\theta + \mathrm{cosec}\,\theta}$.

35. Simplify $\sin(A + 45°)\sin(A - 45°)$.

36. Eliminate θ from the following equations: $x = a \sin\theta$, $y = b\cos\theta$ and $z = a\sin^2\theta + b\cos^2\theta$.

37. If $\sin A = \frac{3}{5}$ and A is not in the first quadrant, then find $\frac{\cos A + \sin 2A}{\tan A + \sec A}$.

38. If $\cos(A - B) = \frac{5}{13}$ and $\sin(A + B) = \frac{4}{5}$, then find $\sin 2B$.

39. If $\operatorname{cosec}\theta - \cot\theta = 2$, then find the value of $\operatorname{cosec}^2\theta + \cot^2\theta$.

40. Prove that $\frac{1+\cos A}{1-\cos A} = (\operatorname{cosec} A + \cot A)^2$.

41. If $\tan 28° = n$, then find the value of $\frac{\tan 152° + \tan 62°}{\tan 242° + \tan 28°}$.

42. If $3\sin A + 4\cos A = 4$, then find $4\sin A - 3\cos A$.

43. A ladder of length 50 m rests against a vertical wall, at a height of 30 m from the ground. Find the inclination of the ladder with the horizontal. Also find the distance between the foot of the ladder and the wall.

44. Eliminate θ from the following equations: $x\sin\alpha + y\cos\alpha = p$ and $x\cos\alpha - y\sin\alpha = q$.

45. Prove that $\left(\frac{1-\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{1+\sin\alpha}\right)\left(\sec\alpha + \frac{1}{\cot\alpha}\right) = 2$.

Essay Type Questions

46. Show that $6(\sin x + \cos x)^4 + 12(\sin x - \cos x)^2 + 8(\sin^6 x + \cos^6 x) = 26$.

47. Prove that $\frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \frac{1+\cos A}{\sin A}$.

48. The angle of depression of the top of the tower from the top of a building is 30° and angle of elevation of the top of the tower from the bottom of the building is 45° and if the height of the tower is 20 m, then find the height of the building.

49. A vertical pole is 60 m high. The angles of depression of two points P and Q on the ground are 30° and 45° respectively. If the points P and Q lie on either side of the pole, then find the distance PQ.

50. Prove that $\sin^8\theta - \cos^8\theta = \cos 2\theta\,(2\sin^2\theta\cos^2\theta - 1)$.

CONCEPT APPLICATION

Level 1

1. If $\sin x° = \sin \alpha x$, then α is
   (a) $\frac{180}{\pi}$ (b) $\frac{\pi}{270}$
   (c) $\frac{270}{\pi}$ (d) $\frac{\pi}{180}$

2. If in a triangle ABC, A and B are complementary, then $\tan C$ is
   (a) ∞ (b) 0
   (c) 1 (d) $\sqrt{3}$

3. If $\alpha = \frac{4}{5}$ and $\beta = \frac{4}{5}$, then which of the following is true?
   (a) $\alpha < \beta$ (b) $\alpha > \beta$
   (c) $\alpha = \beta$ (d) None of these

4. $\sin^2 20 + \sin^2 70$ is equal to ______.
   (a) 1 (b) −1
   (c) 0 (d) 2

5. $\cos 50°50' \cos 9°10' - \sin 50°50' \sin 9°10'$ = ______.
   (a) 0 (b) $\frac{1}{2}$
   (c) 1 (d) $\frac{\sqrt{3}}{2}$

6. $\sin\theta\cos(90° - \theta) + \cos\theta\sin(90° - \theta)$ = ______.
   (a) −1 (b) 2
   (c) 0 (d) 1

7. A wheel makes 20 revolutions per hour. The radians it turns through 25 minutes is ______.

(a) $\frac{50\pi^c}{7}$ (b) $\frac{250\pi^c}{3}$

(c) $\frac{150\pi^c}{7}$ (d) $\frac{50\pi^c}{3}$

8. $\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta}=$ ______.

(a) −1 (b) 2

(c) 0 (d) 1

9. Simplified expression of $(\sec\theta+\tan\theta)(1-\sin\theta)$ is ______.

(a) $\sin^2\theta$ (b) $\cos^2\theta$

(c) $\tan^2\theta$ (d) $\cos\theta$

10. If $a = \sec\theta - \tan\theta$ and $b = \sec\theta + \tan\theta$, then

(a) $a = b$. (b) $\frac{1}{a}=\frac{-1}{b}$.

(c) $a=\frac{1}{b}$. (d) $a - b = 1$.

11. If $\sec\alpha + \tan\alpha = m$, then $\sec^4\alpha - \tan^4\alpha - 2\sec\alpha\tan\alpha$ is ______.

(a) m^2 (b) $-m^2$

(c) $\frac{1}{m^2}$ (d) $\frac{-1}{m^2}$

12. If $\sin^4 A - \cos^4 A = 1$, then $(A/2)$ is ______. $(0 < A \le 90°)$.

(a) 45° (b) 60°

(c) 30° (d) 40°

13. The value of tan 15° tan 20° tan 70° tan 75° is

(a) −1 (b) 2

(c) 0 (d) 1

14. In a ΔABC, $\tan\left(\frac{A+C}{2}\right)=$ ______.

(a) $\tan\frac{B}{2}$ (b) $\cot\frac{B}{2}$

(c) $-\tan B$ (d) $\cot B$

15. If $\tan(A-30°)=2-\sqrt{3}$, then find A.

(a) $\frac{\pi^c}{2}$ (b) $\frac{\pi^c}{4}$

(c) $\frac{\pi^c}{6}$ (d) $\frac{\pi^c}{3}$

16. If $\sin^4\theta - \cos^4\theta = K^4$, then $\sin^2\theta - \cos^2\theta$ is ______.

(a) K^4 (b) K^3

(c) K^2 (d) K

17. $\frac{\tan^3\theta-1}{\tan\theta-1}=$ ______.

(a) $\sec^2\theta+\tan\theta$ (b) $\sec^2\theta-\tan\theta$

(c) 0 (d) $\tan\theta-\sec^2\theta$

18. For all values of θ, $1+\cos\theta$ can be ______.

(a) positive (b) negative

(c) non-positive (d) non-negative

19. If $\sin 3\theta = \cos(\theta - 6°)$, where 3θ and $(\theta - 6°)$ are acute angles then the value of θ is ______.

(a) 42° (b) 24°

(c) 12° (d) 26°

20. $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) =$ ______.

(a) −1 (b) 2

(c) 0 (d) 1

21. If $x = a(\operatorname{cosec}\theta + \cot\theta)$ and $y = b(\cot\theta - \operatorname{cosec}\theta)$, then

(a) $xy - ab = 0$. (b) $xy + ab = 0$.

(c) $\frac{x}{a}+\frac{y}{b}=1$. (d) $x^2y^2 = ab$.

22. The value of $\frac{\cos^4 x+\cos^2 x\sin^2 x+\sin^2 x}{\cos^2 x+\sin^2 x\cos^2 x+\sin^4 x}$ is ______.

(a) 2 (b) 1

(c) 3 (d) 0

23. $\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$ is equal to ______.

(a) $2\sec^2\theta$ (b) $2\cos^2\theta$

(c) 0 (d) 1

24. If $\tan(\alpha+\beta)=\frac{1}{2}$ and $\tan\alpha=\frac{1}{3}$, then $\tan\beta =$ ______.

(a) $\frac{1}{6}$ (b) $\frac{1}{7}$

(c) 1 (d) $\frac{7}{6}$

25. The value of log sin 0° + log sin 1° + log sin 2° + ⋯ + log sin 90° is ______.

(a) 0 (b) 1

(c) −1 (d) undefined

26. Which of the following is not possible?

(a) $\sin\theta = \dfrac{3}{5}$ (b) $\sec\theta = 100$

(c) $\operatorname{cosec}\theta = 0.14$ (d) None of these

27. $\sin^2 20° + \cos^2 160° - \tan^2 45° =$ ______.

(a) 2 (b) 0

(c) 1 (d) −2

28. $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} + \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} =$ ______.

(a) $\dfrac{2}{1 - 2\cos^2\theta}$

(b) $\dfrac{2}{2\sin^2\theta - 1}$

(c) Both (a) and (b)

(d) None of these

29. The length of the side (in cm) of an equilateral triangle inscribed in a circle of radius 8 cm is ______.

(a) $16\sqrt{3}$ (b) $12\sqrt{3}$

(c) $8\sqrt{3}$ (d) $10\sqrt{3}$

30. Which among the following is true?

(a) $\sin 1° > \sin 1^c$

(b) $\sin 1° < \sin 1^c$

(c) $\sin 1° = \sin 1^c$

(d) None of these

Level 2

31. If $2 \sin\alpha + 3 \cos\alpha = 2$, then $3 \sin\alpha - 2\cos\alpha =$ ______.

(a) ±3 (b) ±1

(c) 0 (d) ±2

32. If $\dfrac{\sin^2\alpha - 3\sin\alpha + 2}{\cos^2\alpha} = 1$, then α can be ______.

(a) 60° (b) 45°

(c) 0° (d) 30°

33. If $\cot A = \dfrac{5}{12}$ and A is not in the first quadrant, then $\dfrac{\sin A - \cos A}{1 + \cot A}$ is ______.

(a) $\dfrac{-74}{25}$ (b) $\dfrac{-84}{221}$

(c) $\dfrac{-87}{223}$ (d) None of these

34. If $\dfrac{1 + \sin\alpha}{1 - \sin\alpha} = \dfrac{m^2}{n^2}$, then $\sin\alpha$ is ______.

(a) $\dfrac{m^2 + n^2}{m^2 - n^2}$ (b) $\dfrac{m^2 - n^2}{m^2 + n^2}$

(c) $\dfrac{m^2 + n^2}{n^2 - m^2}$ (d) $\dfrac{n^2 - m^2}{m^2 + n^2}$

35. If $\sin\theta - \cos\theta = \dfrac{3}{5}$, then $\sin\theta\cos\theta =$ ______.

(a) $\dfrac{16}{25}$ (b) $\dfrac{9}{16}$ (c) $\dfrac{9}{25}$ (d) $\dfrac{8}{25}$

36. If $ABCD$ is a cyclic quadrilateral, then the value of $\cos^2 A - \cos^2 B - \cos^2 C + \cos^2 D$ is ______.

(a) 0 (b) 1

(c) −1 (d) 2

37. The length of minute hand of a wall clock is 12 cm. Find the distance covered by the tip of the minute hand in 25 minutes.

(a) $\dfrac{220}{7}$ cm (b) $\dfrac{110}{7}$ cm

(c) $\dfrac{120}{7}$ cm (d) $\dfrac{240}{7}$ cm

38. $\sin^2 2° + \sin^2 4° + \sin^2 6° + \cdots + \sin^2 90° =$ ________.

(a) 22 (b) 23

(c) 44 (d) 45

39. A straight highway leads to the foot of a tower of height 50 m. From the top of the tower, the angles of depression of two cars standing on the highway are 30° and 60°. What is the distance between the two cars? (in metres)

(a) $\dfrac{100}{\sqrt{3}}$ (b) $50\sqrt{3}$

(c) $\dfrac{50}{\sqrt{3}}$ (d) $100\sqrt{3}$

40. The angle of elevation of the top of a hill from the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30°.

If the tower is 50 m high, then what is the height of the hill?

(a) 180 m (b) 150 m

(c) 100 m (d) 120 m

41. $\tan 38° - \cot 22° =$ _____.

(a) $\frac{1}{2}\operatorname{cosec} 38° \sec 22°$

(b) $2 \sin 22° \cos 38°$

(c) $-\frac{1}{2}\operatorname{cosec} 22° \sec 38°$

(d) None of these

42. $\frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{1-\cos\theta} =$ _____.

(a) $2 \sin\theta$ (b) $2 \cos\theta$

(c) $2 \operatorname{cosec}\theta$ (d) $2 \sec\theta$

43. $\sqrt{-4+\sqrt{8+16\operatorname{cosec}^4\alpha \sin^4\alpha}} =$ _____.

(a) $\operatorname{cosec}\alpha - \sin\alpha$

(b) $2\operatorname{cosec}\alpha + \sin\alpha$

(c) $2\operatorname{cosec}\alpha - \sin\alpha$

(d) $\operatorname{cosec}\alpha - 2\sin\alpha$

44. The angles of depression of the top and the bottom of a 7 m tall building from the top of a tower are 45° and 60° respectively. Find the height of the tower in metres.

(a) $7(3+\sqrt{3})$ (b) $\frac{7}{2}(3-\sqrt{3})$

(c) $\frac{7}{2}(3+\sqrt{3})$ (d) $7(3-\sqrt{3})$

45. If $\tan 86° = m$, then $\frac{\tan 176° + \cot 4°}{m + \tan 4°}$ is _____.

(a) $\frac{m^2-1}{m^2+1}$. (b) $\frac{m^2+1}{1-m^2}$.

(c) $\frac{1-m^2}{1+m^2}$. (d) $\frac{m^2+1}{m^2-1}$.

46. The following sentences are the steps involved in proving the result $\frac{\cos x}{1-\tan x} + \frac{\sin x}{1-\cot x} = \cos x + \sin x$. Arrange them in sequential order from first to last.

(A) $\frac{\cos^2 x}{\cos x - \sin x} + \frac{\sin^2 x}{\sin x - \cos x}$

(B) $\frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}$

(C) $\frac{\cos x}{1-\frac{\sin x}{\cos x}} + \frac{\sin x}{1-\frac{\cos x}{\sin x}}$

(a) (A), (B) and (C)

(b) (C), (A) and (B)

(c) (C), (B) and (A)

(d) None of these

47. The following sentences are the steps involved in eliminating θ from the equation $x = y \tan\theta$ and $a = b \sec\theta$. Arrange them in sequential order from first to last.

(A) Subtract $\left(\frac{x}{y}\right)^2$ from $\left(\frac{a}{b}\right)^2$

(B) $\left(\frac{x}{y}\right)^2 - \left(\frac{a}{b}\right)^2 = 1$

(C) Taking squares on both the sides

(D) Find $\frac{x}{y}$ and $\frac{a}{b}$

(a) (D), (A), (C) and (B)

(b) (D), (C), (B) and (A)

(c) (D), (B), (A) and (C)

(d) (D), (C), (A) and (B)

Level 3

48. There is a small island in the river which is 100 m wide and a tall tree stands on the island. P and Q are points directly opposite each other on the two banks and in line with the tree. If the angles of elevation of the top of the tree from P and Q are respectively are 30° and 45°, find the height of the tree (in metres).

(a) $50(\sqrt{3}-1)$ (b) $50(\sqrt{3}+1)$

(c) $100(\sqrt{3}+1)$ (d) $100(\sqrt{3}-1)$

49. A balloon is connected to a meteorological ground station by a cable of length 215 m inclined at 60° to the horizontal. Determine the height of the

PRACTICE QUESTIONS

balloon from the ground. Assume that there is no slack in the cable.

(a) $107.5\sqrt{3}$ m (b) $100\sqrt{3}$ m

(c) $215\sqrt{3}$ m (d) $215/\sqrt{3}$ m

50. If $\sin^2 A = 2\sin A \cos A$ and $\sin 20° = K$, then the value of $\cos 20° \cos 40° \cos 80° \cos 160° =$ _____.

(a) K (b) $-\sqrt{1-K^2}$

(c) $\dfrac{\sqrt{1-K^2}}{8}$ (d) $-\dfrac{\sqrt{1-K^2}}{8}$

51. If $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta = 2\sqrt{2}$, then the value of θ can be _____.

(a) 0° (b) −45°

(c) 30° (d) −60°

52. A circus artist climbs from the ground along a rope which is stretched from the top of a vertical pole and tied at the ground at a certain distance from the foot of the pole. The height of the pole is 12 m and the angle made by the rope with the ground is 30°. Calculate the distance covered by the artist in reaching the top of the pole.

(a) 24 m (b) 6 m

(c) 12 m (d) None of these

53. Find the value of $\sin^2 5° + \sin^2 10° + \sin^2 15° + \cdots + \sin^2 90°$.

(a) 8 (b) 9

(c) $\dfrac{17}{2}$ (d) $\dfrac{19}{2}$

54. If $\sec\theta + \tan\theta = 2$, then find the value of $\sin\theta$.

(a) $\dfrac{3}{5}$ (b) $\dfrac{2}{5}$

(c) $-\dfrac{3}{5}$ (d) $-\dfrac{2}{5}$

55. If $\cos\theta + \left(\dfrac{1}{\sqrt{3}}\right)\sin\theta = \dfrac{2}{\sqrt{3}}$, then find θ in circular measure.

(a) $\dfrac{\pi^c}{10}$ (b) $\dfrac{\pi^c}{9}$

(c) $\dfrac{\pi^c}{6}$ (d) $\dfrac{\pi^c}{3}$

56. $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} =$ _____.

(a) $\sec\theta + \tan\theta$

(b) $\sec\theta - \cot\theta$

(c) $\operatorname{cosec}\theta + \tan\theta$

(d) $\operatorname{cosec}\theta - \tan\theta$

57. If $\dfrac{\sin^2\theta - 5\sin\theta + 3}{\cos^2\theta} = 1$, then θ can be _____.

(a) 30° (b) 45°

(c) 60° (d) 0°

58. If $\cot\theta = \dfrac{24}{7}$ and θ is not in the first quadrant, then find the value of $\tan\theta - \sec\theta$.

(a) 1 (b) $\dfrac{4}{3}$

(c) $\dfrac{3}{2}$ (d) $\dfrac{5}{4}$

59. If $\sin 20° = p$, then find the value of $\left(\dfrac{\sin 380° - \sin 340°}{\cos 380° + \cos 340°}\right)$.

(a) $\sqrt{1-p^2}$ (b) $\sqrt{\dfrac{1-p^2}{p}}$

(c) $\dfrac{p}{\sqrt{1-p^2}}$ (d) None of these

60. Find the value $\tan\left(22\dfrac{1°}{2}\right)$.

(a) $\sqrt{2}-1$ (b) $1+\sqrt{2}$

(c) $2+\sqrt{3}$ (d) $2-\sqrt{3}$

61. If the sun ray's inclination increases from 45° to 60°, the length of the shadow of a tower decreases by 50 m. Find the height of the tower (in m).

(a) $50\left(\sqrt{3}-1\right)$ (b) $75\left(3-\sqrt{3}\right)$

(c) $100\left(\sqrt{3}+1\right)$ (d) $25\left(3+\sqrt{3}\right)$

62. The angles of depression of two points from the top of the tower are 30° and 60°. If the height of the tower is 30 m, then find the maximum possible distance between the two points.

(a) $40\sqrt{3}$ m (b) $30\sqrt{3}$ m

(c) $20\sqrt{3}$ m (d) $10\sqrt{3}$ m

63. From a point on the ground, the angle of elevation of an aeroplane flying at an altitude of 500 m changes from 45° to 30° in 5 seconds. Find the speed of the aeroplane (in kmph).

(a) $720\left(\sqrt{3}-1\right)$ (b) $720\left(\sqrt{3}+1\right)$

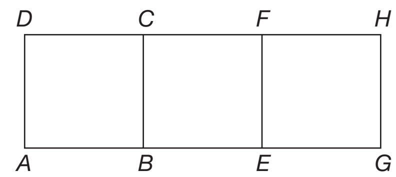
(c) $360\left(\sqrt{3}-1\right)$ (d) $360\left(\sqrt{3}+1\right)$

64. From the top of a building, the angles of elevation and depression of top and bottom of a tower are 60° and 30° respectively. If the height of the building is 5 m, then find the height of the tower.

(a) $10\sqrt{3}$ m (b) 15 m

(c) $15\sqrt{3}$ m (d) 20 m

65. In the figure given below (not to scale), *ABCD*, *BEFC* and *EGHF* are three squares. Find $\angle FAE + \angle HAG$.



(a) 30°

(b) 45°

(c) 60°

(d) 90°

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $30°$ or $150°$
2. $\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$
3. 1
4. 22 cm
5. $0°$
6. $\sin^2\theta$
7. 0
8. 300^g
9. 0
10. 2
11. $a^2 + c^2 = d^2$
12. $\dfrac{5\pi^c}{4}$
13. 0
14. $0°$ and $90°$
15. $\dfrac{19}{4}$
16. 0
17. -1
18. $\theta_1 = \theta_2$
19. $\sqrt{3}$
20. 0
21. 1
22. 0
23. $\dfrac{\tan^2 A - \tan^2 B}{1-\tan^2 A\tan^2 B}$.
24. 40^g
25. 1
26. 1
27. $-\tan\left(\theta + \dfrac{\pi}{4}\right)$
28. $\dfrac{4}{5}$
29. $100\sqrt{3}$ m
30. $\dfrac{-7}{5}$

Short Answer Type Questions

31. $\dfrac{\pi^c}{3}$
32. 2
33. 1
34. $\dfrac{5}{7}$
35. $-\dfrac{1}{2}\cos 2A$
36. $bx^2 + ay^2 = abz$
37. $\dfrac{22}{25}$
38. $\dfrac{-16}{65}$
39. $\dfrac{17}{8}$
41. $\dfrac{1-n^2}{1+n^2}$
42. ± 3
43. 40 m
44. $x^2 + y^2 = p^2 + q^2$

Essay Type Questions

48. $\dfrac{20\left(1+\sqrt{3}\right)}{\sqrt{3}}$ m
49. $60\left(\sqrt{3}+1\right)$ m

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (d)	**9.** (d)	**10.** (c)
11. (c)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (d)
21. (b)	**22.** (b)	**23.** (a)	**24.** (b)	**25.** (d)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (b)	**30.** (b)

Level 2

31. (b)	**32.** (d)	**33.** (c)	**34.** (c)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (b)	**39.** (d)	**40.** (d)
41. (d)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (d)	**46.** (b)	**47.** (d)			

Level 3

48. (d)	**49.** (c)	**50.** (b)	**51.** (b)	**52.** (c)	**53.** (d)	**54.** (a)	**55.** (c)	**56.** (a)	**57.** (a)
58. (b)	**59.** (c)	**60.** (a)	**61.** (d)	**62.** (a)	**63.** (c)	**64.** (d)	**65.** (b)		

CONCEPT APPLICATION

Level 1

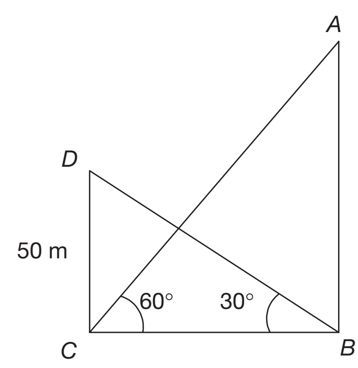
1. Use the relation between degrees and radians.
2. $A + B = 90°$ and $A + B + C = 180°$.
3. Find $\sin\beta$ and compare $\sin\alpha$ and $\sin\beta$.
4. $\sin\theta = \cos(90 - \theta)$.
5. Use $\cos A \cos B - \sin A \sin B = \cos(A + B)$.
6. $\cos(90 - \theta) = \sin\theta$ and $\sin(90 - \theta) = \cos\theta$.
7. Find the angle covered in one hour.
8. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
9. Write $\sec\theta$ and $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.
10. Use the identity $\sec^2\theta - \tan^2\theta = 1$.
11. Simplify the given expression and find $\sec\alpha - \tan\alpha$.
12. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
13. $\tan\theta = \cot(90 - \theta)$ and $\tan\theta \cdot \cot\theta = 1$.
14. $A + B + C = 180°$.
15. Use the identity $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.
16. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
17. Use $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
18. Recall the range of $\cos\theta$.
19. Use complementary angles.
20. Replace $\operatorname{cosec} A$ by $\dfrac{1}{\sin A}$ and $\sec A$ by $\dfrac{1}{\cos A}$.
21. Use $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.
22. Simplify the numerator and denominator by taking common terms appropriately.
23. Take LCM and simplify.
24. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.
25. $\log a + \log b + \log c + \cdots = \log(a \cdot b \cdot c \cdots)$.
26. Recall the ranges of $\sin\theta$ and $\cos\theta$.
27. $\sin(180 - \theta) = \sin\theta$.
28. Simplify the expression.
29. Equal chords subtend equal angles at the centre $(2r\sin 60°)$.
30. (i) $1°$ is always less than 1^c.

    (ii) The value $\sin\theta$ increases from $0°$ to $90°$.

Level 2

31. Put $3\sin\alpha - 2\cos\alpha = k$, square the two equations and add.
32. Use the identity $\cos^2\alpha + \sin^2\alpha = 1$ and convert the equation into quadratic form in terms of $\sin\alpha$ and then solve.
33. (i) $\cot A$ is positive in the first and the third quadrants. As $\cot A$ not in first quadrant, $\cot A$ in third quadrant.

    (ii) Using the right angle triangle find the values of $\sin A$, $\cos A$ and $\cot A$ in third quadrant.
34. Apply componendo–dividendo rule, i.e., if $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.
35. Square the given equation, then use the identity $\cos^2\theta + \sin^2\theta = 1$ to evaluate the value of $\sin\theta\cos\theta$.
36. (i) In a cyclic quadrilateral, sum of the opposite angles is $180°$.

    (ii) Use the result, $\cos(180 - \theta) = -\cos\theta$ and simplify.
37. $l = r\theta$, where l = length, r = radians and θ = angle in radians.
38. Use the results $\sin^2 88 = \cos^2 2$ and $\sin^2\theta + \cos^2\theta = 1$ to simplify the given expression.
39. Use $\tan\theta$ = opposite side/adjacent side in the triangles so formed.
40. 

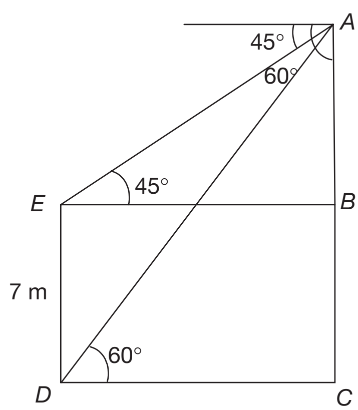
    Use the figure, by taking the values of $\tan 30°$ from ΔBCD and $\tan 60°$ from ΔABC, find CD.
41. (i) Convert the given trigonometric values in terms of $\sin\theta$, $\cos\theta$, then simplify to obtain the required value.

(ii) Use the formula, cost $(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.

42. Take LCM and use the identity $\sin^2\theta + \cos^2\theta = 1$ to simplify the given equation.

43. Express the square root function as $(a+b)^2$ and remove the first square root. Again express the obtained function as $(a-b)^2$ and then simplify.

44. The following figure is drawn as per the description in the problem.



Use the figure, to find DC from ΔADC and AB from ΔAEB as $DC = BE$.

45. Express $\tan 176° = -\cot 86°$, $\cot 4° = \tan 86°$ and $\tan 4° = \cot 86°$. Then substitute these values to simplify the given expression.

46. (C), (A) and (D) are in sequential order from first to last.

47. (D), (C), (A) and (B) are in sequential order from first to last.

Level 3

48. Draw the figure as per the description given in the problem. Two right triangles are formed. Use appropriate trigonometric ratios by considering each of the triangle and find the height of the tree.

49. Draw a right triangle according to the data and take the ratio of $\sin\theta$ to evaluate the height of the balloon from the ground.

50. (i) Multiply and divide the given expression with $\sin 20°$.

(ii) Use $\sin^2 A = 2\sin A \cos A$.

(iii) $\sin(180 - A) = \sin A$.

51. (i) Divide the equation with $2\sqrt{2}$.

Then substitute $\frac{1}{2} = \cos 60°$ and $\frac{\sqrt{3}}{2} = \sin 60°$.

(ii) Now apply the formula,

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ then obtain the value of θ.

52. (i) Draw a figure as per the situation described then use the concept, $\sin\theta = \frac{\text{Opposite side to } \theta}{\text{Hypotenuse}}$.

(ii) Given θ and opposite side, find hypotenuse which is the required length of the rope.

53. $\sin^2 5° + \sin^2 10° + \sin^2 15° + \cdots + \sin^2 90°$

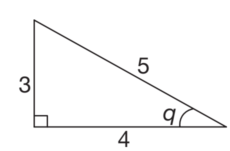
$= (\sin^2 5° + \sin^2 85°) + (\sin^2 10° + \sin^2 80°) + \cdots + \sin^2 45 + \sin^2 90°(\sin 85° = \cos 5°)$

$$= 8 + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 9 + \frac{1}{2} = \frac{19}{2}.$$

54. Given,

$$\sec\theta + \tan\theta = 2 \quad (1)$$

$$\sec\theta - \tan\theta = \frac{1}{2} \quad (2)$$



Adding Eqs. (1) and (2), we get

$$2\sec\theta = \frac{5}{2}$$

$$\Rightarrow \sec\theta = \frac{5}{4}$$

$$\Rightarrow \cos\theta = \frac{4}{5}$$

$$\sin\theta = \frac{3}{5}.$$

55. Given,

$$\cos\theta + \left(\frac{1}{\sqrt{3}}\right)\sin\theta = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}\right)\cos\theta + \left(\frac{1}{2}\right)\sin\theta = 1$$

$$\Rightarrow \sin 60^\circ \cos\theta + \sin\theta \cos 60^\circ = 1$$

$$\Rightarrow \sin(60^\circ + \theta) = \sin 90^\circ$$

$$\Rightarrow 60^\circ + \theta = 90^\circ$$

$$\Rightarrow \theta = 30^\circ = \frac{\pi^c}{6}.$$

56. $$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta.$$

57. $\sin^2\theta - 5\sin\theta + 3 = \cos^2\theta$

$$\Rightarrow \sin^2\theta - 5\sin\theta + 3 = 1 - \sin^2\theta$$

$$\Rightarrow 2\sin^2\theta - 5\sin\theta + 2 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta - 2) = 0$$

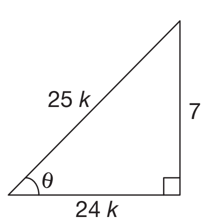
$$\sin\theta = 2 \text{ or } \frac{1}{2}.$$

But $\sin\theta = 2$ is not possible

$$\therefore \sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ.$$

58. Given, $\cot\theta = \dfrac{24}{7}$

As $\tan\theta$ is positive and $\theta \notin Q_1$,



$\theta \in Q_3$.

$$\therefore \sec\theta = -\frac{25}{24}$$

$$\tan\theta = \frac{7}{24}$$

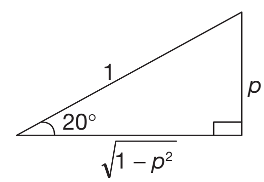
$$\text{Now } \tan\theta - \sec\theta = \frac{7}{24} + \frac{25}{24}$$

$$= \frac{32}{24} = \frac{4}{3}.$$

59. Given, $\sin 20^\circ = p$

$$\sin 380^\circ = \sin(360^\circ + 20^\circ)$$

$$= \sin 20^\circ$$



$$\sin 340^\circ = \sin(360^\circ - 20^\circ)$$

$$= -\sin 20^\circ$$

$$\cos 380^\circ = \cos(360^\circ + 20^\circ)$$

$$= \cos 20^\circ$$

$$\cos 340^\circ = \cos(360^\circ - 20^\circ)$$

$$= \cos 20^\circ$$

$$\therefore \left(\frac{\sin 380 - \sin 340}{\cos 380 + \cos 340}\right) = \frac{\sin 20 - (-\sin 20)}{\cos 20 + \cos 20}$$

$$= \frac{2\sin 20}{2\cos 20}$$

$$= \tan 20^\circ = \frac{p}{\sqrt{1-p^2}}.$$

60. We have, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\text{Let } \theta = \left(22\frac{1}{2}\right)^\circ \Rightarrow 2\theta = 45^\circ$$

$$\therefore \tan 45^\circ = \frac{2\tan\theta}{1-\tan^2\theta}.$$

$$1 = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\Rightarrow 1 - \tan^2\theta = 2\tan\theta$$

$$\Rightarrow \tan^2\theta + 2\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

As θ is $\left(22\frac{1}{2}\right)$, $\tan\left(22\frac{1}{2}\right)^\circ = \sqrt{2} - 1$.

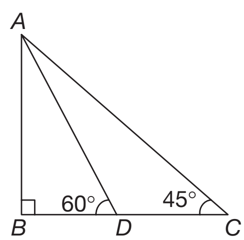
61. Let AB be the height of the tower.

Let $CD = 50$ m (given)

In ΔABC, $\tan 45^\circ = \dfrac{AB}{BC}$

$$\Rightarrow 1 = \frac{AB}{BD + CD} \qquad (1)$$

$$\Rightarrow AB = BD + 50$$



In $\Delta\ ABD$,

$$\tan 60^\circ = \frac{AB}{BD} \Rightarrow \sqrt{3} = \frac{AB}{BD}$$

$$\Rightarrow BD = \frac{AB}{\sqrt{3}} \qquad \text{(substitute } BD \text{ in Eq. (1))}$$

$$AB = \frac{AB}{\sqrt{3}} + 50$$

$$AB\left(1 - \frac{1}{\sqrt{3}}\right) = 50$$

$$AB\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) = 50$$

$$AB = \frac{50\sqrt{3}}{\sqrt{3}-1} = \frac{50\sqrt{3}\left(\sqrt{3}+1\right)}{3-1}$$

$$= 25\left(3 + \sqrt{3}\right) \text{ m.}$$

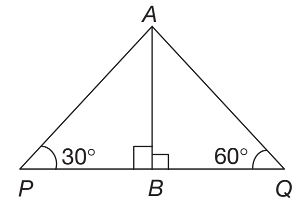
62. For the maximum possible distance, two points lie on either side of the tower.

Let AB be the height of the tower.

$AB = 30$ m (given)

Let P and Q be the given points.

In ΔABP,



$$\tan 30^\circ = \frac{AB}{PB} \Rightarrow \frac{1}{\sqrt{3}} = \frac{30}{PB} \Rightarrow PB = 30\sqrt{3}.$$

In ΔABQ, $\tan 60^\circ = \dfrac{AB}{BQ}$

$$\Rightarrow \sqrt{3} = \frac{30}{BQ} \Rightarrow BQ = \frac{30}{\sqrt{3}} \text{ m.}$$

Now, $PQ = PB + BQ$

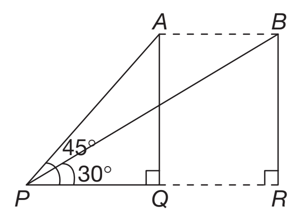
$$= 30\sqrt{3} + \frac{30}{\sqrt{3}} = 30\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$$

$$= 30\left(\frac{4}{\sqrt{3}}\right) = 40\sqrt{3} \text{ m.}$$

63. Let A and B be the initial and final positions of the plane.

Given, $AQ = 500$ m

$\therefore\ BR = 500$ m $(AQ = BR)$



In ΔAPQ,

$$\tan 45^\circ = \frac{AQ}{PQ}$$

$$1 = \frac{500}{PQ}$$

$\therefore PQ = 500$ m.

In ΔBPR,

$$\tan 30^\circ = \frac{BR}{PR}$$

$$\frac{1}{\sqrt{3}} = \frac{500}{PQ + QR} \Rightarrow 500 + QR = 500\sqrt{3}$$

$$QR = 500\left(\sqrt{3}-1\right) \text{ m} \quad \therefore AB = 500\left(\sqrt{3}-1\right) \text{ m.}$$

$$\text{Speed of the plane} = \frac{AB}{\text{Time}}$$

$$= \left(\frac{500\left(\sqrt{3}-1\right)}{5}\right) \text{m/s}$$

$$= 100\left(\sqrt{3}-1\right) \text{m/s}$$

$$= 100\left(\sqrt{3}-1\right)\left(\frac{18}{5}\right) \text{kmph}$$

$$= 360\left(\sqrt{3}-1\right) \text{kmph.}$$

64. AB be the height of the building.

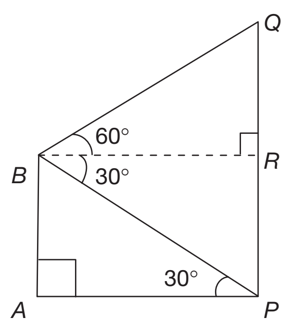
$\therefore$ $AB = 5$ m (given)

Let PQ be the height of the tower.

In ΔABP,

$$\tan 30° = \frac{AB}{AP} \Rightarrow \frac{1}{\sqrt{3}} = \frac{5}{AP}.$$

$$\therefore\ AP = 5\sqrt{3} \Rightarrow \therefore BR = 5\sqrt{3} \text{ m}$$



In ΔBQR, $\tan 60° = \dfrac{QR}{BR}$

$$\sqrt{3} = \frac{QR}{5\sqrt{3}} \Rightarrow QR = 15 \text{ m.}$$

From the figure, $PQ = PR + QR$

$= 5 + 15 = 20$ m.

65. Let $\angle FAE = \theta_1$, and $\angle HAG = \theta_2$

$$\tan\theta_1 = \frac{FE}{AE} = \frac{1}{2}$$

$$\tan\theta_2 = \frac{HG}{AG} = \frac{1}{3}$$

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$$

$$= \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$\tan(\theta_1 + \theta_2) = \tan 45°$

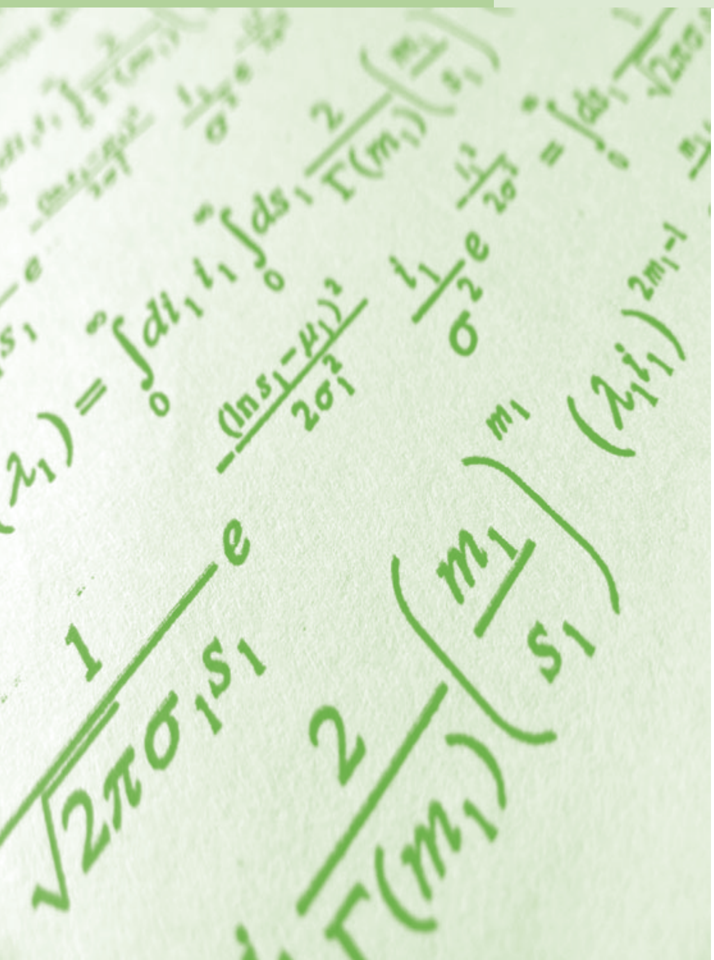
$\Rightarrow \theta_1 + \theta_2 = 45°$

$\angle FAE + \angle HAG = 45°$.

Chapter

9

Limits



REMEMBER

Before beginning this chapter, you should be able to:

- State the term function
- Apply basic operations on functions

KEY IDEAS

After completing this chapter, you would be able to:

- Identify limit of a function
- Understand meaning of symbols used in limits
- Study evaluation of limits by different methods
- Obtain limits of a function using a formula

INTRODUCTION

Limit of a Function

Let $y = f(x)$ be a function of x and let 'a' be any real number.

We first understand what a 'limit' is. A limit is the value, a function approaches, as the independent variable of the function (usually 'x') gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, say a, what is $f(x)$ very close to? It may be equal to $f(a)$ but may be different. It may exist even when $f(a)$ is not defined.

Meaning of '$x \to a$'

Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a', then we say that 'x tends to a' or 'x approaches a' and is written as '$x \to a$'. By $x \to a$, we mean that $x \neq a$ and x may approach 'a' from left or right, which is explained in the example given below.

What is the limit of the function $f(x) = x^2$ as x approaches 3? The expression 'the limit as x approaches to 3' is written as: $\lim\limits_{x \to 3}$. Let us check out some values of $\lim\limits_{x \to 3}$, as x increases and gets closer to 3, without ever exactly getting there.

When $x = 2.9$, $f(x) = 8.4100$

When $x = 2.99$, $f(x) = 8.9401$

When $x = 2.999$, $f(x) = 8.9940$

When $x = 2.9999$, $f(x) = 8.9994$

As x increases and approaches 3, $f(x)$ gets closer and closer to 9 and since x tends to 3 from the left this is called the 'left-hand limit' and is written as $\lim\limits_{x \to 3^-}$.

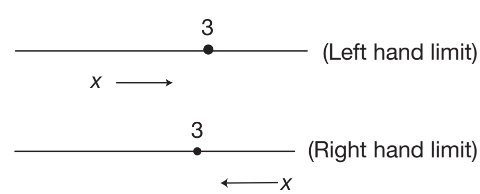
Now, let us see what happens when x is greater than 3.

When $x = 3.1$, $f(x) = 9.6100$

When $x = 3.01$, $f(x) = 9.0601$

When $x = 3.001$, $f(x) = 9.0060$

When $x = 3.0001$, $f(x) = 9.0006$



As x decreases and approaches 3, $f(x)$ still approaches 9. This is called the 'right-hand limit' and is written as $\lim\limits_{x \to 3^+}$.

As $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 9$, we write that $\lim\limits_{x \to 3} x^2 = 9$.

Meaning of the Symbol: $\lim\limits_{x \to a} f(x) = l$

Let $f(x)$ be a function of x where x takes values closer and closer to 'a' ($\neq a$), and let $f(x)$ assume values nearer and nearer to l.

We say, $f(x)$ tends to the limit 'l' as x tends to 'a'.

The following are some simple algebraic rules of limits.

1. $\lim\limits_{x \to a} k\, f(x) = k \lim\limits_{x \to a} f(x)$

2. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

3. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (where $\lim_{x\to a} g(x) \neq 0$)

Notes

1. If the left hand limit of a function is not equal to the right hand limit of the function, then the limit does not exist.
2. A limit equal to infinity does not imply that the limit does not exist.

Example: The limit of a chord of a circle passing through a fixed point Q and a variable point P as P approaches Q is the tangent to the circle at P.

If PQ is chord of a circle, when P approaches Q along the circle, then the chord becomes the tangent to the circle at P.

Problems While evaluating the limits we use the following methods:

1. Method of substitution
2. Method of factorization
3. Method of rationalization
4. Using the formula $\lim_{x\to a}\left[\dfrac{x^n - a^n}{x-a}\right] = na^{n-1}$

Method of Substitution

In this method we directly substitute the value of x in the given function to obtain the limit value.

Examples:

1. $\lim_{x\to 2}[x^2 + 5x + 6] = (2)^2 + 5(2) + 6 = 20.$

2. $\lim_{x\to 3}\left[\dfrac{x^2 + 4x + 1}{x+5}\right] = \dfrac{(3)^2 + 4(3) + 1}{3+5} = \dfrac{22}{8} = \dfrac{11}{4}.$

Indeterminate Form If $f(x) = \dfrac{x^2 - 9}{x-3}$, then $f(3) = \dfrac{3^2 - 9}{3-3} = \dfrac{0}{0}$, which is not defined. By substituting $x = 3$, the function assumes a form whose value cannot be determined, this form is called an indeterminate form.

Method of Factorization

If $\dfrac{f(x)}{g(x)}$ assumes an indeterminate form when $x = a$, then there exists a common factor for $f(x)$ and $g(x)$.

We remove the common factors, and use the substitution method to find the limit.

EXAMPLE 9.1

Evaluate $\lim_{x\to 2}\left[\dfrac{x^2-5x+6}{x^2-3x+2}\right]$.

SOLUTION

When $x = 2$, $\dfrac{4-10+6}{4-6+2} = \dfrac{0}{0}$, which is an indeterminate form.

Now, $x^2 - 5x + 6 = (x - 3)(x - 2)$

$x^2 - 3x + 2 = (x - 1)(x - 2)$

$$\lim_{x\to 2}\left[\frac{x^2-5x+6}{x^2-3x+2}\right] = \lim_{x\to 2}\left[\frac{(x-3)(x-2)}{(x-1)(x-2)}\right]$$

$$\lim_{x\to 2}\left[\frac{x-3}{x-1}\right] = \left[\frac{2-3}{2-1}\right] = -1.$$

Method of Rationalization

If $\dfrac{f(x)}{g(x)}$ assumes an indeterminate form when $x = a$ and $f(x)$ or $g(x)$ are irrational functions, then we rationalize $f(x)$, $g(x)$ and cancel the common factor, then use the substitution method to find the limit.

EXAMPLE 9.2

Evaluate $\lim_{x\to 0}\left[\dfrac{x}{1-\sqrt{1-x}}\right]$.

SOLUTION

When $x = 0$, the given function assumes the form $\dfrac{0}{1-\sqrt{1-0}} = \dfrac{0}{0}$, which is an indeterminate form.

Since $g(x)$, the denominator is an irrational form, we multiply the numerator and denominator with the rationalizing factor of $g(x)$.

$$\lim_{x\to 0}\left[\frac{x}{1-\sqrt{1-x}}\right] = \lim_{x\to 0}\frac{x}{\left(1-\sqrt{1-x}\right)} \times \frac{\left(1+\sqrt{1-x}\right)}{\left(1+\sqrt{1-x}\right)}$$

$$= \lim_{x\to 0}\frac{x\left(1+\sqrt{1-x}\right)}{1-1+x}$$

$$= \lim_{x\to 0}\frac{x\left(1+\sqrt{1-x}\right)}{x}$$

$$= \lim_{x\to 0}\left(1+\sqrt{1-x}\right) = 1+\sqrt{1} = 2.$$

EXAMPLE 9.3

Evaluate $\lim\limits_{x\to 3}\dfrac{\left(3-\sqrt{6+x}\right)}{x-3}$.

SOLUTION

When $x=3, \dfrac{\left(3-\sqrt{6+3}\right)}{3-3}=\dfrac{0}{0}$, which is an indeterminate form. As $f(x)$ is irrational, we multiply the numerator and denominator with the rationalizing factor of $f(x)$.

$$\lim_{x\to 3}\frac{\left(3-\sqrt{6+x}\right)}{x-3}=\lim_{x\to 3}\frac{\left(3-\sqrt{6+x}\right)}{x-3}\times\frac{\left(3+\sqrt{6+x}\right)}{\left(3+\sqrt{6+x}\right)}=\lim_{x\to 3}\frac{9-(6+x)}{(x-3)\left(3+\sqrt{6+x}\right)}$$

$$=\lim_{x\to 3}\frac{3-x}{(x-3)\left(3+\sqrt{6+x}\right)}=\lim_{x\to 3}\frac{-1}{\left(3+\sqrt{6+x}\right)}=\frac{-1}{\left(3+\sqrt{6+3}\right)}$$

$$=\frac{-1}{(3+3)}=-\frac{1}{6}.$$

EXAMPLE 9.4

Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{\sqrt{9+x}-\sqrt{9-x}}$.

SOLUTION

When $x=0, \dfrac{\sqrt{4}-\sqrt{4}}{\sqrt{9}-\sqrt{9}}=\dfrac{0}{0}$, which an indeterminate form.

Here $f(x)$ and $g(x)$ both are irrational functions. We multiply both the numerator and the denominator with their rationalizing factors.

$$\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{\sqrt{9+x}-\sqrt{9-x}}\times\frac{\sqrt{4+x}+\sqrt{4-x}}{\sqrt{9+x}+\sqrt{9-x}}\times\frac{\sqrt{9+x}-\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}$$

$$\lim_{x\to 0}\frac{(4+x)-(4-x)}{(9+x)-(9-x)}\times\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}=\lim_{x\to 0}\frac{2x}{2x}\times\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}$$

$$=\lim_{x\to 0}\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}=\frac{\sqrt{9}+\sqrt{9}}{\sqrt{4}+\sqrt{4}}=\frac{2\sqrt{9}}{2\sqrt{4}}=\frac{3}{2}.$$

Using the Formula

1. $\lim\limits_{x\to a}\left[\dfrac{x^n-a^n}{x-a}\right]=na^{n-1}$ (where 'n' is a any rational number).

Proof:

Case 1: n is a positive integer.

$$\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=\lim_{x\to a}\left[\frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+a^{n-1})}{x-a}\right]$$

$$=\lim_{x\to a}[x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+a^{n-1}]$$

$$=a^{n-1}+a^{n-2}\ a+a^{n-3}\ a^2+\cdots+a^{n-1}(n\text{ terms})$$

$$=a^{n-1}+a^{n-1}+a^{n-1}+\cdots+a^{n-1}(n\text{ terms})=n\cdot a^{n-1}.$$

Case 2: n is negative integer.
Let $n=-m$ (m is positive integer)

$$\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=\lim_{x\to a}\left[\frac{x^{-m}-a^{-m}}{x-a}\right]$$

$$=\lim_{x\to a}\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a}=\lim_{x\to a}\frac{\frac{a^m-x^m}{x^m a^m}}{x-a}=\lim_{x\to a}\frac{-(x^m-a^m)}{(x-a)x^m a^m}=-\lim_{x\to a}\frac{x^m-a^m}{x-a}\times\lim_{x\to a}\frac{1}{x^m a^m}$$

$$=-ma^{m-1}\frac{1}{a^m a^m}\qquad(\because\ m\text{ is positive interget using case (1)})$$

$$=-m\frac{a^{m-1}}{a^{2m}}=-ma^{m-1-2m}=(-m)a^{-m-1}$$

$$\Rightarrow\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=na^{n-1}.\qquad(\because\ n=-m)$$

Case 3: n is a rational number.

Let $n=\frac{p}{q}$ where p, q are integers and $q\neq 0$.

$$\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=\lim_{x\to a}\left[\frac{x^{p/q}-a^{p/q}}{x-a}\right]=\lim_{x\to a}\left[\frac{(x^{1/q})^p-(a^{1/q})^p}{x-a}\right]$$

Let $x^{1/q}=y$ then $x=y^q$ and $a^{1/q}=b$ then $a=b^q$.

$x\to a\Rightarrow y^q\to b^q\Rightarrow y\to b$.

$$\lim_{x\to a}\left[\frac{(x^{1/q})^p-(a^{1/q})^p}{x-a}\right]=\lim_{x\to a}\left[\frac{y^p-b^p}{y^q-b^q}\right]$$

$$=\lim_{y\to b}\frac{\frac{y^p-b^p}{y-b}}{\frac{y^q-b^q}{y-b}}=\frac{\lim_{y\to b}\frac{y^p-b^p}{y-b}}{\lim_{y\to b}\frac{y^q-b^q}{y-b}}=\frac{p\cdot b^{p-1}}{q\cdot b^{q-1}}\qquad(\because\ p,q\text{ are integers}).$$

$$=\left(\frac{p}{q}\right)b^{p-1-(q-1)}=\left(\frac{p}{q}\right)b^{p-q}=\left(\frac{p}{q}\right)b^{q((p/q)-1)}=\left(\frac{p}{q}\right)(b^q)^{((p/q)-1)}$$

$$\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=na^{n-1}.\qquad\left(\because\ n=\frac{p}{q};b^q=a\right)$$

Hence, when n is rational number,

$$\lim_{x \to a}\left[\frac{x^n - a^n}{x - a}\right] = na^{n-1}.$$

Note $\lim_{x \to a}\left[\frac{x^m - a^m}{x^n - a^n}\right] = \frac{m}{n}a^{m-n}.$

EXAMPLE 9.5

Evaluate $\lim_{x \to 4}\left[\frac{x^4 - 256}{x - 4}\right]$.

SOLUTION

$$\lim_{x \to 4}\left[\frac{x^4 - 256}{x - 4}\right]$$

$$= \lim_{x \to 4}\left[\frac{x^4 - 4^4}{x - 4}\right]$$

$$= 4 \times 4^{4-1} = 4 \times 4^3 = 256.$$

EXAMPLE 9.6

Evaluate $\lim_{x \to 3}\left[\frac{x^5 - 243}{x^2 - 9}\right]$.

SOLUTION

$$\lim_{x \to 3}\left[\frac{x^5 - 243}{x^2 - 9}\right] = \lim_{x \to 3}\left[\frac{x^5 - 3^5}{x^2 - 3^2}\right]$$

$$= \frac{5}{2}3^{5-2} \qquad \left(\because \lim_{x \to a}\left[\frac{x^m - a^m}{x^n - a^n}\right] = \frac{m}{n}a^{m-n}\right)$$

$$= \frac{5}{2} \cdot 3^3 = \frac{135}{2}.$$

EXAMPLE 9.7

If $\lim_{x \to 2}\left[\frac{x^n - 2^n}{x - 2}\right] = 32,$ then find the value of n.

SOLUTION

$$\lim_{x \to 2}\left[\frac{x^n - 2^n}{x - 2}\right] = n \cdot 2^{n-1}$$

$$32 = n \cdot 2^{n-1}$$

$$4 \times 8 = n \cdot 2^{n-1}$$

$$4 \times 2^3 = n \cdot 2^{n-1}$$

$$4 \times 2^{4-1} = n \times 2^{n-1}$$

$\therefore\ n = 4.$

Limits as x Tends to Infinity

We know, when x tends to infinity $\frac{1}{x}$ tends to 0. While evaluating limits at infinity, put $x = \frac{1}{y}$.

EXAMPLE 9.8

Evaluate $\lim\limits_{x\to\infty} \frac{2x+3}{x-5}$.

SOLUTION

Put $x = \frac{1}{y}$; if $x \to \infty$, $y \to 0$

$$\therefore \lim_{x\to\infty} \frac{2x+3}{x-5} \Rightarrow \lim_{y\to 0} \frac{\frac{2}{y}+3}{\frac{1}{y}-5} = \lim_{y\to 0} \frac{2+3y}{1-5y} = \frac{2+0}{1-0} = 2.$$

EXAMPLE 9.9

Evaluate $\lim\limits_{x\to\infty} \frac{3x^2+4x+5}{4x^2+7}$.

SOLUTION

Put $x = \frac{1}{y}$, when $x \to \infty$, $y \to 0$

$$\lim_{x\to\infty} \frac{3x^2+4x+5}{4x^2+7} = \lim_{y\to 0} \frac{3\cdot\frac{1}{y^2}+4\cdot\frac{1}{y}+5}{4\cdot\frac{1}{y^2}+7}$$

$$= \lim_{y\to 0} \frac{3+4y+5y^2}{4+7y^2} = \frac{3+0+0}{4+7(0)} = \frac{3}{4}.$$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. When the number of sides of a polygon tends to infinity, it approaches _______.
2. The scientists who put calculus in mathematical form are _______ and _______.
3. The radius of circle tends to zero then it approaches a _______.
4. _______ helps in finding the areas of the curves.
5. Evaluate $\lim\limits_{x\to 2}(2x-2)$.
6. The limiting position of a secant is _______.
7. $\lim\limits_{x\to a} x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n =$ _______.
8. $\lim\limits_{x\to 3} \log(4x-11) =$ _______.
9. A circle is inscribed in a polygon. As the number of sides increases, the difference in areas of circle and polygon _______.
10. $\lim\limits_{x\to 0} \dfrac{x^2+8x}{x} =$ _______.
11. $\lim\limits_{x\to\infty} \dfrac{n(n+1)}{n^2} =$ _______.
12. $\lim\limits_{x\to\infty} \dfrac{1}{n} =$ _______.
13. $\lim\limits_{x\to 1} \dfrac{\sqrt[5]{x}-1}{\sqrt[4]{x}-1} =$ _______.
14. $\lim\limits_{x\to -a} \dfrac{x^n+a^n}{x+a}$ (where n is an odd natural number) $=$ _______.
15. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} =$ _______.
16. $\lim\limits_{x\to 3} \sqrt{9-x^2} =$ _______.
17. Evaluate: $\lim\limits_{x\to 3} \dfrac{|x-3|}{x-3}$.
18. Evaluate: $\lim\limits_{x\to a} \left[\dfrac{x^n-a^n}{x^m-a^m}\right]$.
19. $\lim\limits_{x\to 5} \dfrac{\sqrt{x}-\sqrt{5}}{x-5} =$ _______.
20. If $\lim\limits_{x\to 0} \left[\dfrac{2x^2+3x+b}{x^2+4x+3}\right] = 2$, then the value of b is _______.
21. What is the value of $\lim\limits_{x\to 1}(x^3+1)(x^2-2x+4)$?
22. Evaluate: $\lim\limits_{x\to 3} \left[\dfrac{x^3-27}{x-3}\right]$.
23. Evaluate: $\lim\limits_{x\to\infty} \left[\dfrac{x^2+x+6}{x+1}\right]$.
24. Evaluate: $\lim\limits_{x\to\infty} \left[\dfrac{4x^n+1}{5x^{n+1}+5}\right]$.
25. Evaluate: $\lim\limits_{x\to a} \dfrac{x^{1/4}-a^{1/4}}{x^4-a^4}$.
26. Evaluate: $\lim\limits_{x\to -2} \dfrac{x^7+128}{x+2}$.
27. Evaluate: $\lim\limits_{x\to 1} \left[\dfrac{x^2+3x+2}{x^2-5x+3}\right]$.
28. Evaluate: $\lim\limits_{x\to 5} \sqrt{25-x^2}$.
29. In finding $\lim\limits_{x\to a} f(x)$, we replace x by $\dfrac{1}{n}$, then the limit becomes _______.
30. $\lim\limits_{x\to\infty} \dfrac{7x-3}{8x-10} =$ _______.

Short Answer Type Questions

31. Evaluate: $\lim\limits_{x\to\infty} \dfrac{11|x|+7}{8|x|-9}$.
32. Evaluate: $\lim\limits_{x\to 2} \left[\dfrac{2x^2-9x+10}{5x^2-5x-10}\right]$.
33. Evaluate: $\lim\limits_{x\to 3} \dfrac{x^5-243}{x-3}$.
34. Evaluate: $\lim\limits_{x\to\infty} \dfrac{x^5+3x^4-4x^3-3x^2+2x+1}{2x^5+4x^2-9x+16}$.

PRACTICE QUESTIONS

35. Evaluate: $\lim_{x\to a}\dfrac{x^{14}-a^{14}}{x^{-7}-a^{-7}}$.

36. Evaluate: $\lim_{x\to 2}\dfrac{x-2}{\sqrt{x+2}-2}$.

37. Evaluate: $\lim_{x\to 0}\dfrac{\sqrt{5+x}-\sqrt{5-x}}{\sqrt{10+x}-\sqrt{10-x}}$.

38. Evaluate: $\lim_{x\to 0}\dfrac{\sqrt{1+x+x^2+x^3}-1}{x}$.

39. $\lim_{x\to\infty}\dfrac{x^n+a^n}{x^n-a^n}=$ ______.

40. Evaluate: $\lim_{x\to 1}\left[\dfrac{x^4-2x^3-x^2+2x}{x-1}\right]$.

41. Evaluate: $\lim_{x\to a}\dfrac{\sqrt{x+a}-\sqrt{2a}}{x-a}$.

42. Evaluate: $\lim_{n\to\infty}\left(\sum_{r=0}^{n}\dfrac{1}{2^r}\right)$.

43. If $\lim_{x\to -3}\dfrac{x^k+3^k}{x+3}=405$, where k is an odd natural number then, $k=$ ______.

44. $\lim_{x\to 4^-}\dfrac{|x-4|}{x-4}=$ ______.

45. $\lim_{x\to 3}\dfrac{\log(2x-3)-\log(3x+2)}{\log(2x+1)}=$ ______.

Essay Type Questions

46. Evaluate: $\lim_{x\to\infty}\dfrac{n(1+4+9+16+\cdots+n^2)}{n^4+8n^3}$.

47. Evaluate: $\lim_{x\to\infty}\dfrac{n^2(1+2+3+4+\cdots+n)}{n^4+4n^2}$.

48. $\lim_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{5x-3}}{\sqrt{2x+3}-\sqrt{4x+1}}=$ ______.

49. $\lim_{n\to\infty}\dfrac{1+3+5+7+\ldots n \text{ terms}}{2+4+6+8+\ldots n \text{ terms}}=$ ______.

50. $\lim_{x\to 3}\left[\dfrac{2}{x-3}+\dfrac{2}{x^2-7x+12}\right]=$ ______.

CONCEPT APPLICATION

Level 1

1. Evaluate: $\lim_{x\to 3}(4x^2+3)=$ ______.

(a) 36 (b) 39

(c) 40 (d) None of these

2. $\lim_{x\to\infty}\dfrac{2x+4}{x-2}=$ ______.

(a) 1 (b) 0

(c) 2 (d) 6

3. $\lim_{x\to 5^+}\dfrac{|x-5|}{x-5}=$ ______.

(a) 1 (b) −1

(c) 0 (d) Cannot be determined

4. Evaluate: $\lim_{x\to 1}\dfrac{2x^2+4x+4}{2x-1}$.

(a) 1 (b) 10

(c) 20 (d) 5

5. Evaluate: $\lim_{x\to 20}\dfrac{\sqrt{x+5}+5}{\sqrt{x+5}-5}$.

(a) 1 (b) 2

(c) 4 (d) ∞

6. $\lim_{x\to 0}\dfrac{\sqrt{8-3x}+\sqrt{8+4x}}{\sqrt{2-3x}}=$ ______.

(a) 5 (b) 3

(c) 2 (d) 4

7. $\lim_{x\to\infty}\dfrac{(x-1)(2x-1)}{(x-4)(x-7)}=$ ______.

(a) 0 (b) 1

(c) 2 (d) −1

8. $\lim_{x\to 2}\dfrac{x^{-8}-\dfrac{1}{256}}{x-2}=$ ______.

PRACTICE QUESTIONS

(a) $-\frac{1}{32}$ (b) $-\frac{1}{128}$

(c) $-\frac{1}{256}$ (d) $-\frac{1}{64}$

9. $\lim_{x\to 2}\frac{x^2+2x-8}{2x^2-3x-2} =$ _______.

(a) $\frac{1}{5}$ (b) $\frac{6}{5}$

(c) $\frac{3}{2}$ (d) $-\frac{1}{6}$

10. $\lim_{x\to 4^-}\sqrt{16-x^2}$ is a/an

(a) complex number.
(b) real number.
(c) natural number.
(d) integer.

11. $\lim_{x\to 0}\log\left(\frac{2x+1}{5x+4}\right) =$ _______.

(a) $-2\log 2$ (b) $\log 4$

(c) $-\log\left(\frac{1}{2}\right)$ (d) $-3\log 2$

12. $\lim_{x\to 3}\frac{\sqrt{x+3}+\sqrt{x+6}}{\sqrt{x+1}-2} =$ _______.

(a) 1 (b) 7

(c) 14 (d) None of these

13. $\lim_{x\to 5}\frac{\sqrt{x+20}-\sqrt{3x+10}}{5-x} =$ _______.

(a) $-\frac{2}{5}$ (b) $\frac{2}{5}$

(c) $\frac{1}{5}$ (d) $\frac{-1}{5}$

14. $\lim_{x\to -3}\frac{x^2-2x-15}{x^2+2x-3} =$ _______.

(a) 1 (b) 2

(c) -3 (d) -4

15. $\lim_{x\to\infty}\frac{(x-5)(x+7)}{(x+2)(5x+1)} =$ _______.

(a) $-\frac{1}{5}$ (b) 25

(c) 5 (d) $\frac{1}{5}$

16. Evaluate: $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$, where $f(x) = x^2 - 4x$.

(a) -1 (b) 2

(c) 0 (d) 4

17. Evaluate: $\lim_{x\to 9}\frac{2x-7\sqrt{x}+3}{3x-11\sqrt{x}+6}$.

(a) $\frac{3}{4}$ (b) $\frac{5}{3}$

(c) $\frac{5}{7}$ (d) $\frac{3}{7}$

18. $\lim_{x\to b}\frac{\sqrt{x}-\sqrt{b}}{x-b} =$ _______.

(a) $\frac{1}{2b}$ (b) $\frac{\sqrt{2}}{b}$

(c) $\frac{1}{2\sqrt{b}}$ (d) $\frac{1}{\sqrt{2}\,b}$

19. $\lim_{x\to 3}\frac{x-3}{\sqrt{x+6}-\sqrt{2x+3}} =$ _______.

(a) -5 (b) -6

(c) 9 (d) 6

20. $\lim_{x\to -2}\frac{x+2}{\sqrt{2x+8}-\sqrt{2-x}} =$ _______.

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{4}{3}$

21. Evaluate: $\lim_{x\to 1}\left[\frac{\log x^3+\log\left(\frac{1}{x^2}\right)+\log 2}{\log(x^3+3)}\right]$.

(a) $\frac{3}{2}$ (b) 1

(c) $\frac{1}{2}$ (d) 2

22. $\lim_{x\to 3}\frac{x^{-5}-\frac{1}{243}}{x-3} =$ _______.

(a) $-\frac{5}{729}$ (b) $-\frac{5}{243}$

(c) $\frac{5}{81}$ (d) $\frac{5}{729}$

PRACTICE QUESTIONS

23. $\lim_{x\to\infty} \frac{4x-3}{(2x+3)} =$ _______.

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) 2

24. Evaluate: $\lim_{d\to 3} \frac{d^3-27}{d-3}$.

(a) 3 (b) 9

(c) 27 (d) 6

25. $\lim_{x\to 3^-} \sqrt{9-x^2}$ is a/an

(a) natural number.

(b) real number.

(c) imaginary number.

(d) integer.

26. Evaluate: $\lim_{x\to 4} \frac{3x-8\sqrt{x+4}}{5x-9\sqrt{x}-2}$.

(a) $\frac{1}{5}$ (b) $\frac{4}{11}$

(c) $\frac{3}{10}$ (d) 5

27. For some real number k, the value of $\lim_{x\to -k} \frac{x^5+k^5}{x+k}$ can be _______.

(a) 50 (b) 60

(c) 70 (d) 80

28. Evaluate: $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$, where $f(x) = x^2 - 2x$.

(a) −1 (b) 0

(c) 1 (d) 2

29. Evaluate: $\lim_{x\to a} \frac{x^{1/5}-a^{1/5}}{x^{-4/5}-a^{-4/5}}$.

(a) $\frac{-a}{4}$ (b) $\frac{-1}{4a}$

(c) $\frac{1}{4a}$ (d) $\frac{a}{4}$

30. $\lim_{x\to\infty} \frac{6x^4+7x^3+2x+1}{x^4+1} =$ _______.

(a) 1 (b) 2

(c) 3 (d) 6

Level 2

31. Evaluate: $\lim_{x\to -2} \frac{2x^3-x^2-13x-6}{3x^2+x-10}$.

(a) $-\frac{5}{7}$ (b) $-\frac{17}{5}$

(c) $-\frac{15}{11}$ (d) $-\frac{10}{3}$

32. Evaluate: $\lim_{x\to -1} \frac{\log x^2 - \log\left(\frac{1}{x^4}\right) + \log 3}{\log\left(\frac{x^3}{-3}\right)}$.

(a) 1 (b) 0

(c) −1 (d) Does not exist

33. $\lim_{x\to\infty} \left\{\frac{3x}{\sqrt{x^2+5x-6}+2x}\right\} =$ _______.

(a) −1 (b) 1

(c) 0 (d) ∞

34. $\lim_{x\to\infty} \frac{(4x+5)(2x-1)}{(27x^2+1)} =$ _______.

(a) $\frac{4}{27}$ (b) $\frac{8}{27}$

(c) $\frac{2}{27}$ (d) $\frac{6}{27}$

35. $\lim_{x\to 1} \frac{x^3-6x^2+11x-6}{x^2-5x+4} =$ _______.

(a) $\frac{2}{3}$ (b) $-\frac{5}{3}$

(c) $\frac{5}{3}$ (d) $\frac{-2}{3}$

36. If $\lim_{x\to a} \frac{x^5-a^5}{x-a} = 5$, then find the sum of the possible real values of a.

(a) 0 (b) 2

(c) 3 (d) 5

PRACTICE QUESTIONS

37. $\lim_{x\to 0} \dfrac{x}{\sqrt{3+x}-\sqrt{3-x}} =$ ______.

(a) $\sqrt{3}$ (b) $\sqrt{-3}$

(c) $2\sqrt{3}$ (d) $-2\sqrt{3}$

38. $\lim_{x\to 1} \dfrac{4-\sqrt{15+x}}{1-x} =$ ______.

(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{10}$ (d) $\dfrac{1}{12}$

39. $\lim_{x\to 0} \dfrac{\sqrt{1+x+x^2+x^3}-1}{x} =$ ______.

(a) 1 (b) 2

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$

40. $\lim_{x\to 0} \dfrac{\sqrt[4]{x+16}-2}{x} =$ ______.

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{32}$ (d) $\dfrac{1}{20}$

41. $\lim_{x\to\infty} \dfrac{(x+2)^{10}+(x+4)^{10}+\cdots+(x+20)^{10}}{x^{10}+1} =$ ______.

(a) 12 (b) 20

(c) 40 (d) 10

42. $\lim_{x\to\infty} \dfrac{2+6+10+14+\ldots 2n \text{ terms}}{2+4+6+8+\ldots n \text{ terms}} =$ ______.

(a) 4 (b) 16

(c) 8 (d) 10

43. Evaluate: $\lim_{x\to -a} \dfrac{1}{x}\left[\dfrac{1}{x+a}+\dfrac{b+a}{x^2-bx+ax-ab}\right]$.

(a) $\dfrac{1}{a}$ (b) $\dfrac{1}{a+b}$

(c) $\dfrac{1}{a(a+b)}$ (d) $\dfrac{1}{a(a-b)}$

44. $\lim_{x\to 5} \dfrac{\sqrt{2x-3}-\sqrt{3x-8}}{\sqrt{2x-1}-\sqrt{3x-6}} =$ ______.

(a) $\dfrac{2\sqrt{7}}{3}$ (b) $\dfrac{3}{2\sqrt{7}}$

(c) $\dfrac{4}{3}\sqrt{7}$ (d) $\dfrac{3}{\sqrt{7}}$

45. If $\lim_{x\to a} \dfrac{x^7-a^7}{x-a} = 7$, then find the number of possible real values of a.

(a) 0 (b) 1

(c) 2 (d) 7

Level 3

46. $\lim_{n\to\infty} \dfrac{1+4+9+16+\cdots+n^2}{4n^3+1} =$ ______.

(a) $\dfrac{1}{12}$ (b) $\dfrac{1}{24}$

(c) $\dfrac{1}{8}$ (d) $\dfrac{1}{16}$

47. $\lim_{x\to 0} \dfrac{\sqrt[5]{x+32}-2}{x} =$ ______.

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{45}$

(c) $\dfrac{1}{80}$ (d) $\dfrac{1}{100}$

48. $\lim_{x\to 2} \dfrac{\sqrt{x^3-2x^2+2x-3}-1}{x-2} =$ ______.

(a) 4 (b) 8

(c) 6 (d) 3

49. Evaluate: $\lim_{x\to\infty} \dfrac{(1^2+2^2+3^2+\cdots+n^2)}{n^2(1+3+5+7+\cdots+n \text{ terms})}$.

(a) 0 (b) 1

(c) ∞ (d) Does not exist

50. Evaluate: $\lim_{x\to 0} \dfrac{\sqrt{6+x}-\sqrt{6-x}}{\sqrt{8+x}-\sqrt{8-x}}$.

(a) $\sqrt{3}$ (b) $\dfrac{-5\sqrt{3}}{3}$

(c) $\dfrac{4\sqrt{3}}{3}$ (d) $\dfrac{2\sqrt{3}}{3}$

PRACTICE QUESTIONS

51. Evaluate: $\lim_{\theta\to 0} \cot\theta - \operatorname{cosec}\theta$.

(a) 0 (b) 2

(c) 4 (d) None of these

52. $\lim_{x\to\infty} \dfrac{(x+1)^{100}+(x+2)^{100}+\cdots+(x+50)^{100}}{x^{100}+x^{99}+\cdots+x+1}$ = ______.

(a) 100 (b) 1

(c) 21 (d) 50

53. $\lim_{x\to 3^-} \dfrac{|2x-6|}{2x-6}$ is ______.

(a) 1 (b) −1

(c) 0 (d) ∞

54. If $\lim_{x\to -2} \dfrac{x^p+2^p}{x+2} = 80$ (where p is an odd number), then p can be ____.

(a) 3 (b) 5

(c) 7 (d) 9

55. If $\lim_{x\to m} \dfrac{x^3-m^3}{x-m} = 3$, then find the number of possible values of m.

(a) 0 (b) 2

(c) 1 (d) 3

56. $\lim_{x\to 3} \dfrac{\sqrt{5x+1}-\sqrt{7x-5}}{\sqrt{7x+4}-\sqrt{5x+10}}$ = _______.

(a) 0 (b) $-\dfrac{5}{4}$

(c) $\dfrac{5}{4}$ (d) ∞

57. $\lim_{x\to a}(x-a)\left(\dfrac{1}{x-a}-\dfrac{1}{x^2-(a+b)x+ab}\right)$ = ______.

(a) $\dfrac{a-b-1}{a-b}$ (b) 0

(c) 1 (d) $\dfrac{a+b}{a-b}$

58. $\lim_{x\to \frac{2}{5}} \dfrac{1}{|5x-2|}$ = _______.

(a) 0 (b) ∞

(c) 1 (d) Does not exist

59. $\lim_{x\to\infty} \dfrac{5x^2}{\sqrt{25x^4+13x^3+14x^2+17x+6}}$ is ______.

(a) 1 (b) $\dfrac{1}{5}$

(c) $\dfrac{5}{14}$ (d) None of these

60. $\lim_{x\to -1} \dfrac{x^5+1}{x+1}$ = _______.

(a) 1 (b) −5

(c) 5 (d) None of these

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. circle
2. Newton and Leibnitz
3. point
4. integration
5. 2
6. tangent
7. na^n
8. 0
9. decreases
10. 8
11. 1
12. 0
13. $\frac{4}{5}$
14. $n(-a)^{n-1}$
15. -1
16. does not exist
17. Limit does not exist
18. $\frac{n}{m}a^{n-m}$
19. $\frac{1}{2\sqrt{5}}$
20. 6
21. 6
22. 27
23. ∞
24. 0
25. $\frac{1}{16}\cdot a^{-15/4}$
26. 448
27. -6
28. Limit does not exist
29. $\lim\limits_{\frac{1}{n}\to a} f\left(\frac{1}{n}\right)$
30. $\frac{7}{8}$

Short Answer Type Questions

31. $\frac{11}{8}$
32. $\frac{-1}{15}$
33. 405
34. $\frac{1}{2}$
35. $-2a^{21}$
36. 4
37. $\sqrt{2}$
38. $\frac{1}{2}$
39. 1
40. -2
41. $\frac{1}{2\sqrt{2a}}$
42. 2
43. -1
44. Cannot be determined
45. $\log_7\left(\frac{6}{11}\right)$

Essay Type Questions

46. $\frac{1}{3}$
47. $\frac{1}{2}$
48. $\sqrt{10}$
49. 1
50. 0

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (d)	**6.** (d)	**7.** (c)	**8.** (d)	**9.** (b)	**10.** (b)
11. (a)	**12.** (d)	**13.** (c)	**14.** (b)	**15.** (d)	**16.** (c)	**17.** (c)	**18.** (c)	**19.** (b)	**20.** (d)
21. (c)	**22.** (a)	**23.** (d)	**24.** (c)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (d)

Level 2

31. (c)	**32.** (c)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (a)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (c)
41. (d)	**42.** (c)	**43.** (c)	**44.** (d)	**45.** (c)					

Level 3

46. (a)	**47.** (c)	**48.** (d)	**49.** (a)	**50.** (d)	**51.** (a)	**52.** (d)	**53.** (b)	**54.** (b)	**55.** (b)
56. (b)	**57.** (a)	**58.** (b)	**59.** (a)	**60.** (c)					

CONCEPT APPLICATION

Level 1

1. Substitute $x = 3$ in the given function.
2. Divide both numerator and denominator by x, then use if $x \to \infty$ then $\frac{1}{x} \to 0$.
3. When $x > 5, |x - 5| = x - 5$.
4. Substitute $x = 1$ in the given expression.
5. Substitute $x = 20$.
6. Substitute $x = 0$.
7. Divide both numerator and denominator by x^2. Now substitute, $\frac{1}{x} = 0$.
8. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.
9. Factorize the numerator and denominator and remove common factor. Now substitute $x = 2$.
10. Apply the concept of left hand limit.
11. Substitute $x = 0$.
12. Rationalize the denominator.
13. Rationalize the numerator and cancel common factor. Then substitute, $x = 5$.
14. Factorize the numerator and denominator and cancel the common factor and then substitute $x = -3$.
15. Divide both the numerator and the denominator by x^2. Then substitute, $\frac{1}{x} = 0$.
16. Substitute then factorize numerator $f(x)$, then cancel the common factor in both the numerator and the denominator, then substitute $x = 2$.
17. Factorize numerator and denominator, eliminate the common factor then substitute $x = 9$.
18. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.
19. Rationalize the denominator and remove the common factor. Now substitute $x = 3$.
20. Rationalize the denominator and cancel the common factor and then substitute $x = -2$.
21. Put $x = 1$ in the given limit.
22. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, by converting the given limit into the form.
23. Divide both the numerator and the denominator by x and apply the concept hat, as $x \to \infty$, $\frac{1}{x} \to 0$.
24. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.
25. Apply the concept of left hand limit.
26. Factorize the numerator and denominator and cancel the common factor which is in both numerator and denominator then substitute $x = 4$.
27. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.
28. Calculate $f(1)$ substitute $f(x)$ and $f(1)$, factorise numerator, then cancel the common factor and substitute $x = 1$.
29. Use the formula, $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$.
30. (i) The highest power of x is to be taken common from numerator and denominator.

    (ii) Put $\frac{1}{x}, \frac{1}{x^2}, \ldots = 0$ as $x \to \infty \frac{1}{x}, \frac{1}{x^2} \to 0$.

Level 2

31. Factorize the numerator and the denominator and cancel the common factor then substitute $x = -2$.
32. Put $x = -1$ in the given limit.
33. Numerator and denominator are of the same degree.

    $\therefore$ Limiting value of the given function

    $= \frac{\text{Coefficient of } x \text{ in numerator}}{\text{Coefficient of } x \text{ in denominator}}$.

HINTS AND EXPLANATION

34. Divide both the numerator and the denominator by x^2 and then use when

$x \to \infty \Rightarrow \frac{1}{x} \to 0.$

35. Factorize the numerator and the denominator, and cancel the common factor, then substitute $x = 1$.

36. (i) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

(ii) Equating na^{n-1} with 5 obtain the value of a.

(iii) Add all the possible real values of a.

37. Rationalize the denominator and cancel common factor and then substitute $x = 0$.

38. Rationalize the numerator and cancel the common factor and then substitute $x = 1$.

39. Rationalize the numerator and cancel the common factor x, then substitute $x = 0$.

40. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

41. Take as the highest power of x common and cancel it, then substitute $\frac{1}{x} = 0$ as $x \to \infty, \frac{1}{x} \to 0$.

42. Find the sum of $2n$ terms in numerator and sum of n terms in denominator and take the highest power of n as common in both the numerator and the denominator and then substitute $\frac{1}{n} = 0$ as $n \to \infty, \frac{1}{n} \to 0$.

43. Simplify and then factorize the numerator. Cancel the common factor and then substitute $x = -a$.

44. Rationalize the numerator and denominator and cancel the common factor and then substitute $x = 5$.

45. (i) Use the formula, $\lim_{x\to a} \frac{x^m - a^m}{n^n - a^n} = \frac{m}{n} a^{m-n}$.

(ii) $7a^6 = 7$.

(iii) Now, find the number of possible real values of a.

Level 3

46. Write formula for sum of the squares of first n natural numbers, i.e., $\frac{n(n+1)(2n+1)}{6}$.

Divide the numerator and the denominator by n^3.

Then substitute $\frac{1}{n} = 0$, as $n \to \infty, \frac{1}{n} = 0$.

47. Use the formula, $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

48. (i) Rationalize the numerator and find its factors.

(ii) Cancel the common factor of numerator and denominator and substitute the value of limit.

49. Substitute the formulae for sum of the squares of n natural numbers and sum of n natural number and sum of n odd natural numbers and simplify.

50. Rationalize both numerator and denominator.

51. $\lim_{\theta\to 0} \cot\theta - \operatorname{cosec}\theta$

$= \infty - \infty$, i.e., indeterminate form.

$\cot\theta - \operatorname{cosec}\theta$

$$= \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \frac{\cos\theta - 1}{\sin\theta}$$

$$= \frac{\cos^2\theta - 1}{\sin\theta(\cos\theta + 1)}$$

$$= \frac{-\sin\theta}{\sin\theta(\cos\theta+1)} = \frac{-\sin\theta}{\cos\theta + 1}$$

$\therefore \lim_{\theta\to 0} \cot\theta - \operatorname{cosec}\theta$

$$= \lim_{\theta\to 0} \frac{-\sin\theta}{\cos\theta + 1} = \frac{-\sin 0}{\cos 0 + 1} = \frac{-0}{1+1} = 0.$$

52. $\lim_{x\to\infty} \frac{(x+1)^{100} + (x+2)^{100} + \cdots + (x+50)^{100}}{x^{100} + x^{99} + \cdots + x + 1}$

$$\lim_{\frac{1}{x}\to\infty} \frac{x^{100}\left(\left(1+\frac{1}{x}\right)^{100} + \left(1+\frac{2}{x}\right)^{100} + \cdots + \left(1+\frac{50}{x}\right)^{100}\right)}{x^{100}\left(1 + \frac{1}{x} + \cdots + \frac{1}{x^{99}} + \frac{1}{x^{100}}\right)}$$

$$= \frac{1 + 1 + \cdots + 1 \text{ (upto 50 terms)}}{1} = 50.$$

53. $\lim_{x\to 3^-} \frac{|2x-6|}{2x-6}$

$$= \lim_{x\to 3^-} \frac{-(2x-6)}{(2x-6)} \text{(for } x < 3, 2x - 6 < 0) = -1.$$

54. $\lim_{x\to -2} \dfrac{x^p + 2^p}{x+2} = 80$

$\Rightarrow \lim_{x\to -2} \dfrac{x^p - (-2)^p}{x-(-2)} = 80$

$\Rightarrow p(-2)^{p-1} = 5(-2)^{5-1} \therefore p = 5.$

55. $\lim_{x\to m} \dfrac{x^3 - m^3}{x-m} = 3$

$\Rightarrow 3m^2 = 3 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

$\therefore$ There are two values for m.

56. $\lim_{x\to 3} \dfrac{\sqrt{5x+1} - \sqrt{7x-5}}{\sqrt{7x+4} - \sqrt{5x+10}} = \dfrac{0}{0}$

That is, indeterminate form.
Multiply both the numerator and the denominator with $\left(\sqrt{5x+1} + \sqrt{7x-5}\right)\left(\sqrt{7x+4} + \sqrt{5x+10}\right)$

$\Rightarrow \lim_{x\to 3} \dfrac{[5x+1-(7x-5)]\left(\sqrt{7x+4} + \sqrt{5x+10}\right)}{(7x+4-5x-10)\left(\sqrt{5x+1} + \sqrt{7x-5}\right)}$

$\Rightarrow \lim_{x\to 3} \dfrac{(-2x+6)\left(\sqrt{7x+4} + \sqrt{5x+10}\right)}{(2x-6)\left(\sqrt{5x+1} + \sqrt{7x-5}\right)}$

$= \lim_{x\to 3} \dfrac{-\left(\sqrt{7x+4} + \sqrt{5x+10}\right)}{\left(\sqrt{5x+1} + \sqrt{7x-5}\right)}$

$= \dfrac{-\left(\sqrt{25} + \sqrt{25}\right)}{\left(\sqrt{16} + \sqrt{16}\right)} = \dfrac{-10}{8} = \dfrac{-5}{4}.$

57. $\lim_{x\to a} (x-a)\left(\dfrac{1}{x-a} - \dfrac{1}{x^2 - (a+b)x + ab}\right)$

$= \lim_{x\to a} (x-a)\left(\dfrac{1}{x-a} - \dfrac{1}{(x-a)(x-b)}\right)$

$= \lim_{x\to a} (x-a)\left(\dfrac{x-b-1}{(x-a)(x-b)}\right)$

$\lim_{x\to a} (x-a)\left(\dfrac{x-b-1}{x-b}\right) = \dfrac{a-b-1}{a-b}.$

58. $\lim_{x\to \frac{2}{5}} \dfrac{1}{|5x-2|}$

As $|5x-2| \geq 0$

$\lim_{x\to \frac{2}{5}} \dfrac{1}{|5x-2|}$ exists

$\therefore \lim_{x\to \frac{2}{5}} \dfrac{1}{|5x-2|} = \dfrac{1}{0} = \infty.$

59. $\lim_{x\to \infty} \dfrac{5x^2}{\sqrt{25x^4 + 13x^3 + 14x^2 + 17x + 6}}$

$= \lim_{x\to \infty} \dfrac{5x^2}{x^2\sqrt{25 + 13\left(\frac{1}{x}\right) + 14\left(\frac{1}{x^2}\right) + 17\left(\frac{1}{x^3}\right) + 6\left(\frac{1}{x^4}\right)}}$

As $x \to \infty, \dfrac{1}{x} \to 0$

$= \lim_{\frac{1}{x}\to 0} \dfrac{5}{\sqrt{25 + 13\left(\frac{1}{x}\right) + 14\left(\frac{1}{x}\right)^2 + 17\left(\frac{1}{x}\right)^3 + 6\left(\frac{1}{x}\right)^4}}$

$= \dfrac{5}{\sqrt{25+10}} = \dfrac{5}{5} = 1.$

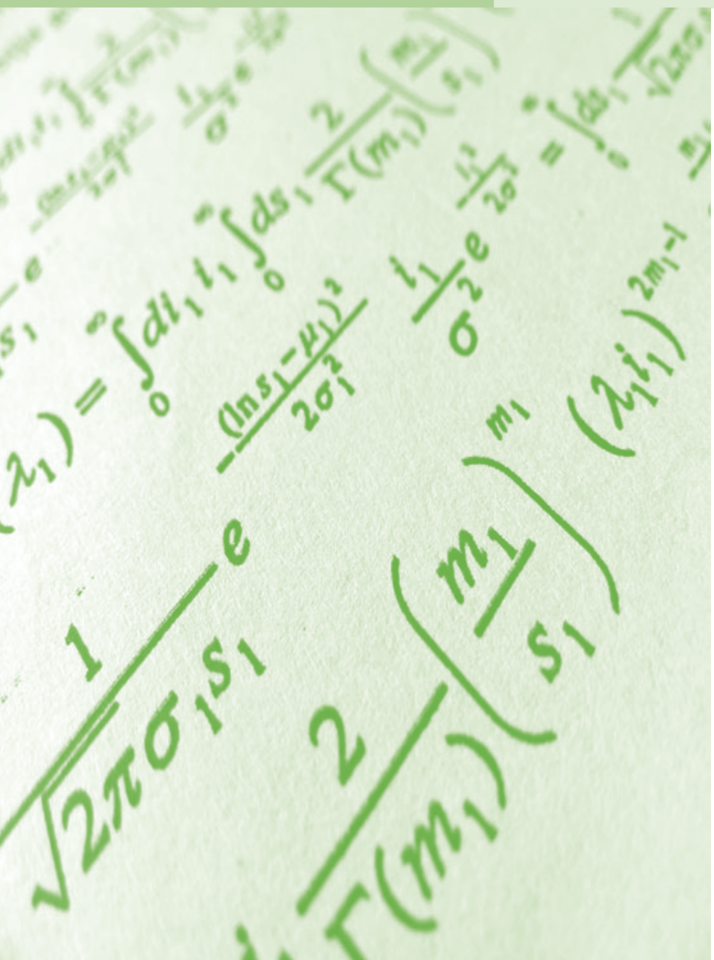
60. $\lim_{x\to -1} \dfrac{x^5 + 1}{x+1}$

$\lim_{x\to -1} \dfrac{x^5 - (-1)^5}{x-(-1)} = 5(-1)^{5-1} = 5(-1)^4 = 5.$

Chapter

10

Matrices



REMEMBER

Before beginning this chapter, you should be able to:

- State terms such as matrix, order of a matrix, etc.
- Apply basic operations on matrices

KEY IDEAS

After completing this chapter, you would be able to:

- Obtain order of a matrix
- Apply operations on matrices such as addition, subtraction, multiplication and study their properties
- Calculate determinant
- Obtain solution of simultaneous linear equations of two variables using matrix method

INTRODUCTION

A *matrix* is a rectangular arrangement of a set of elements in the form of horizontal and vertical lines. The elements can be numbers (real or complex) or variables. Matrices is the plural of matrix.

Horizontal line of elements is called a row and the vertical line of elements is called a column. The rectangular array of elements in a matrix are enclosed by brackets [ ] or parenthesis ( ).

Generally, we use capital letters to denote matrices.

Examples:

1. $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \end{bmatrix}$ is a matrix having 2 rows and 3 columns.

Here, the elements of matrix are numbers.

2. $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix having 2 rows and 2 columns.

Here, the elements of matrix are variables.

ORDER OF A MATRIX

If a matrix A has 'm' rows and 'n' columns, then $m \times n$ is called the order (or type) of matrix, and is denoted as $A_{m\times n}$.

Examples:

1. $A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 1 \end{bmatrix}$ is a matrix consisting of 2 rows and 3 columns. So its order is 2×3.

2. $B = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is a matrix consisting of 3 rows and 1 column. So its order is 3×1.

So in general, a set of mn elements can be arranged as a matrix having m rows and n columns as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdot & a_{2n} \\ a_{31} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}_{m\times n} \quad \text{or} \quad A = [a_{ij}]_{m\times n}$$

In the above matrix a_{ij} represents an element of the matrix occurring in ith row and jth column.

In general, a_{ij} is called (i, j)th element of the matrix.

Hence, in a particular matrix, we can note that (3, 4)th element is the element occurring in 3rd row and 4th column.

(1, 3)rd element is the element occurring is 1st row and 3rd column.

Example: Let $P = \begin{bmatrix} 2 & 3 & 51 \\ 4 & -2 & -3 \\ 5 & -31 & 1 \end{bmatrix}$

In this matrix we have,

(1, 1)th element = 2; (1, 2)th element = 3; (1, 3)th element = 51

(2, 1)th element = 4; (2, 2)th element = −2; (2, 3)th element = −3

(3, 1)th element = 5; (3, 2)th element = −31; (3, 3)th element = 1.

In compact form any matrix A can be represented as

$$A = [a_{ij}]_{m\times n}, \text{ where } 1 \le i \le m,$$
$$1 \le j \le n.$$

Various Types of Matrices

Rectangular Matrix

In a matrix if number of rows is not equal to number of columns, then the matrix is called a rectangular matrix.

Examples:

1. $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}_{3\times 2}$; A has 3 rows and 2 columns.
2. $B = \begin{bmatrix} -2 & 3 & 1 & -4 \\ 5 & -1 & 0 & 1 \end{bmatrix}_{2\times 4}$; B has 2 rows and 4 columns.

Row Matrix

A matrix which has only one row is called a row matrix.

Examples:

1. $[5 \quad -3 \quad 2 \quad 1]$ is a row matrix of order 1×4.
2. $[4 \quad 3 \quad 13 \quad -4 \quad -31]$ is a row matrix of order 1×5.

In general, order of any row matrix is $1 \times n$, where n is number of columns and $n = 2, 3, 4, \ldots$.

Column Matrix

A matrix which has only one column is called a column matrix.

Examples:

1. $\begin{bmatrix} 5 \\ -3 \\ 2 \\ -1 \end{bmatrix}_{4\times 1}$ is a column matrix of order 4×1.

2. $\begin{bmatrix} -3 \\ 5 \\ 20 \\ -2006 \\ 2 \\ -1 \end{bmatrix}_{6\times 1}$ is a column matrix of order 6×1.

In general, order of any column matrix is $m \times 1$, where m is number of rows in the matrix and $m = 2, 3, 4, \ldots$.

Null Matrix or Zero Matrix

If every element of a matrix is zero, then the matrix is called null matrix or zero matrix. A zero matrix of order $m \times n$ is denoted by $O_{m\times n}$ or in short by O.

Examples:

1. $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{2\times 4}$ **2.** $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$ **3.** $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

Square Matrix

In a matrix, if number of rows is equal to number of columns, then the matrix is called a square matrix. A matrix of order $n \times n$ is termed as a square matrix of order n.

Examples:

1. $\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$ is a square matrix of order 2.

2. $\begin{bmatrix} a & b & -3 \\ 4 & c & -2 \\ y & x & z \end{bmatrix}$ is a square matrix of order 3.

Principal Diagonal of a Square Matrix In a square matrix A of order n, the elements a_{ii} (i.e., a_{11}, a_{22}, …, a_{nn}) constitute principal diagonal. The elements a_{ii} are called elements of principal diagonal.

Examples:

1. $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

The elements 2 and 5 constitute the principal diagonal of A.

2. $P = \begin{bmatrix} -3 & 4 & 5 \\ 6 & -2 & 1 \\ a & 3 & b \end{bmatrix}$

The elements -3, -2 and b constitute principal diagonal of P.

Diagonal Matrix

In a square matrix, if all the non-diagonal elements are zeroes and at least one principal diagonal element is non-zero, then the matrix is called diagonal matrix.

Examples:

1. $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ is a diagonal matrix of order 2.

2. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a diagonal matrix of order 3.

Scalar Matrix

In a matrix, if all the diagonal elements are equal and rest of the elements are zeroes, then the matrix is called scalar matrix.

Examples:

1. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a scalar matrix of order 2.

2. $\begin{bmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{bmatrix}$ is a scalar matrix of order 3.

Identity Matrix or Unit Matrix

In a square matrix, if all the principal diagonal elements are unity and rest of the elements are zeroes, then the square matrix is called identity matrix or unit matrix. It is denoted by I.

Examples:

1. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2.

2. $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 4.

Comparable Matrices

Two matrices A and B can be compared, only when they are of same order.

Example: Consider two matrices A and B, given by

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 21 \\ 10 & -4 \end{bmatrix}_{3\times 2} \quad \text{and } B = \begin{bmatrix} -3 & 10 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}_{3\times 2}$$

Matrices A and B can be compared as both of them are of order 3×2.

Equality of Two Matrices

Two matrices are said to be equal only when,

1. they are of same order and
2. corresponding elements of both the matrices are equal.

Example: If $\begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 6 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ a & 2 \\ 6 & b \end{bmatrix}$ are equal matrices, then $a = 1$ and $b = -2$.

OPERATIONS ON MATRICES

Multiplication of a Matrix by a Scalar

If every element of a matrix A is multiplied by a number (real or complex) k, the matrix obtained is k times of A and is denoted by kA and the operation is called scalar multiplication.

EXAMPLE 10.1

If $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix}$, then find **(a)** $-A$ **(b)** $3A$ **(c)** $\frac{1}{4}A$.

SOLUTION

(a) $-A = -\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -1\times 2 & -1\times 3 & -1\times(-1) \\ -1\times 5 & -1\times 6 & -1\times 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -5 & -6 & -1 \end{bmatrix}$.

(b) $3A = 3\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 2 & 3\times 3 & 3(-1) \\ 3\times 5 & 3\times 6 & 3\times 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & -3 \\ 15 & 18 & 3 \end{bmatrix}$.

(c) $\frac{1}{4}A = \frac{1}{4}\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\times 2 & \frac{1}{4}\times 3 & \frac{1}{4}\times(-1) \\ \frac{1}{4}\times 5 & \frac{1}{4}\times 6 & \frac{1}{4}\times 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{-1}{4} \\ \frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$.

Notes

1. If a and b are any two scalars and P is a matrix, then $a(bP) = (ab)P$.
2. If m and n are any two scalars and A is a matrix, then $(m + n)\,A = mA + nA$.

Addition of Matrices

1. Two matrices A and B can be added only when they are of same order.

Example: Let $A = \begin{bmatrix} -3 & 2 & 1 \\ 5 & 6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -13 & 21 & 33 \\ -52 & 4 & 49 \end{bmatrix}$.

Here, both matrices A and B are of order 2×3. So, they can be added.

2. The **sum** matrix of two matrices A and B is obtained by adding the corresponding elements of A and B and the **sum** matrix is of same order as that of A or B.

3. Here, $A+B=\begin{bmatrix} -3+(-13) & 2+21 & 1+33 \\ 5+(-52) & 6+4 & -5+49 \end{bmatrix}=\begin{bmatrix} -16 & 23 & 34 \\ -47 & 10 & 44 \end{bmatrix}$.

Note If two matrices are of different orders, then their sum is not defined.

Properties of Matrix Addition

1. Matrix addition is commutative, i.e., if A and B are two matrices of same order, then $A+B=B+A$.
2. Matrix addition is associative, i.e., if A, B and C are three matrices of same order, then $A+(B+C)=(A+B)+C$.
3. **Additive identity:** If $O_{m\times n}$ is a null matrix of order $m \times n$ and A is any matrix of order $m \times n$, then $A+O=O+A=A$.

   So, O is called additive identity.
4. **Additive inverse:** If $A_{m\times n}$ is any matrix of order $m \times n$, then $A+(-A)=(-A)+A=O$.

   So, $-A$ is called additive inverse of the matrix A.
5. If k is a scalar and A and B are two matrices of same order, then $k(A+B)=kA+kB$.

Matrix Subtraction

1. Matrix subtraction is possible only when both the matrices are of same order.
2. The difference of two matrices of same type (or order) A and B, i.e., $A-B$, is obtained by subtracting corresponding element of B from that of A.
3. The difference matrix is of the same order as that of A or B.

EXAMPLE 10.2

If $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix}$ then find $A-B$.

SOLUTION

$$A-B=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}-\begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 2-(-3) & 3-1 \\ -1-4 & 4-(-2) \end{bmatrix}=\begin{bmatrix} 5 & 2 \\ -5 & 6 \end{bmatrix}.$$

Transpose of a Matrix

For a given matrix A, the matrix obtained by interchanging its rows and columns is called transpose of the matrix A and is denoted by A^T.

Examples:

1. If $A=\begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & -6 & 11 & -1 \end{bmatrix}$, then

$$\text{Transpose of } A = A^T = \begin{bmatrix} 2 & 5 \\ -1 & -6 \\ 3 & 11 \\ 4 & -1 \end{bmatrix}.$$

Here, we can note that order of A is 2×4, while that of A^T is 4×2.

2. $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, then $A^T = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix}$.

3. $A = \begin{bmatrix} -1 \\ 2004 \\ 53 \\ -47 \end{bmatrix}$, then $A^T = [-1 \quad 2004 \quad 53 \quad -47]$.

4. If $A = [5003]$, then $A^T = [5003]$.

Notes

1. If the order of a matrix is $m \times n$, then order of transpose of the matrix is $n \times m$.
2. $(A^T)^T = A$.

Example: Let $A = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} -2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix} = A.$$

3. If A and B are two matrices of same order, then $(A + B)^T = A^T + B^T$.
4. If k is a scalar and A is any matrix, then $(kA)^T = kA^T$.

Symmetric Matrix

A square matrix is said to be symmetric if the transpose of the given matrix is equal to the matrix itself.

Hence, a square matrix A is symmetric.

$$\Rightarrow A = A^T.$$

Examples:

1. If $A = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}$.

   Thus, we can observe that $A^T = A$, so A is a symmetric matrix.

2. Similarly for $P = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & 4 \\ 43 & 4 & 2 \end{bmatrix}$,

$$P^T = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & 4 \\ 43 & 4 & 2 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & 4 \\ 43 & 4 & 2 \end{bmatrix} = P.$$

So, P is a symmetric matrix.

Skew-symmetric Matrix

A square matrix A is said to be skew-symmetric if $A^T = -A$, i.e., transpose of the matrix is equal to its additive inverse.

Example: If $A = \begin{bmatrix} 0 & 2003 \\ -2006 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & -2006 \\ 2006 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2006 \\ -2006 & 0 \end{bmatrix} = -A.$

So, A is a skew-symmetric matrix.

Notes

1. For a square matrix A, $\frac{1}{2}(A + A^T)$ is always a symmetric matrix.

2. For a square matrix A, $\frac{1}{2}(A - A^T)$ is always a skew-symmetric matrix.

Multiplication of Matrices

Two matrices A and B can be multiplied only if the number of columns in A is equal to the number of rows in B.

Suppose order of matrix A is $m \times q$. Then order of matrix B, such that AB exists, should be of the form $q \times n$. Further order of the product matrix AB will be $m \times n$.

Now, consider a matrix A of order 2×3 and another matrix B of the order 3×4. As the number of columns in $A(=3)$ is equal to number of rows in $B(=3)$. So, AB exists and it is of the order 2×4. We can obtain the product matrix AB as follows:

(1, 1)th element of AB

= Sum of products of elements of first row of A with the corresponding elements of first column of B.

(1, 2)th element of AB

= Sum of products of elements of first row of A with the corresponding elements of second column of B.

(2, 1)th element of AB

= Sum of products of elements of second row of A with the corresponding elements of first column of B and so on.

In general, (i, j)th element of AB

= Sum of products of elements of ith row in A with the corresponding elements of jth column in B.

Following example will clearly illustrate the method:

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \\ 3 & 5 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} -5 & 6 & 4 \\ 9 & 11 & 8 \end{bmatrix}_{2\times 3}$

As A is of order 3×2 and B is of order 2×3, AB will be of the order 3×3.

$$\therefore AB = \begin{bmatrix} 2\times(-5)+(-1)\times 9 & 2\times 6+(-1)\times 11 & 2\times 4+(-1)\times 8 \\ 1\times(-5)+7\times 9 & 1\times 6+7\times 11 & 1\times 4+7\times 8 \\ 3\times(-5)+5\times 9 & 3\times 6+5\times 11 & 3\times 4+5\times 8 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -10+(-9) & 12+(-11) & 8+(-8) \\ -5+63 & 6+77 & 4+56 \\ -15+45 & 18+55 & 12+40 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -19 & 1 & 0 \\ 58 & 83 & 60 \\ 30 & 73 & 52 \end{bmatrix}_{3\times 3}.$$

In general, if $A = [a_{ip}]$ is a matrix of order $m \times q$ and $B = [b_{pj}]$ is a matrix of order $q \times n$, then the product matrix $AB = Q = [x_{ij}]$ will be of the order $m \times n$ and is given by $x_{ij} = \sum_{p=1}^{q} a_{ip} b_{pj}$.

This evaluation is made clear in the following:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdot & x_{1n} \\ x_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{31} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_{ij} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{m1} & x_{m2} & x_{m3} & \cdot & \cdot & x_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1q} & \cdot & a_{1n} \\ a_{21} & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & \cdot & \cdot & a_{iq} & \cdot & a_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdot & b_{1q} & \cdot & b_{1n} \\ b_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i1} & \cdot & \cdot & b_{iq} & \cdot & b_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1} & \cdot & \cdot & b_{mq} & \cdot & b_{mn} \end{bmatrix}.$$

Properties of Matrix Multiplication

1. In general, matrix multiplication is not commutative, i.e., $AB \neq BA$.

Example: Let $A = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} -3\times 2 + 1\times 0 & -3\times(-1) + 1\times 1 \\ 0\times 2 + 2\times 0 & 0(-2) + 2\times 1 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 0 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-3) + (-1)\times 0 & 2\times 1 + (-1)\times 2 \\ 0(-3) + 1\times 0 & 0\times 1 + 1\times 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}.$$

So, we can observe that $AB \neq BA$.

2. Matrix multiplication is associative, i.e., $A(BC) = (AB)C$.
3. Matrix multiplication is distributive over addition, i.e.,
   (i) $A(B + C) = AB + AC$,
   (ii) $(B + C)A = BA + CA$.
4. For any two matrices A and B if $AB = O$, then it is not necessarily imply that $A = O$ or $B = O$ or both A and B are zero.

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -12 \\ -4 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 8 & -12 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 2(8) + 4(-4) & 2(-12) + 4(6) \\ 4(8) + 8(-4) & 4(-12) + 8(6) \end{bmatrix}$$

$$= \begin{bmatrix} 16 - 16 & -24 + 24 \\ 32 - 32 & -48 + 48 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we can observe that, though $AB = O$, $A \neq O$ and $B \neq O$.

5. For any three matrices A, B and C if $AB = AC$, then it is not necessarily imply that $B = C$ or $A = O$ (But in case of any three real numbers a, b and c, if $ab = ac$ and $a \neq 0$, then it is necessary that $b = c$).

Example: Let $A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$, $B = \begin{bmatrix} 10 & 5 \\ 15 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} -10 & 35 \\ 25 & -5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} 10 & 5 \\ 15 & 10 \end{bmatrix} = \begin{bmatrix} 5\times 10 + 10\times 15 & 5\times 5 + 10\times 10 \\ 10\times 10 + 20\times 15 & 10\times 5 + 20\times 10 \end{bmatrix}$$

$$= \begin{bmatrix} 50 + 150 & 25 + 100 \\ 100 + 300 & 50 + 200 \end{bmatrix} = \begin{bmatrix} 200 & 125 \\ 400 & 250 \end{bmatrix}$$

$$AC = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} -10 & 35 \\ 25 & -5 \end{bmatrix} = \begin{bmatrix} 5(-10) + 10(25) & 5(35) + 10(-5) \\ 10(-10) + 20(25) & 10(35) + 20(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -50 + 250 & 175 - 50 \\ -100 + 500 & 350 - 100 \end{bmatrix} = \begin{bmatrix} 200 & 125 \\ 400 & 250 \end{bmatrix}$$

Here, $AB = AC$, but $B \neq C$.

6. If A is a square matrix of order n and I is the identity matrix of order n, then $AI = IA = A$, i.e., I is the multiplicative identity matrix.

Example:

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = A.$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = A.$$

$\therefore$ $AI = IA = A$; here, I is called identity matrix.

7. If matrix A is multiplied by a null matrix, then the resultant matrix is null matrix, i.e., $AO = OA = O$.
8. If A and B are two matrices such that AB exists, then $(AB)^T = B^T A^T$.

Note If $A_1, A_2, A_3, \ldots, A_n$ are n matrices, then $(A_1A_2A_3\ldots A_n)^T = A_n{}^T A^T{}_{n-1}\ldots A_1{}^T$

9. If A is any square matrix, then $(A^T)^n = (A^n)^T$.
10. If A and B are any two square matrices, then
    (i) $(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B)$
    $= A^2 + AB + BA + B^2$.
    (ii) $(A - B)^2 = (A - B)(A - B) = A(A - B) - B(A - B)$
    $= A^2 - AB - BA + B^2$.
    (iii) $(A + B)(A - B) = A(A - B) + B(A - B)$
    $= A^2 - AB + BA - B^2$.

EXAMPLE 10.3

If $A = \begin{pmatrix} 0 & 2009 \\ -2009 & 0 \end{pmatrix}$, then find $[A^{2009} + (A^T)^{2009}]$.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 & 2009 \\ -2009 & 0 \end{pmatrix}$ **(d)** None of these

SOLUTION

Find A^2 then calculate A^{2009}.

EXAMPLE 10.4

If $A = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $AB = -13I$, then find the value of $a + b - c + d$.

(a) 5 **(b)** 3 **(c)** 2 **(d)** 1

SOLUTION

$AB = -13I$

$$\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -13\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a+3c & 2b+3d \\ 5a+c & 5b+d \end{pmatrix} = \begin{pmatrix} -13 & 0 \\ 0 & -13 \end{pmatrix}$$

$$\Rightarrow 2a + 3c = -13, \quad (1)$$

$$5a + c = 0 \quad (2)$$

$$2b + 3d = 0 \quad (3)$$

$$5b + d = -13 \quad (4)$$

Solving Eqs. (1) and (2), we get $a = 1$ and $c = -5$. Solving Eqs. (3) and (4), we get $b = -3$ and $d = 2$.

$$a + b - c + d = 1 - 3 + 5 + 2 = 5.$$

EXAMPLE 10.5

If $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ are two matrices, then find $AB + BA$.

(a) I **(b)** O **(c)** A **(d)** B

SOLUTION

$$AB = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6-6 & 18-18 \\ -2+2 & -6+6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$BA = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6-6 & -12+12 \\ 3-3 & -6+6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AB + BA = O.$$

We have learnt about the order of matrix, different kinds of matrices, some operations like transpose, addition, subtraction, multiplication of matrices. We also learnt different properties of matrix multiplication.

In this chapter, we will learn how to find determinant and inverse of a 2×2 square matrix. We also learn about how to apply the concept of matrices to solve a system of linear equations in two variables.

DETERMINANT

For a given 2×2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the real number $(ad - bc)$ is defined as the determinant of A and is denoted by $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Example: If $A = \begin{bmatrix} 2 & -5 \\ 6 & 3 \end{bmatrix}$, then determinant of $A = |A| = \begin{vmatrix} 2 & -5 \\ 6 & 3 \end{vmatrix} = 2(3) - (-5) \times 6 = 36.$

Singular Matrix

If determinant of a square matrix is zero, then the matrix is called a singular matrix.

Example: For the square matrix $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$, $|A| = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix} = 6 \times 3 - 9 \times 2 = 18 - 18 = 0.$

So, A is a singular matrix.

Non-singular Matrix

If determinant of a square matrix is not equal to zero, then the matrix is called non-singular matrix.

Example: For the square matrix $A = \begin{bmatrix} 2 & -4 \\ 5 & 3 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = 2(3) - (-4) \times 5 = 6 + 20 = 26 \neq 0.$

So, A is a non-singular matrix.

EXAMPLE 10.6

If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $ps = 15$ and det $A = 21$, then find the value of qr.

(a) 6 **(b)** −6 **(c)** 5 **(d)** −8

SOLUTION

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

det $A = ps - qr = 21 \Rightarrow qr = ps - 21$

$qr = 15 - 21 \Rightarrow qr = -6.$

Multiplicative Inverse of a Square Matrix

For every non-singular square matrix A of order n, there exists a non-singular square matrix B of same order, such that $AB = BA = I$. (Note that I is unit matrix of order n). Here, B is called multiplicative inverse of A and is denoted as $A^{-1} \Rightarrow B = A^{-1}$.

Note If $AB = KI$, then $A^{-1} = \frac{1}{K}B.$

Multiplicative Inverse of a 2 × 2 Square Matrix

For a 2×2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can show that $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Notes

1. For a singular square matrix $|A| = 0$, and so its multiplicative inverse does not exist. Conversely, if a matrix A does not have multiplicative inverse, then $|A| = 0$.

2. If A is a square matrix and K is any scalar, then $(KA)^{-1} = \frac{1}{K} A^{-1}$.

3. For any two square matrices A and B of same order $(AB)^{-1} = B^{-1} A^{-1}$.

Method for Finding Inverse of a 2 × 2 Square Matrix

We know that for a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

From this formula we can find A^{-1} using the following steps:

1. Find whether $|A| = 0$ or not. If $|A| = 0$, then the given matrix is singular, so A^{-1} does not exist. If $|A| \neq 0$, then the matrix has a multiplicative inverse and can be found by the following steps (2), (3) and (4).
2. Interchange the elements of principal diagonal.
3. Multiply the other two elements by -1.
4. Multiply each element of the matrix by $\frac{1}{|A|}$.

EXAMPLE 10.7

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -4 \\ 3 & -5 \end{bmatrix}$.

SOLUTION

$$|A| = \begin{vmatrix} 2 & -4 \\ 3 & -5 \end{vmatrix} = -10 + 12 = 2 \neq 0.$$

$\therefore$ A is non-singular and A^{-1} exists.

$$\therefore A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & 4 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{4}{2} \\ -\frac{3}{2} & \frac{2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

EXAMPLE 10.8

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, then find $A^{-1} + A$.

(a) I **(b)** $2I$ **(c)** $3I$ **(d)** $4I$

SOLUTION

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} + A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I.$$

EXAMPLE 10.9

If $A = \begin{bmatrix} x^2 & y \\ 5 & -4 \end{bmatrix}$ and $A = A^{-1}$, then find $\begin{bmatrix} x^3 & y+x \\ 1 & 2x^2+y \end{bmatrix}^{-1}$.

(a) $\frac{1}{41}\begin{bmatrix} 8 & 1 \\ -1 & 5 \end{bmatrix}$ **(b)** $\frac{1}{41}\begin{bmatrix} 5 & 8 \\ 1 & -1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$ **(d)** $\frac{1}{41}\begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$

HINTS

(i) Use $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A = A^{-1}$, then $a = d$ and $ad - bc = -1$.

(ii) Substitute the values of x and y. Find the inverse of that matrix.

Solution of Simultaneous Linear Equations in Two Variables

The concept of matrices and determinants can be applied to solve a system of linear equations in two or more variables. Here, we present two such methods. First one is matrix inversion method and the second one is Cramer's method.

Matrix Inversion Method

Let us try to understand the method through an example:

EXAMPLE 10.10

Solve the simultaneous linear equations

$$2x - 5y = 1,\ 5x + 3y = 18.$$

SOLUTION

Given system of linear equations can be written in matrix form as shown below:

$$\begin{bmatrix} 2x - 5y \\ 5x + 3y \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

The LHS matrix can be further written as product of two matrices as shown below:

$$\begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix} \quad \text{or} \quad AX = B. \tag{1}$$

Here, $A = \begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}$ is called coefficient matrix, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is called variable matrix and $B = \begin{bmatrix} 1 \\ 18 \end{bmatrix}$ is called constant matrix. Now, we need to find values of x and y, i.e., the matrix X.

To find X, pre-multiplying both the sides of Eq. (1) with A^{-1}.

$$\Rightarrow A^{-1}(AX) = A^{-1}B$$

or $(A^{-1}A)\, X = A^{-1}B$ [since $A(BC) = (AB)C$]

or $I\,X = A^{-1}B$, [$\because A^{-1}A = I$] or $X = A^{-1}B$, [$IX = X$]

$$X = A^{-1}B$$

So, to find X we have to find inverse of coefficient matrix (i.e., A) and multiply it with B.

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

$$= \frac{1}{2 \times 3 - (-5) \times 5} \begin{bmatrix} 3 & 5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

$$= \frac{1}{31} \begin{bmatrix} 3 \times 1 + 5 \times 18 \\ -5 \times 1 + 2 \times 18 \end{bmatrix}$$

$$= \frac{1}{31} \begin{bmatrix} 93 \\ 31 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Thus, $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow x = 3, y = 1.$

Thus, in general any system of linear equations $px + qy = a$, and $rx + sy = b$ can be represented in matrix form (i.e., $AX = B$) as $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Here, A is coefficient matrix $= \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, X is variables matrix $= \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \end{bmatrix}$ is constant matrix.

$$X = A^{-1}B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Notes

1. Matrix inversion method is applicable only when the coefficient matrix A is non-singular, i.e., $|A| \neq 0$. If $|A| = 0$, then A^{-1} does not exist and so the method is not applicable.
2. This method can be extended for a system of linear equations in more than 2 variables.

Cramer's Rule or Cramer's Method

This is another method of solving system of linear equations using concept of determinants. Unlike matrix inversion method, in this method we don't need to find the inverse of coefficient matrix.

EXAMPLE 10.11

Solve the system of linear equations $3x + 4y = 2$, $5x - 3y = 13$ by Cramer's method.

SOLUTION

The system of equations can be written in matrix form (i.e., $AX = B$) as shown below:

$$\begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}.$$

Here, $A = \begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$.

To find the solution by Cramer's method we define two matrices B_1 and B_2. The matrix B_1 is obtained by replacing first column of matrix A by the column in B. similarly B_2 is obtained by replacing column 2 of matrix A by the column in B.

That is, $B_1 = \begin{bmatrix} 2 & 4 \\ 13 & -3 \end{bmatrix}$, $B_2 = \begin{bmatrix} 3 & 2 \\ 5 & 13 \end{bmatrix}$.

Now, $x = \dfrac{|B_1|}{|A|}$

$$= \frac{\begin{vmatrix} 2 & 4 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}} = \frac{2(-3) - 4(13)}{3(-3) - 4(5)}$$

$$= \frac{-6 - 52}{-9 - 20} = \frac{58}{29} = 2$$

and,

$$y = \frac{|B_2|}{|A|} = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 13 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}}$$

$$= \frac{3 \times 13 - 2 \times 5}{3(-3) - 4 \times 5} = \frac{29}{-29} = -1$$

Thus, in general for a system of linear equations $px + qy = a$, $rx + sy = b$, solution by Cramer's method is

$$x = \frac{\begin{vmatrix} a & q \\ b & s \end{vmatrix}}{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} p & a \\ r & b \end{vmatrix}}{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}.$$

Notes

1. If the coefficient matrix A is singular, then $|A| = 0$, and so the method is not applicable.
2. This method can be extended to system of linear equations in more than two variables.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Who gave the name 'matrix' to a rectangular arrangement of certain numbers in some rows and columns?

2. If $a_{ij} = 0$ $(i \neq j)$ and $a_{ij} = 4(\ i = j)$, then the matrix $A = [a_{ij}]_{n\times n}$ is a _______ matrix.

3. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & m \end{bmatrix}$ is a scalar matrix, then $x + m$ = _______.

4. The order of column matrix containing n rows is _________.

5. If $P = \begin{bmatrix} 3 & 0 \\ 0 & \lambda \end{bmatrix}$ is scalar matrix then $\lambda =$ _______.

6. If $\begin{bmatrix} 4 & -3 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 2^t \end{bmatrix}$ then $t =$_______.

7. If $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, then find $|A|$.

8. The product of two matrices, i.e., $AB = I$, then B is called the _______ of A and written as _______.

9. If $(A + B^T)^T$ is a matrix of order 4×3, then the order of matrix B is _______.

10. If $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} + \begin{bmatrix} -y & 4 \\ 7 & x \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}$ then $x =$ _______ and $y =$ _______.

11. Is $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ singular?

12. If A is any square matrix, then $\frac{1}{2}(A - A^T)$ is a _______ matrix.

13. If the determinant of a square matrix is non-zero, then the matrix is called a _______ matrix.

14. $(AB)^{-1} =$ _______.

15. The inverse of matrix A, if $A^2 = I$, is _______.

16. The additive inverse of $\begin{bmatrix} -1 & 3 & 4 \\ 5 & -7 & 8 \end{bmatrix}$ is _______.

17. If $A \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, then the order of A is _______.

18. If the order of matrices A, B and C are 3×4, 7×3 and 4×7 respectively, then the order of $(AC)B$ is _______.

19. Express the equations $2x - y + 6 = 0$ and $6x + y + 8 = 0$, in the matrix equation form.

20. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} =$ _______.

21. If $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 4$, then $d =$ _______.

22. If $AB = KI$, where $K \in R$, then $A^{-1} =$ _______.

23. If $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $P^{-1} =$ _______.

24. The value of $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} =$ _______.

25. If K is real number, then $(KA)^{-1} =$ _______.

26. The matrix $A = \begin{bmatrix} a & d \\ c & a \end{bmatrix}$ is singular then $a =$ _______.

27. If A and B commute, then $(A + B)^2 =$ _______.

28. If $|A| = 5$, $|B_1| = 5$ and $|B_2| = 25$, then find the values of x and y in Cramer's method.

29. If $A = [s\ 2]$ and $B = \begin{bmatrix} x \\ y \end{bmatrix}$ then $AB =$ _______.

30. The matrix obtained by multiplying each of the given matrix A with -1 is called the _______ of A and is denoted by _______.

Short Answer Type Questions

31. If $A = [a_{ij}]_{2\times 2}$ such that $a_{ij} = i - j + 3$, then find A.

32. If $A + B^T = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ and $A^T - B = \begin{bmatrix} 7 & 8 \\ -1 & 3 \end{bmatrix}$, then find matrices A and B.

33. If $\begin{pmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{4}{6} & -\frac{1}{7} \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then find a, b, c and d.

34. If $B = \begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix}$ and $f(x) = x^2 - 4x + 5$, then find $f(B)$.

35. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix}$ then find $A(B + C)$.

36. If $A \times \begin{bmatrix} -3 & 4 \\ 5 & 10 \end{bmatrix} = [13 \quad 6]$, then find A.

37. Two friends Jack and Jill attend IIT entrance test which has three sections; Mathematics, Physics and Chemistry. Each question in Mathematics, Physics and Chemistry carry 5 marks, 8 marks and 3 marks respectively. Jack attempted 10 questions in Mathematics, 12 in Physics and 6 in Chemistry while Jill attempted 18, 5 and 9 questions in Mathematics, Physics and Chemistry respectively. Assuming that all the questions attempted were correct, find the individual marks obtained by the boys by showing the above information as a matrix product.

38. If A and B are two matrices such that $A + B = \begin{bmatrix} 3 & 8 \\ 11 & 6 \end{bmatrix}$ and $A - B = \begin{bmatrix} 5 & 2 \\ -3 & -6 \end{bmatrix}$, then find the matrices A and B.

39. Compute the product
$$\begin{pmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 4 & -5 & -1 \\ 5 & 6 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$$

40. If $A = \begin{pmatrix} 7 & 2 \\ 18 & 5 \end{pmatrix}$, then show that $A - A^{-1} = 12I$.

41. If $A = \begin{pmatrix} 9 & -7 \\ -4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & -7 \\ -4 & -9 \end{pmatrix}$, then find AB and hence find A^{-1}.

42. Given $A = \begin{pmatrix} 3 & p \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix}$. If $AB = BA$, then find p.

43. If $A = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find the matrix X such that $4A - 2X + I = O$.

44. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 9 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 6 \\ 2 & 1 \end{bmatrix}$, then find $2A + 3B - 4C$.

45. Find the possible orders for matrices A and B if they have 18 and 19 elements respectively.

Essay Type Questions

46. If $A = \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$, then find $A + 10\,A^{-1}$.

47. Solve the following simultaneous equations using Cramer's method:
$$\frac{3x - 5y}{18} = 1,\ 2y - 4x + 10 = 0.$$

48. If $A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$ find AB, $(AB)^{-1}$, A^{-1}, B^{-1} and $B^{-1}A^{-1}$. What do you notice?

49. Solve the following system of linear equations using matrix inversion method:
$$5x - 3y = -13,\ 2x + 5y = 1.$$

50. If $A = \begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$, then show that $A + 23\,A^{-1} = 6I$.

CONCEPT APPLICATION

Level 1

1. If $\begin{vmatrix} 2 & -3 \\ p-4 & 2p-1 \end{vmatrix} = -6$, then $p =$ ________.

(a) $\frac{8}{7}$ (b) $\frac{7}{8}$

(c) 5 (d) 0

2. If $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b+c \\ b-c & d \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 3 & 2 \end{pmatrix}$, then $(a-b) + (c-d) =$ ________.

(a) −2 (b) 9

(c) 2 (d) −1

3. If $\begin{vmatrix} 5 & -3 \\ 6 & -a \end{vmatrix} = 4$, then $5a - 4 =$ ________.

(a) 0 (b) 10

(c) 14 (d) $\frac{14}{5}$

4. If $\begin{bmatrix} 2 & -3 \\ 5x+4 & 4 \end{bmatrix}$ has a multiplicative inverse, then x cannot be ________.

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$

(c) $\frac{-3}{4}$ (d) $\frac{-4}{3}$

5. If $A = \begin{bmatrix} 8 & 7 \\ -9 & -8 \end{bmatrix}$, then $A^{-1} =$ ________.

(a) A (b) $-A$

(c) $2A$ (d) $\begin{bmatrix} 8 & 7 \\ -(-9) & -8 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 4 & -2 \\ 2a-1 & 5a-3 \end{bmatrix}$ and if A does not have multiplicative inverse, then $12a - 13 =$ ________.

(a) 6 (b) $\frac{7}{12}$

(c) $\frac{12}{7}$ (d) −6

7. If $A = \begin{pmatrix} 7 & 2 \\ -3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} p & 2 \\ -3 & 5 \end{pmatrix}$ and $AB = BA$, then find p.

(a) −2 (b) 1

(c) 3 (d) p does not have a unique value

8. Given $A = \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$, then $A^{-1} =$ ________.

(a) $\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -3 \\ -2 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & -3 \\ 2 & -5 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 3 \\ 2 & -5 \end{pmatrix}$

9. If $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$, then which of the following is true?

(a) $14a + 3c = 5b$ (b) $14a - 3c = 5b$

(c) $14a + 3c = 10b$ (d) $14a + 10b = 3c$

10. If a square matrix A is skew-symmetric, then which of the following is correct?

(a) A^T is skew-symmetric

(b) A^{-1} is skew-symmetric

(c) A^{2007} is skew-symmetric

(d) All of these

11. If $|A| = 47$, then find $|A^T|$.

(a) −47 (b) 47

(c) 0 (d) 1

12. There are 25 software engineers and 10 testers in Infosys and 15 software engineers and 8 testers in Wipro. In both the companies, a software engineer is paid ₹5000 per month and a tester is paid ₹3000 per month. Find the total amount paid by each of the companies per month by representing the data in matrix form.

(a) $\begin{pmatrix} 155000 \\ 99000 \end{pmatrix}$ (b) $\begin{pmatrix} 23000 \\ 24000 \end{pmatrix}$

(c) $\begin{pmatrix} 50000 \\ 30000 \end{pmatrix}$ (d) $\begin{pmatrix} 155000 \\ 100000 \end{pmatrix}$

13. If det(A) = 5, then find det(15A) where A is of order 2×2.

(a) 225 (b) 75

(c) 375 (d) 1125

14. If $A = \begin{bmatrix} \text{cosce}\,\alpha & \tan\alpha \\ \cot\alpha & -\sin\alpha \end{bmatrix}$, then A is a/an ________.

(a) singular matrix

(b) scalar matrix

(c) symmetric matrix

(d) non-singular matrix

15. Which of the following statements is true?

(a) A singular matrix has an inverse.

(b) If a matrix does not have multiplicative inverse, it need not be a singular matrix.

(c) If a, b are non-zero real numbers, then $\begin{bmatrix} a+b & a-b \\ b-a & a+b \end{bmatrix}$ is a non-singular matrix.

(d) $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$ is a singular matrix.

16. What is the condition that is to be satisfied for the identity $(P+Q)(P-Q) = P^2 - Q^2$ to be true for any two square matrices P and Q?

(a) The identity is always true.

(b) $PQ \neq QP$.

(c) Both PQ and QP are not null matrices.

(d) P, Q and PQ are symmetric.

17. Solve the simultaneous equations:

$$2x - 3y = 11 \text{ and } 5x + 4y = 16$$

(a) $x = 5, y = -\frac{1}{3}$ (b) $x = 2, y = \frac{2}{3}$

(c) $x = -1, y = 4$ (d) $x = 4, y = -1$

18. If I is a 2×2 identity matrix, then $|(3I)^{30}|^{-1}$ = ________.

(a) $\frac{1}{3^{30}}$ (b) $\frac{1}{3^{60}}$

(c) 3^{30} (d) 3^{60}

19. If A is a 2×2 square matrix, such that det $A = 9$, then det(9A) = ___________.

(a) $\frac{1}{3^{30}}$ (b) 9

(c) 81 (d) 729

20. Which of the following statement (s) is true?

(a) Inverse of a square matrix is not unique.

(b) If A and B are two square matrices, then $(AB)^T = A^T B^T$.

(c) If A and B are two square matrices, then $(AB)^{-1} = A^{-1} B^{-1}$.

(d) If A is a non-singular square matrix, then its inverse can be uniquely expressed as sum of a symmetric and a skew-symmetric matrix.

21. If A and B are two square matrices such that $AB = A$ and $BA = B$, then find $(A^{2006} B^{2006})^{-1}$.

(a) $A^{-1} B^{-1}$ (b) $B^{-1} A^{-1}$

(c) AB (d) Cannot be determined

22. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$, then find the determinant of AB.

(a) 10 (b) 20

(c) 12 (d) 15

23. If the trace of the matrix A is 4 and the trace of matrix B is 7, then find the trace of matrix AB.

(a) 4 (b) 7

(c) 28 (d) Cannot be determined

24. If the trace of the matrix A is 5, and the trace of the matrix B is 7, then find trace of the matrix $(3A + 2B)$.

(a) 12 (b) 29

(c) 19 (d) None of these

25. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then find A^n. (where $n \in N$)

(a) $\begin{bmatrix} 3n & 0 \\ 0 & 3n \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $I_{2 \times 2}$

PRACTICE QUESTIONS

26. If A is a 2×2 scalar matrix and 7 is the one of the elements in its principal diagonal, then the inverse of A is ________.

(a) $\begin{bmatrix} \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -7 \\ -7 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{bmatrix}$

27. $A_1, A_2, A_3, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ are non-singular square matrices of order n such that $A_1B_1 = I_n$, $A_2B_2 = I_n$, $A_3B_3 = I_n$, $\ldots$, $A_n B_n = I_n$, then $(A_1 A_2 A_3 \ldots A_n)^{-1} =$ ________.

(a) $B_1 B_2 B_3 \ldots B_n$

(b) $B_1^{-1} B_2^{-1} B_3^{-1} \ldots B_n^{-1}$

(c) $B_n B_{n-1} B_{n-2} \ldots B_1$

(d) $B_{n-1} B_{n-1}^{-1} B_{n-2}^{-1} \ldots B_1^{-1}$

28. If $A = \begin{bmatrix} 5 & 6 \\ 9 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ p & 3 \end{bmatrix}$ and $AB = BA$, then $p =$ ________.

(a) $\frac{9}{2}$ (b) $\frac{-2}{9}$

(c) $\frac{-9}{2}$ (d) $\frac{2}{9}$

29. The inverse of a scalar matrix A of order 2×2, where one of the principal diagonal elements is 5, is ________.

(a) $5I$ (b) I

(c) $\frac{1}{5}I$ (d) $\frac{1}{25}I$

30. If $A = \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$, then $AA^{-1} =$ ________.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Level 2

31. If $A = \begin{pmatrix} 4 & 22 \\ -1 & -6 \end{pmatrix}$, then find $A + A^{-1}$.

(a) $\begin{bmatrix} 8 & -11 \\ -1 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 33 \\ \frac{1}{2} & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 33 \\ -\frac{3}{2} & -8 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & 33 \\ -\frac{3}{2} & -4 \end{bmatrix}$

32. If $A = \begin{bmatrix} 4 & p \\ 3 & -4 \end{bmatrix}$ and $A - A^{-1} = 0$, then $p =$ ________.

(a) 4 (b) 3

(c) −5 (d) 5

33. If $\begin{pmatrix} 11 & -4 \\ 8 & -3 \end{pmatrix}\begin{pmatrix} -x & 4 \\ -8 & y \end{pmatrix} = -\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, then find $2x - y$.

(a) −5 (b) 5

(c) 0 (d) 14

34. If $\begin{bmatrix} a^x \\ a^{-x} \end{bmatrix}[1 \quad 2] = \begin{bmatrix} p & a^{-2} \\ q & \log_2 2 \end{bmatrix}, (a > 0)$, then $a^{p-q} =$ ________.

(a) $2^{\frac{3}{2}}$ (b) $2^{\frac{-3}{2}}$

(c) 1 (d) $4^{\frac{3}{2}}$

35. If the matrix $\begin{bmatrix} 2^a & 32 \\ 36 & 12^b \end{bmatrix}$ is singular and if $k = \dfrac{2a}{ca + 1}$, then find c.

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$

(c) $\frac{4}{3}$ (d) $\frac{3}{4}$

36. If $A = \begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}$, then find $(A^{12345})^{-1}$.

(a) A^T (b) A

(c) I (d) Cannot be determined

37. The inverse of a diagonal matrix, whose principal diagonal elements are l, m is ________.

(a) $\begin{bmatrix} \frac{1}{l} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$ (b) $\begin{bmatrix} l & 0 \\ 0 & m \end{bmatrix}$

(c) $\begin{bmatrix} l^2 & 0 \\ 0 & m^2 \end{bmatrix}$ (d) $\begin{bmatrix} 2l & 0 \\ 0 & 2m \end{bmatrix}$

38. If $\begin{bmatrix} 4^b & 288 \\ 72 & 18^a \end{bmatrix}$ is a singular matrix and $2b = a + \frac{1}{c}$, then c is ________.

(a) 4 (b) $\frac{1}{4}$

(c) $\frac{1}{6}$ (d) 6

39. If A is a non-singular square matrix such that $A^2 - 7A + 5I = 0$, then $A^{-1} =$ ________.

(a) $7A - I$ (b) $\frac{7}{5}I - \frac{1}{5}A$

(c) $\frac{7}{5}I + \frac{1}{5}A$ (d) $\frac{A}{5} - \frac{7}{5}$

40. If $A = a\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$nd $B = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, then find B^{-1}. A^{-1}.

(a) $\begin{bmatrix} \frac{-1}{2} & \frac{3}{4} \\ \frac{-1}{2} & \frac{-5}{4} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{-5}{4} & \frac{3}{4} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-5}{4} & \frac{3}{4} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & \frac{-3}{4} \\ \frac{-1}{2} & \frac{-5}{4} \end{bmatrix}$

41. If

$P = \begin{bmatrix} \sec\alpha & \tan\alpha \\ -\cot\alpha & \cos\alpha \end{bmatrix}$ and $Q = \begin{bmatrix} -\cos\alpha & \tan\alpha \\ -\cot\alpha & -\sec\alpha \end{bmatrix}$, then $2P^{-1} + Q =$ ________.

(a) $\begin{bmatrix} \cos\alpha & -\tan\alpha \\ \cot\alpha & \sec\alpha \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -\cos\alpha & \tan\alpha \\ -\cot\alpha & -\sec\alpha \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

42. If A is a skew-symmetric matrix such that $AB = aI$, then find $(A^{-1})^T$.

(a) $-1B$ (b) $(-aB^T)$

(c) $\frac{B}{a}$ (d) $-\frac{B}{a}$

43. If $A = \begin{bmatrix} 8 & -7 \\ 9 & -8 \end{bmatrix}$, then $(A^{2007})^{-1} =$ ______.

(a) I (b) $2A$

(c) A (d) $2007I$

44. If the matrix $\begin{pmatrix} 10 & -9 \\ 5x+7 & 5 \end{pmatrix}$ is non-singular, then the range of x.

(a) $\frac{113}{45}$ (b) $R - \left\{\frac{-113}{45}\right\}$

(c) $R - \left\{\frac{113}{45}\right\}$ (d) $\frac{-113}{45}$

45. If $AB = BA$, then prove that $ABAB = A^2B^2$. The following are the steps involved in proving the above result. Arrange them in the sequential order.

(A) $ABAB = A(BA)B$

(B) $(AA)(BB)$

(C) $A(AB)B$

(D) A^2B^2

(a) ABCD

(b) ACBD

(c) BCAD

(d) ADBC

46. The following are the steps in finding the matrix B, if $B + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$. Arrange them in sequential order.

(A) $\therefore \begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$

(B) Let $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

(C) $\begin{pmatrix} p+2 & q+3 \\ r+4 & s+5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$

(D) $p + 2 = 5,\ q + 3 = 4,\ r + 4 = 3,\ s + 5 = 2$
$\Rightarrow p = 3,\ q = 1,\ r = -1,\ s = -3$

(E) $\therefore B = \begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix}$

(a) BACDE

(b) BADCE

(c) BDCAE

(d) BADEC

47. $(AB)^{-1} =$ ________.

(a) $A^{-1}B^{-1}$

(b) $B^{-1}A$

(c) AB^{-1}

(d) $B^{-1}A^{-1}$

48. If $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 4$, then $d =$ ________.

(a) 13 (b) 26

(c) −13 (d) −26

49. If $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, then find $|A|$.

(a) 0 (b) 1

(c) 2 (d) 3

Level 3

50. The number of integral values of x for which the determinant of the matrix $\begin{bmatrix} 5x+14 & -2 \\ 7x+8 & x \end{bmatrix}$ is always less than 1 is ________.

(a) 3 (b) 4

(c) 5 (d) 6

51. If $A = \begin{bmatrix} x^2 & y^2 \\ \log_{1024} a & -9 \end{bmatrix}$, $a = 16^{25}$ and if $A = A^{-1}$, then $\begin{bmatrix} x^2 & y \\ 1 & x^2 + y \end{bmatrix}^{-1} =$ ________.

(a) $\frac{1}{65}\begin{bmatrix} 7 & 2 \\ -1 & 9 \end{bmatrix}$ (b) $\frac{1}{65}\begin{bmatrix} 7 & -2 \\ 1 & 9 \end{bmatrix}$

(c) $\frac{1}{65}\begin{bmatrix} 7 & 2 \\ 1 & 9 \end{bmatrix}$ (d) $\frac{1}{65}\begin{bmatrix} 9 & 2 \\ -1 & 7 \end{bmatrix}$

52. If $A = \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 5 & 5 \end{pmatrix}$ and $A^n = \begin{pmatrix} 5^{200} & 5^{200} \\ 0 & 0 \end{pmatrix}$, then find n.

(a) 100 (b) 50

(c) 25 (d) None of these

53. If $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008} = \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010}$, then $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2009}$ is ________.

(a) $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008} + \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010}$

(b) $\frac{1}{2}\left[\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010} - \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008}\right]$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$

54. If $A = \begin{bmatrix} \sin\theta & \tan\theta \\ \tan\theta & \sin\theta \end{bmatrix}$ has no multiplicative inverse, then ________.

(a) $q = 0°$ (b) $q = 45°$

(c) $q = 60°$ (d) $q = 30$

55. A and B are two square matrices of same order. If $AB = B^{-1}$, then $A^{-1} =$ ________.

(a) BA (b) A^2

(c) B^2 (d) B

56. $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where p, q, r and s are positive integers. If A is symmetric matrix, $|A| = 20$ and $p = q$, then find how many values are possible for s.

(a) 1 (b) 2

(c) 3 (d) 4

57. If $a = \begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$, then find the trace of $(AB^T)^T$.

(a) 10 (b) 14

(c) −4 (d) −18

58. If A is a 2×3 matrix and B is 3×2 matrix, then the order of $(AB)^T$ is equal to the order of ________.

(a) AB (b) $A^T B^T$

(c) BA (d) All of these

59. If $A_{2\times3}$, $B_{4\times3}$ and $C_{2\times4}$ are three matrices, then which of the following is/are defined?

(a) AC^TB (b) B^TC^TA

(c) AB^TC (d) A^TBC

60. If $A = \begin{bmatrix} 2 & 4 \\ k & -2 \end{bmatrix}$ and $A^2 = O$, then find the value of k.

(a) −4 (b) −3

(c) −2 (d) −1

61. If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$ is a skew-symmetric matrix, then which of the following is equal to $x + y + z$?

(a) $a + b + c$ (b) $l + m + n$

(c) $a - b - m$ (d) $c - l - n$

62. If $P = \begin{bmatrix} 0 & 4 & -2 \\ x & 0 & -y \\ 2 & -8 & 0 \end{bmatrix}$ is a skew-symmetric matrix, then $x - y =$ ______.

(a) 8 (b) 4

(c) −12 (d) −8

63. In solving simultaneous linear equations by Crammer's method, $B_1 = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then det A = ________. (A is the coefficient matrix).

(a) −1 (b) −2

(c) 3 (d) 4

64. $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is a singular matrix. Its determinant is equal to its trace, then $p =$ ________.

(a) $-q$ (b) r

(c) 0 (d) $-s$

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. James Joseph Sylvester
2. scalar
3. -2
4. $n \times 1$
5. $\lambda = 3$
6. 4.
7. 0
8. multiplicative inverse, A^{-1}
9. 4×3
10. 5 and 1
11. Given matrix is singular matrix
12. skew-symmetric
13. non-singular
14. $B^{-1}A^{-1}$
15. A
16. $\begin{bmatrix} 1 & -3 & -4 \\ -5 & 7 & -8 \end{bmatrix}$
17. 3×2
18. 3×3
19. $\begin{bmatrix} 2 & -1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$
20. $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
21. -13
22. $\frac{1}{K}B$
23. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
24. 1
25. $\frac{1}{K}A^{-1}$
26. $\sqrt{bc}$
27. $A^2 + 2AB + B^2$
28. $x, y = 3, 5$ respectively
29. $(sx + 2y)_{1\times 1}$
30. additive inverse, $(-A)$

Short Answer Type Questions

31. $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$
32. $A = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}; B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$
33. $a = -\frac{1}{2}, b = -\frac{3}{5}, c = \frac{4}{6}$ and $d = \frac{1}{7}$
34. $\begin{bmatrix} 10 & 0 \\ -2 & 5 \end{bmatrix}$
35. $\begin{bmatrix} -13 & -5 \\ 9 & 25 \end{bmatrix}$
36. $\begin{bmatrix} -2 & \frac{7}{5} \end{bmatrix}$
37. Marks obtained by Jack = 164

    Marks obtained by Jill = 157
38. $A = \begin{bmatrix} 4 & 5 \\ 4 & 0 \end{bmatrix}; B = \begin{bmatrix} -1 & 3 \\ 7 & 6 \end{bmatrix}$
39. $\begin{bmatrix} -193 \\ 232 \\ -78 \end{bmatrix}$
41. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

    $A^{-1} = B = \begin{bmatrix} -3 & -7 \\ -4 & -9 \end{bmatrix}$

42. $p = 0$

43. $\begin{bmatrix} \frac{9}{2} & -10 \\ 0 & \frac{5}{2} \end{bmatrix}$

44. $\begin{bmatrix} 3 & 7 \\ 3 & 16 \end{bmatrix}$

45. The orders possible for A are 1×18, 2×9, 3×6, 6×3, 9×2, 18×1 the orders possible for B are 1×19 and 19×1

Essay Type Questions

46. $5I$

47. $x = 1$, $y = -3$

48. $(AB)^{-1} = B^{-1} \cdot A^{-1}$

49. $x = 1$, $y = -3$

CONCEPT APPLICATION

Level 1

1. (a)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (b)	**9.** (c)	**10.** (d)
11. (b)	**12.** (a)	**13.** (d)	**14.** (d)	**15.** (c)	**16.** (d)	**17.** (d)	**18.** (b)	**19.** (d)	**20.** (d)
21. (b)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (a)	**29.** (c)	**30.** (c)

Level 2

31. (c)	**32.** (c)	**33.** (a)	**34.** (b)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (b)	**40.** (c)
41. (b)	**42.** (d)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (a)	**47.** (d)	**48.** (c)	**49.** (a)	

Level 3

50. (b)	**51.** (a)	**52.** (a)	**53.** (d)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (b)	**58.** (a)	**59.** (b)
60. (d)	**61.** (c)	**62.** (b)	**63.** (a)	**64.** (d)					

CONCEPT APPLICATION

Level 1

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

2. Apply matrix multiplication concept and then equate the corresponding elements.

3. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

4. If, multiplicative inverse of A exists, then determinant of $A \neq 0$.

5. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

6. If, multiplicative inverse of A does not exist, then $|A| = 0$.

7. Apply matrix multiplication concept.

8. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

9. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

10. Apply the properties of skew-symmetric.

11. det A = det A^T.

12. Write the given data in matrix form.

13. $\det(kA) = k^2|A|$ (where A is 2×2 matrix).

14. Find determinant of A.

16. If P, Q and PQ are symmetric, then $PQ = QP$.

17. Write the equations in matrix form.

18. If A is an $n \times n$ matrix, then $|kA| = k^n|A|$ and det $I = 1$.

19. If A is an $n \times n$ matrix, then $|kA| = k^n|A|$.

20. Recall the properties of matrices.

21. (i) If $AB = A$, $BA = B$ then $A^2 = A$; $B^2 = B$.

    (ii) $(AB)^{-1} = B^{-1}A^{-1}$.

22. det (AB) = det $A \cdot$ det B.

23. Trace$(AB) \neq$ Trace$(A) \cdot$ Trace(B).

24. Trace $(kA + mB) = k(\text{Trace } A) + m(\text{Trace } B)$.

25. (i) $A = 3I$.

    (ii) $A^n = 3^n I^n$.

26. (i) Refer the definition of scalar matrix and write A.

    (ii) $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

27. (i) $AB = I \Rightarrow A^{-1} = B$.

    (ii) $(AB)^{-1} = B^{-1} A^{-1}$.

28. Find AB and BA.

29. Write matrix A and find A^{-1}.

30. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Level 2

31. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

32. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

33. Apply matrix multiplication concept.

34. Multiply the left side matrices and equate the corresponding elements.

35. (i) If the matrix is singular, then its determinant is zero. Using this condition we can obtain the values of a and b.

    (ii) Substitute the values of a and b in $b = \dfrac{2a}{ca + 1}$, then find the value of 'c'.

HINTS AND EXPLANATION

36. (i) Calculate A^2.

(ii) Find A^2, A^3, … and observe.

37. The inverse of diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$.

38. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a singular matrix, then $ad - bc = 0$.

39. Pre-multiply the given equation with A^{-1} and use the relation, $A^{-1}A = I$, $A^{-1}I = A^{-1}$.

40. First find AB, after that find $(AB)^{-1}$.

We know that $(AB)^{-1} = B^{-1}A^{-1}$.

41. Find P^{-1} and then proceed.

42. (i) A is skew-symmetric matrix, $A^T = -A$.

(ii) According to the problem, the inverse of A is $\frac{1}{a}B$.

43. Calculate A^2, then find $(A^{2007})^{-1}$.

44. Determinant is non-zero.

45. The required sequential order is ACBD.

46. The required sequential order is BACDE.

47. $(AB)^{-1} = B^{-1}A^{-1}$.

48. Given $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 2(d-3) - (-4 \times 9) = 4$

$$2d - 6 + 36 = 4,$$

$$2d = -26,\ d = -13.$$

49. Given $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = (18 - 18)$

$$= 0 \left(\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc) \right).$$

Level 3

50. (i) Find the determinant of matrix.

(ii) Solve the inequation.

51. (i) Put $a = 16^{25}$ in $\log_{1024} a$ and simplify.

(ii) Then, find x and y values using the relation $A = A^{-1}$.

(iii) Find the inverse of the given matrix.

52. (i) Simplify the matrix A.

(ii) Find A^2, A^3, …, A^n.

53. (i) Let $= A = \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$.

(ii) Calculate A^2.

54. Since A has no multiplicative inverse, $|A| = 0$.

$\Rightarrow \sin^2\theta - \tan^2\theta = 0$

$\Rightarrow \sin^2\theta = \tan^2\theta$

$\Rightarrow \sin\theta = \tan\theta \Rightarrow \theta = 0°$.

55. $AB = B^{-1}$

$(AB)\ B = B^{-1} \cdot B$

$AB^2 = I$

$A^{-1} \cdot A \cdot B^2 = A^{-1} \cdot I$

$I \cdot B^2 = A^{-1}$

$\therefore A^{-1} = B^2$.

56. Given $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, A is symmetric, $p = \theta$.

$\therefore A = \begin{bmatrix} p & p \\ p & s \end{bmatrix}$

$|A| = ps - p^2 = 20$

$p(s - p) = 20$.

S No	$p \times (s - p)$	s	$ps - p^2$
1	1×20	21	20
2	2×10	12	20
3	4×5	9	20

$\therefore$ There are 3 possible values for s.

57. $A = \begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

$$B^T = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$A \times B^T = \begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4+0 & 6-0 \\ -10-3 & 15+3 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -13 & 18 \end{pmatrix}$$

$$(AB^T)^T = \begin{pmatrix} -4 & -13 \\ 6 & 18 \end{pmatrix}$$

Trace of $(AB^T)^T = -4 + 18 = 14$.

58. The product AB is of the order 2×2. Since AB is a square matrix, its transpose $(AB)^T$ is also of the same order. The product BA is of the 3×3. Order of A^T is 3×2 and that of B^T is 2×3.

$\therefore$ The order of $B^T \cdot A^T$ is 2×2.

59. Given A is 2×3 matrix

B is 4×3 matrix

C is 2×4 matrix

C^T is 4×2 matrix

$\therefore$ C^TB is not possible

$C^T = 4 \times 2$ $A = 2 \times 3$

$\therefore$ C^TA is a matrix of order 4×3

B^T is of order 3×4; C^TA is of order 4×3

$B^T(C^TA)$ is a matrix of order 3×3

Option (b) follows.

60. $A = \begin{bmatrix} 2 & 4 \\ k & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & 4 \\ k & -2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ k & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4k & 8-8 \\ 2k-2k & 4k+4 \end{bmatrix}$$

$A^2 = 0$.

Given, $\begin{bmatrix} 4+4k & 0 \\ 0 & 4k+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore 4 + 4k = 0 \Rightarrow k = -1$.

61. If $A^T = -A$ then A is called skew-symmetric matrix.

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$$

$$-A = \begin{bmatrix} -a & -b & -c \\ -x & -y & -z \\ -l & -m & -n \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & x & l \\ b & y & m \\ c & z & n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & x & l \\ b & y & m \\ c & z & n \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -x & -y & -z \\ -l & -m & -n \end{bmatrix}$$

$\Rightarrow x = -b, z = -m.$

In a skew-symmetric matrix principle diagonal elements should be zero.

$\therefore$ $a = y = n = 0$

$\therefore$ $x + y + z = -b + a - m.$

62. $P^T = -P$

$$\Rightarrow \begin{bmatrix} 0 & x & 2 \\ 4 & 0 & -8 \\ -2 & -y & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 2 \\ -x & 0 & y \\ -2 & 8 & 0 \end{bmatrix}$$

$\therefore$ $x = -4$, $y = -8$

$\Rightarrow x - y = -4 + 8 = 4.$

63. Given $B_1 = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow |A| = 2 - 3 = -1.$$

64. $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

trace of $M = p + s$

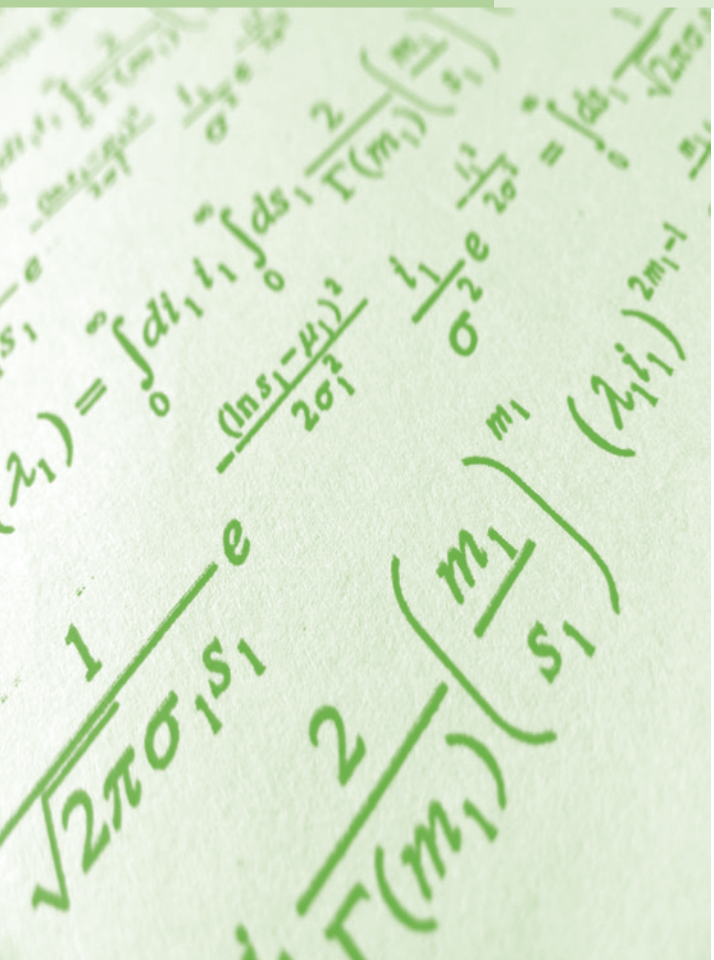
$p + s = 0$ (Trace $= |M| = 0$)

$p = -s.$

HINTS AND EXPLANATION

Chapter 11

Remainder and Factor Theorems



REMEMBER

Before beginning this chapter, you should be able to:

- Define polynomials and its functions

KEY IDEAS

After completing this chapter, you would be able to:

- Obtain zero of a polynomial
- Prove remainder theorem
- State and prove factor theorem
- Do factorization of polynomials using factor theorem
- Apply Horner’s process for synthetic division of polynomials

INTRODUCTION

A real valued function $f(x)$ of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n$, $(a_0 \neq 0)$ is called as a polynomial of degree n, where n is a non-negative integer. Here $a_0, a_1, \ldots, a_n$ are the coefficients of various powers of x.

Examples:

1. $4x^6 + 5x^5 + x^4 + x^2 - 1$ is a polynomial in x of degree 6.
2. $2x^3 + x^2 + 1$ is a polynomial in x of degree 3.

Note A constant is considered to be a polynomial of zero degree.

In earlier classes, we have learnt the different operations on polynomials like addition, subtraction, multiplication and division. Here, we shall learn two important theorems on polynomials.

REMAINDER THEOREM

If $p(x)$ is any polynomial and 'a' is any real number, then the remainder when $p(x)$ is divided by $(x - a)$ is given by $p(a)$.

Proof: Let $q(x)$ and $r(x)$ be the quotient and the remainder respectively when $p(x)$ is divided by $x - a$.

$\therefore$ By division algorithm

Dividend = Quotient × Divisor + Remainder,

i.e., $p(x) = q(x)(x - a) + r(x)$

If $x = a$, then

$p(a) = q(a)(a - a) + r(a) \Rightarrow r(a) = p(a)$,

i.e., $p(x) = (x - a)\, q(x) + p(a)$

Thus, the remainder is $p(a)$.

Notes

1. If $p(a) = 0$, we say that 'a' is a zero of the polynomial $p(x)$.
2. If $p(x)$ is a polynomial and 'a' is a zero of $p(x)$, then $p(x) = (x - a)\, q(x)$.
3. If $p(x)$ is divided by $ax + b$, then the remainder is given by $p\left(\frac{-b}{a}\right)$.
4. If $p(x)$ is divided by $ax - b$, then the remainder is given by $p\left(\frac{b}{a}\right)$.

EXAMPLE 11.1

Find the remainder when the polynomial $p(z) = z^3 - 3z + 2$ is divided by $z - 2$.

SOLUTION

Given, $p(z) = z^3 - 3z + 2$

The remainder when $p(z)$ is divided by $z - 2$ is given by $p(2)$.

Now, $p(2) = (2)^3 - 3(2) + 2$

$= 8 - 6 + 2 = 4$

Hence, when $p(z)$ is divided by $z - 2$ the remainder is 4.

FACTOR THEOREM

If $p(x)$ is a polynomial of degree n (≥ 1) and a be any real number such that $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Proof: Let $q(x)$ be the quotient and $(x - a)(a \in R)$ be a divisor of $p(x)$.

Given $p(a) = 0$

$\therefore$ By division algorithm,

Dividend = Quotient × Divisor + Remainder

$p(x) = q(x)(x - a) + p(a)$

$\Rightarrow p(x) = q(x)(x - a)$ ($\because p(a) = 0$)

Therefore, $(x - a)$ is a factor of $f(x)$, which is possible only if $p(a) = 0$.

Hence, $(x - a)$ is a factor of $p(x)$ ($\because p(a) = 0$).

Notes

1. If $p(-a) = 0$, then $(x + a)$ is a factor of $p(x)$.
2. If $p\left(\dfrac{-b}{a}\right) = 0$, then $(ax + b)$ is a factor of $p(x)$.
3. If $p\left(\dfrac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $p(x)$.
4. If sum of all the coefficients of a polynomial is zero, then $(x - 1)$ is one of its factors.
5. If sum of the coefficients of odd powers of x is equal to the sum of the coefficients of even powers of x, then one of the factors of the polynomial is $(x + 1)$.

Examples:

1. Determine whether $x - 3$ is a factor of $f(x) = x^2 - 5x + 6$.

   Given, $f(x) = x^2 - 5x + 6$

   Now, $f(3) = (3)^2 - 5(3) + 6$

   $= 9 - 15 + 6$

   $= 0 \Rightarrow f(3) = 0.$

   Hence, by factor theorem we can say that $(x - 3)$ is a factor of $f(x)$.

2. Determine whether $(x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$.

   Let $f(x) = x^3 - 6x^2 + 11x - 6$

   Now $f(1) = (1)^3 - 6(1)^2 + 11(1) - 6$

   $= 1 - 6 + 11 - 6 = 0 \Rightarrow f(1) = 0.$

   Hence, by factor theorem we can say that $(x - 1)$ is a factor of $f(x)$.

Factorization of Polynomials Using Factor Theorem

1. Factorize $x^2(y - z) + y^2(z - x) + z^2(x - y)$.

   Let us assume the given expression as a polynomial in x, say $f(x)$

   $f(x) = x^2(y - z) + y^2(z - x) + z^2(x - y)$

Now, put $x = y$ in the given expression

$\Rightarrow f(y) = y^2(y - z) + y^2(z - y) + z^2(y - y)$

$= y^3 - zy^2 + y^2z - y^3 + 0 = 0 \Rightarrow f(y) = 0$

$\Rightarrow x - y$ is a factor of the given expression.

Similarly, if we consider the given expression as a polynomial in y, we get $y - z$ is a factor of the given expression and we also get $z - x$ is a factor of the expression when we consider it as an expression in z.

Let $x^2(y - z) + y\,(z - x) + z^2(x - y) = k(x - y)(y - z)(z - x)$

For $x = 0$, $y = 1$ and $z = 2$, we get

$0^2(1 - 2) + 1^2(2 - 0) + 2^2(0 - 1) = k(0 - 1)(1 - 2)(2 - 0)$

$\Rightarrow -2 = 2k \Rightarrow k = -1$

$\therefore$ The factors of the given expression are $x - y$, $y - z$ and $z - x$.

2. Use factor theorem to factorize $x^3 + y^3 + z^3 - 3xyz$.

Given expression is $x^3 + y^3 + z^3 - 3xyz$.

Consider the expression as a polynomial in variable x say $f(x)$.

That is, $f(x) = x^3 + y^3 + z^3 - 3xyz$

Now, $f[-(y + z)] = [-(y + z)]^3 + y^3 + z^3 - 3[-(y + z)]yz$

$$= -(y + z)^3 + y^3 + z^3 + 3yz(y + z)$$

$$= -(y + z)^3 + (y + z)^3 = 0 \Rightarrow f[-(y + z)] = 0$$

$\Rightarrow$ According to factor theorem $x - [-(y + z)]$, i.e., $x + y + z$ is a factor of $x^3 + y^3 + z^3 - 3xyz$.

Now, using the long division method we get the other factor as,

$x^2 + y^2 + z^2 - xy - yz - zx$

$\therefore x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$.

Horner's Process for Synthetic Division of Polynomials

When a polynomial $f(x) = p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$ is divided by a binomial $x - \alpha$, let the quotient be $Q(x)$ and remainder be r.

We can find quotient $Q(x)$ and remainder r by using Horner's synthetic division process as explained below.

α	p_0	p_1	p_2	$\ldots$	p_{n-1}	p_n	…1st row
left (corner)		$q_0\alpha$	$q_1\alpha$	$\ldots$	$q_{n-2}\,\alpha$	$q_{n-1}\,\alpha$	…2nd row
	q_0	q_1	q_2	$\ldots$	q_{n-1}	r	…3rd row

Step 1: Write all the coefficients $p_0, p_1, p_2, \ldots, p_n$ of the given polynomial $f(x)$ in the order of descending powers of x as in the first row. When any term in $f(x)$ (as seen with descending powers of x) is missing we write zero for its coefficient.

Step 2: Divide the polynomial $f(x)$ by $(x - \alpha)$ by writing α in the left corner as shown above $(x - \alpha = 0 \Rightarrow x = \alpha)$.

Step 3: Write the first term of the third row as $q_0 = p_0$, then multiply q_0 by α to get $q_0\alpha$ and write it under p_1, as the first element of the second row.

Step 4: Add $q_0\alpha$ to p_1 to get q_1, the second element of the third row.

Step 5: Again multiply q_1 with α to get $q_1\alpha$ and write $q_1\alpha$ under p_2 and add $q_1\alpha$ to p_2 to get q_2 which is the third element of the third row.

Step 6: Continue this process till we obtain q_{n-1} in the third row. Multiply q_{n-1} with α and write $q_{n-1}\alpha$ under p_n and add $q_{n-1}\alpha$ to p_n to get r in the third row as shown in previous page.

In the above process, the elements of the third row, i.e., $q_0, q_1, q_2, \ldots, q_{n-1}$ are the coefficients of the quotient $Q(x)$ in the same order of descending powers starting with x^{n-1}.

$\therefore$ $Q(x) = q_0x^{n-1} + q_1x^{n-2} + \cdots + q_{n-2}x + q_{n-1}$ and the remainder is r, i.e., the last element of the third row.

Note If the remainder $r = 0$ then α is one of the roots of $f(x) = 0$ or $x - \alpha$ is a factor of $f(x)$.

Example: Factorize $x^4 - 10x^2 + 9$.

Let $p(x) = x^4 - 10x^2 + 9$

Here, sum of coefficients = 0, and also sum of coefficients of even powers of x = sum of coefficients of odd powers of x.

$\therefore$ $(x - 1)$ and $(x + 1)$ are the factors of $p(x)$.

Multiplier of $x - 1$ is 1 and $x + 1$ is -1

$\therefore$ The quotient is $x^2 - 9$

Hence, $p(x) = (x - 1)(x + 1)(x^2 - 9)$

$\Rightarrow p(x) = (x - 1)(x + 1)(x - 3)(x + 3)$.

1	1	0	−10	0	9
	0	1	1	−9	−9
−1	1	1	−9	−9	0
	0	−1	0	9	
	1	0	−9	0	

EXAMPLE 11.2

Find the value of a if $ax^3 - (a + 1)x^2 + 3x - 5a$ is divisible by $(x - 2)$.

SOLUTION

Let $p(x) = ax^3 - (a + 1)\,x^2 + 3x - 5a$

If $p(x)$ is divisible by $(x - 2)$, then its remainder is zero, i.e., $p(2) = 0$

$$\Rightarrow a(2)^3 - (a + 1)(2)^2 + 3(2) - 5a = 0$$
$$\Rightarrow 8a - 4a - 4 + 6 - 5a = 0$$
$$\Rightarrow -a + 2 = 0$$
$$\Rightarrow a = 2.$$

$\therefore$ The required value of a is 2.

EXAMPLE 11.3

If the polynomial $x^3 + ax^2 - bx - 30$ is exactly divisible by $x^2 - 2x - 15$. Find a and b and also the third factor.

SOLUTION

Let $p(x) = x^3 + ax^2 - bx - 30$

Given $p(x)$ is exactly divisible by $x^2 - 2x - 15$, i.e., $(x - 5)(x + 3)$

$\Rightarrow p(x)$ is divisible by $(x + 3)$ and $(x - 5)$

$\therefore\ p(-3) = 0$ and $p(5) = 0$

Consider $p(-3) = 0$

$$\Rightarrow (-3)^3 + a(-3)^2 - b(-3) - 30 = 0$$

$$\Rightarrow -27 + 9a + 3b - 30 = 0$$

$$\Rightarrow 9a + 3b - 57 = 0$$

$$\Rightarrow 3a + b - 19 = 0 \tag{1}$$

Now, consider $p(5) = 0$

That is, $53 + a(5)^2 - b(5) - 30 = 0$

$$\Rightarrow 125 + 25a - 5b - 30 = 0$$

$$\Rightarrow 25a - 5b + 95 = 0$$

$$\Rightarrow 5a - b + 19 = 0 \tag{2}$$

Adding Eqs. (1) and (2), we get

$$8a = 0$$

$$\Rightarrow a = 0$$

Substituting a in Eq. (1), we get $b = 19$.

$\therefore$ The required values of a and b are 0 and 19 respectively.

$$\Rightarrow p(x) = x^3 + 0(x^2) - 19x - 30.$$

That is, $p(x) = x^3 - 19x - 30$.

Thus, the third factor is $x + 2$.

−3	1	0	−19	−30
	0	−3	9	30
5	1	−3	−10	0
	0	5	10	
	1	2	0	

EXAMPLE 11.4

Find the linear polynomial in x which when divided by $(x - 3)$ leaves 6 as remainder and is exactly divisible by $(x + 3)$.

SOLUTION

Let the linear polynomial be $p(x) = ax + b$

Given, $p(3) = 6$ and $p(-3) = 0$.

$$\Rightarrow a(3) + b = 6 \text{ and } a(-3) + b = 0$$

$$\Rightarrow 3a + b = 6 \quad (1)$$

$$\text{and } -3a + b = 0 \quad (2)$$

Adding Eqs. (1) and (2),

$$2b = 6 \Rightarrow b = 3$$

Substituting the value of b in Eq. (1), we get $a = 1$

$\therefore$ The required linear polynomial is $x + 3$.

EXAMPLE 11.5

A quadratic polynomial in x leaves remainders as 4 and 7 respectively when divided by $(x + 1)$ and $(x - 2)$. Also it is exactly divisible by $(x - 1)$. Find the quadratic polynomial.

SOLUTION

Let the quadratic polynomial be $p(x) = ax^2 + bx + c$.

Given, $p(-1) = 4$, $p(2) = 7$ and $p(1) = 0$

$$p(-1) = a(-1)^2 + b(-1) + c = 4$$

$$\Rightarrow a - b + c = 4 \quad (1)$$

Now, $p(1) = 0$ and $p(2) = 7$

$\therefore$

$$a(1)^2 + b(1) + c = 0 \text{ and}$$

$$a(2)^2 + b(2) + c = 7$$

$$\Rightarrow a + b + c = 0 \quad (2)$$

$$4a + 2b + c = 7 \quad (3)$$

Subtracting Eq. (1) from Eq. (2), we have

$2b = -4 \Rightarrow b = -2$.

Subtracting Eq. (2) from Eq. (3), we have

$$3a + b = 7$$

$$\Rightarrow 3a - 2 = 7 \ (\because b = -2)$$

$$\Rightarrow 3a = 9 \Rightarrow a = 3$$

Substituting the values of a and b in Eq. (1), we get $c = -1$

Hence, the required quadratic polynomial is $3x^2 - 2x - 1$.

EXAMPLE 11.6

Find a common factor of the quadratic polynomials $3x^2 - x - 10$ and $2x^2 - x - 6$.

SOLUTION

Consider, $p(x) = 3x^2 - x - 10$ and $q(x) = 2x^2 - x - 6$

Let $(x - k)$ be a common factor of $p(x)$ and $q(x)$.

$\therefore\ p(k) = q(k) = 0$

$$\Rightarrow 3k^2 - k - 10 = 2k^2 - k - 6$$
$$\Rightarrow k^2 - 4 = 0$$
$$\Rightarrow k^2 = 4$$
$$\Rightarrow k = \pm 2$$

$\therefore$ The required common factor is $(x - 2)$ or $(x + 2)$.

EXAMPLE 11.7

Find the remainder when x^{999} is divided by $x^2 - 4x + 3$.

SOLUTION

Let $q(x)$ and $mx + n$ be the quotient and the remainder respectively when x^{999} is divided by $x^2 - 4x + 3$.

$\therefore$
$$x^{999} = (x^2 - 4x + 3)\ q(x) + mx + n.$$

If $x = 1$,

$$1^{999} = (1 - 4 + 3)\ q(x) + m(1) + n$$
$$\Rightarrow 1 = 0 \times q(x) + m + n$$
$$\Rightarrow m + n = 1 \qquad (1)$$

If $x = 3$,

$$3^{999} = (3^2 - 4(3) + 3)\ q(x) + 3m + n$$
$$\Rightarrow 3^{999} = 0 \times q(x) + 3m + n$$
$$\Rightarrow 3m + n = 3^{999} \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), we get

$$2m = 3^{999} - 1$$
$$m = \frac{1}{2}(3^{999} - 1)$$

Substituting m in Eq. (1), we have

$$n = 1 - \frac{1}{2}(3^{999} - 1) = 1 - \frac{1}{2}3^{999} + \frac{1}{2} = \frac{3}{2} - \frac{1}{2}3^{999}$$
$$n = \frac{3}{2}(1 - 3^{998})$$

$\therefore$ The required remainder is $\frac{1}{2}(3^{999} - 1)x + \frac{3}{2}(1 - 3^{998})$.

EXAMPLE 11.8

Find the remainder when x^5 is divided by $x^3 - 4x$.

SOLUTION

Let $q(x)$ be the quotient and $lx^2 + mx + n$ be the remainder, when x^5 is divided by $x^3 - 4x$.

That is, $x(x - 2)(x + 2)$

$\therefore\ x^5 = (x^3 - 4x)\ q(x) + lx^2 + mx + n$

Put, $x = 0$

$$\Rightarrow 0 = 0 \times q(x) + l(0) + m(0) + n$$
$$\Rightarrow n = 0$$

Put, $x = 2$

$$\Rightarrow 2^5 = (8 - 8)\ q(x) + l(2)^2 + m(2) + n$$
$$\Rightarrow 32 = 4l + 2m + n$$
$$\Rightarrow 4l + 2m = 32\ (\because n = 0)$$
$$\Rightarrow 2l + m = 16 \qquad (1)$$

Put, $x = -2$

$$(-2)^5 = (-8 + 8)\ q(x) + l(-2)^2 + m\ (-2) + n$$
$$\Rightarrow -32 = 4l - 2m + n$$
$$\Rightarrow 4l - 2m = -32\ (\because n = 0)$$
$$\Rightarrow 2l - m = -16 \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$4l = 0$$
$$\Rightarrow l = 0$$

Substituting l in Eq. (1), we get

$$2(0) + m = 16$$
$$\Rightarrow m = 16.$$

$\therefore$ The required remainder is $0(x^2) + 16x + 0$, i.e., $16x$.

EXAMPLE 11.9

If $f(x + 2) = x^2 + 7x - 13$, then find the remainder when $f(x)$ is divided by $(x + 2)$.

(a) -25 **(b)** -12 **(c)** -23 **(d)** -11

SOLUTION

Given, $f(x + 2) = x^2 + 7x - 13$ (1)

The remainder, when $f(x)$ is divided by$(x + 2)$ is $f(-2)$.

$\therefore$ Put $x = -4$ in Eq. (1)

$f(-4 + 2) = (-4)^2 + 7(-4) -13$

$\Rightarrow f(-2) =16 - 28 - 13 = -25.$

EXAMPLE 11.10

If $(x - 2)$ and $(x - 3)$ are two factors of $f(x) = x^3 + ax + b$, then find the remainder when $f(x)$ is divided by $x - 5$.

(a) 0 **(b)** 15 **(c)** 30 **(d)** 60

SOLUTION

Given $f(x) = x^3 + ax + b$

$f(2) = 0$ and $f(3) = 0$

$$(2)^3 + 2a + b = 0 \Rightarrow 2a + b = -8 \qquad (1)$$

$$(3)^3 + 3a + b = 0 \Rightarrow 3a + b = -27 \qquad (2)$$

On solving Eqs. (1) and (2), we get

$$a = -19 \text{ and } b = 30$$

$$f(x) = x^3 - 19x + 30$$

Now, $f(5) = 5^3 - 19(5) + 30 = 125 - 95 + 30 = 60$.

EXAMPLE 11.11

If the polynomials $f(x) = x^2 + 5x - p$ and $g(x) = x^2 - 2x + 6p$ have a common factor, then find the common factor.

(a) $x + 2$ **(b)** x **(c)** $x + 4$ **(d)** Either (b) or (c)

SOLUTION

Given, $f(x) = x^2 + 5x - p$ and $g(x) = x^2 - 2x + 6p$

Let $x - k$ be the common factor of $f(x)$ and $g(x)$.

$\therefore f(k) = 0$ and $g(k) = 0$

$$\Rightarrow k^2 + 5k - p = 0 \qquad (1)$$

$$k^2 - 2k + 6p = 0 \qquad (2)$$

From Eqs. (1) and (2), we get

$k = p$

Substitute $k = p$ in Eq. (1), we have

$$p^2 + 5p - p = 0$$

$$p^2 + 4p = 0 \Rightarrow p = 0$$

or $p = -4$

$\therefore$ x or $x + 4$ is a common factor of $f(x)$ and $g(x)$.

EXAMPLE 11.12

When a fourth degree polynomial $f(x)$ is divided by $(x + 6)$, the quotient is $Q(x)$ and the remainder is -6. And when $f(x)$ is divided by $[Q(x) + 1]$, the quotient is $(x + 6)$ and the remainder is $R(x)$. Find $R(x)$.

(a) $12 + x$ **(b)** $-(x + 12)$ **(c)** 0 **(d)** 3

SOLUTION

Given,

$$f(x) = Q(x)(x + 6) - 6$$

$$\Rightarrow Q(x)(x + 6) = f(x) + 6 \qquad (1)$$

And also given,

$$f(x) = (x + 6)[Q(x) + 1] + R(x)$$

$$\Rightarrow f(x) = (x + 6)\ Q(x) + x + 6 + R(x)$$

$$\Rightarrow f(x) = f(x) + 6 + x + 6 + R(x) \text{ (from Eq. (1))}$$

$$\Rightarrow R(x) = -(x + 12).$$

EXAMPLE 11.13

Given $f(x)$ is a cubic polynomial in x. If $f(x)$ is divided by $(x + 3)$, $(x + 4)$, $(x + 5)$ and $(x + 6)$, then it leaves the remainders 0, 0, 4 and 6 respectively. Find the remainder when $f(x)$ is divided by $x + 7$.

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3

SOLUTION

From the given data $x + 3$ and $x + 4$ are two factors of $f(x)$.

Let other factor be $ax + p$

$\therefore f(x) = (x + 3)(x + 4)(ax + p)$

And also given,

$f(-5) = 4$ and $f(-6) = 6$

$$\Rightarrow (-2)(-1)(-5a + p) = 4$$

$$\Rightarrow -5a + p = 2 \tag{1}$$

and $(-3)(-2)\,(-6a + p) = 6$

$$\Rightarrow -6a + p = 1 \tag{2}$$

On solving Eqs. (1) and (2), we get

$a = 1$ and $p = 7$.

$\therefore f(x) = (x + 3)(x + 4)(x + 7)$

$\therefore f(-7) = 0.$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ $(a_0 \neq 0)$ be a polynomial of degree n. If $x + 1$ is one of its factors, then ______.
2. If a polynomial $f(x)$ is divided by $(x + a)$, then the remainder obtained is ______.
3. If $a - b$ is a factor of $a^n - b^n$, then n is ______.
4. If $f(x) = x^3 + 2$ is divided by $x + 2$, then the remainder obtained is ______.
5. The condition for which $ax^2 + bx + a$ is exactly divisible by $x - a$ is ______.
6. If $x + 1$ is a factor of $x^m + 1$, then m is ______.
7. The remainder when $f(x) = x^3 + 5x^2 + 2x + 3$ is divided by x is ______.
8. The remainder when $(x - a)^2 + (x - b)^2$ is divided by x is ______.
9. The remainder when $x^6 - 4x^5 + 8x^4 - 7x^3 + 3x^2 + 2x - 7$ is divided by $x - 1$ is ______.
10. For two odd numbers x and y, if $x^3 + y^3$ is divisible by 2^k, $k \in N$, then $x + y$ is divisible by 2^k. [True/False]
11. One of the factors of $2x^{17} + 3x^{15} + 7x^{23}$ is ______. $(x^{17}/x^{15}/x^{23})$
12. If $(x - 2)^2$ is the factor of an expression of the form $x^3 + bx + c$, then the other factor is ______.
13. What should be added to $3x^3 + 5x^2 - 6x + 3$ to make it exactly divisible by $x - 1$?
14. The remainder when $2x^6 - 5x^3 - 3$ is divided by $x^3 + 1$ is ______.
15. The remainder when $f(x)$ is divided by $g(x)$ is $f\left(-\frac{3}{2}\right)$, then $g(x)$ is necessarily $2x + 3$. [True/False]
16. Find the remainder when the polynomial $x^2 + 13x + 11$ is divided by $x - 1$.
17. Find the value of the polynomial $a^2 - \frac{1}{6}a + \frac{3}{2}$ when $a = \frac{1}{2}$.
18. The polynomial $7x^2 - 11x + a$ when divided by $x + 1$ leaves a remainder of 8. Then find the value of 'a'.
19. If $x + 2$ is a factor of $f(x)$ and $f(x) = x^3 + 4x^2 + kx - 6$, then find the value of k.
20. Find the values of a if $x^3 - 5x(a - 1) - 3(x + 1) + 5a$ is divisible by $x - a$.
21. Find the value of a if $x - a$ is a factor of the polynomial $x^5 - ax^4 + x^3 - ax^2 + 2x + 3a - 2$.
22. Find the remainder when $x^3 + 3px + q$ is divided by $(x^2 - a^2)$ without actual division.
23. The remainder obtained when $x^2 + 3x + 1$ is divided by $(x - 5)$ is ______.
24. If the polynomial $3x^4 - 11x^2 + 6x + k$ is divided by $x - 3$, it leaves a remainder 7. Then the value of k is ______.
25. $(7x - 1)$ is a factor of $7x^3 + 6x^2 - 15x + 2$. (True/False)
26. If $ax^2 + bx + c$ is exactly divisible by $2x - 3$, then the relation between a, b and c is ______.
27. If $x^2 + 5x + 6$ is a factor of $x^3 + 9x^2 + 26x + 24$, then find the remaining factor.
28. If $(2x - 1)$ is a factor of $2x^2 + px - 2$, then the other factor is ______.
29. The expression $x^{m^n} - 1$ is divisible by $x + 1$, only if M is ______. (even/odd)
30. If $x + m$ is one of the factors of the polynomial $x^2 + mx - m + 4$, then the value of m is ______.

Short Answer Type Questions

31. For what values of m and n is $2x^4 - 11x^3 + mx + n$ is divisible by $x^2 - 1$?
32. Find a linear polynomial which when divided by $(2x + 1)$ and $(3x + 2)$ leaves remainders 3 and 4, respectively.
33. Prove that $x^m + 1$ is a factor of $x^{mn} - 1$ if n is even.
34. The remainders of a polynomial $f(x)$ in x are 10 and 15 respectively when $f(x)$ is divided by $(x - 3)$ and $(x - 4)$. Find the remainder when $f(x)$ is divided by $(x - 3)(x - 4)$.

PRACTICE QUESTIONS

35. If x^{555} is divided by $x^2 - 4x + 3$, then find its remainder.

36. If $(x^2 - 1)$ is a factor of $ax^3 - bx^2 - cx + d$, then find the relation between a and c.

37. When $x^4 - 3x^3 + 4x^2 + p$ is divided by $(x - 2)$, the remainder is zero. Find the value of p.

38. Find the common factors of the expressions $a_1x^2 + b_1x + c_1$ and $a_2x^2 + b_2x + c_1$ where $c_1 \neq 0$.

39. If $(x - 3)$ is a factor of $x^2 + q$ (where $q \in Q$), then find the remainder when $(x^2 + q)$ is divided by$(x - 2)$.

40. If $p + q$ is a factor of the polynomial $p^n - q^n$, then n is ______.

41. The expression $x^{4005} + y^{4005}$ is divisible by ______.

42. The value of a for which $x - 7$ is a factor of $x^2 + 11x - 2a$, is ______.

43. If a polynomial $f(x)$ is divided by $(x - 3)$ and $(x - 4)$ it leaves remainders as 7 and 12 respectively, then find the remainder when $f(x)$ is divided by $(x - 3)(x - 4)$.

44. Find the remainder when $5x^4 - 11x^2 + 6$ is divided by $5x^2 - 6$.

45. If $f(x - 2) = 2x^2 - 3x + 4$, then find the remainder when $f(x)$ is divided by $(x - 1)$.

Essay Type Questions

46. Factorize $x^4 - 2x^3 - 9x^2 + 2x + 8$ using remainder theorem.

47. Find the remainder when x^{29} is divided by $x^2 - 2x - 3$.

48. If $x^2 - 2x - 1$ is a factor of $px^3 + qx^2 + 1$, (where p, q are integers) then find the value of $p + q$.

49. If $x^2 - x + 1$ is a factor of $x^4 + ax^2 + b$, then the values of a and b are respectively ______.

50. If $lx^2 + mx + n$ is exactly divisible by $(x - 1)$ and $(x + 1)$ and leaves a remainder 1 when divided by $x + 2$, then find m and n.

CONCEPT APPLICATION

Level 1

1. The value of a for which the polynomial $y^3 + ay^2 - 2y + a + 4$ in y has $(y + a)$ as one of its factors is ______.
   (a) $\frac{-3}{4}$ (b) $\frac{4}{3}$
   (c) $\frac{3}{4}$ (d) $\frac{-4}{3}$

2. If the expression $2x^3 - 7x^2 + 5x - 3$ leaves a remainder of $5k - 2$ when divided by $x + 1$, then find the value of k.
   (a) 3 (b) -3
   (c) 5 (d) -5

3. Find the remainder when $x^{2003} + y^{6009}$ is divided by $x + y^3$.
   (a) y^{4006} (b) 1
   (c) 0 (d) y^{4000}

4. Find the remainder when $x^6 - 7x^3 + 8$ is divided by $x^3 - 2$.
   (a) -2 (b) 2
   (c) 7 (d) 1

5. If both the expressions $x^{1248} - 1$ and $x^{672} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ______.
   (a) 48 (b) 96
   (c) 54 (d) 112

6. When $x^2 - 7x + 2$ is divided by $x - 8$, then the remainder is ______.
   (a) 122 (b) 4
   (c) 45 (d) 10

7. If $ax^2 + bx + c$ is exactly divisible by $4x + 5$, then
   (a) $25a - 5b + 16c = 0$.
   (b) $25a + 20b + 16c = 0$.
   (c) $25a - 20b - 16c = 0$.
   (d) $25a - 20b + 16c = 0$.

8. The expression $2x^3 + 3x^2 - 5x + p$ when divided by $x + 2$ leaves a remainder of $3p + 2$. Find p.
   (a) -2 (b) 1
   (c) 0 (d) 2

9. $3x - 4$ is a factor of ______.

(a) $18x^4 - 3x^3 - 28x^2 - 3x + 4$

(b) $3x^4 - 10x^3 - 7x^2 + 38x - 24$

(c) $9x^4 - 6x^3 + 5x^2 - 15$

(d) $9x^4 + 36x^3 + 17x^2 - 38x - 24$

10. Which of the following is a factor of $5x^{20} + 7x^{15} + x^9$?

(a) x^{20} (b) x^{15}

(c) x^9 (d) x^{24}

11. If $(x + 3)^2$ is a factor of $f(x) = ex^3 + kx + 6$, then find the remainder obtained when $f(x)$ is divided by $x - 6$.

(a) 1 (b) 0

(c) 5 (d) 4

12. The expression $x^{mn} + 1$ is divisible by $x + 1$, only if

(a) n is odd.

(b) m is odd.

(c) both m and n are even.

(d) Cannot say

13. If both the expressions $x^{1215} - 1$ and $x^{945} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ______.

(a) 135 (b) 270

(c) 945 (d) None of these

14. If $(x - 2)$ is a factor of $x^2 + bx + 1$ (where $b \in Q$), then find the remainder when $(x^2 + bx + 1)$ is divided by $2x + 3$.

(a) 7 (b) 8

(c) 1 (d) 0

15. When $x^3 + 3x^2 + 4x + a$ is divided by $(x + 2)$, the remainder is zero. Find the value of a.

(a) 4 (b) 6

(c) −8 (d) −12

16. If $(x + 1)$ and $(x - 1)$ are the factors of $ax^3 + bx^2 + cx + d$, then which of the following is true?

(a) $a + b = 0$ (b) $b + c = 0$

(c) $b + d = 0$ (d) $a + c = 0$

17. Find the remainder when x^5 is divided by $x^2 - 9$.

(a) $81x$ (b) $81x + 10$

(c) $3^5x + 34$ (d) 81

18. The remainder when $x^{45} + x^{25} + x^{14} + x^9 + x$ divided by $x^2 - 1$ is ______.

(a) $4x - 1$ (b) $4x + 2$

(c) $4x + 1$ (d) $4x - 2$

19. For what values of a and b, the expression $x^4 + 4x^3 + ax^2 - bx + 3$ is a multiple of $x^2 - 1$?

(a) $a = 1, b = 7$ (b) $a = 4, b = -4$

(c) $a = 3, b = -5$ (d) $a = -4, b = 4$

20. When the polynomial $p(x) = ax^2 + bx + c$ is divided by $(x - 1)$ and $(x + 1)$, the remainders obtained are 6 and 10 respectively. If the value of $p(x)$ is 5 at $x = 0$, then the value of $5a - 2b + 5c$ is ______.

(a) 40 (b) 44

(c) 21 (d) 42

21. If $p - q$ is a factor of the polynomial $p^n - q^n$, then n is ______.

(a) a prime number

(b) an odd number

(c) an even number

(d) All of these

22. When the polynomial $f(x) = ax^2 + bx + c$ is divided by x, $x - 2$ and $x + 3$, remainders obtained are 7, 9 and 49 respectively. Find the value of $3a + 5b + 2c$.

(a) −2 (b) 2

(c) 5 (d) −5

23. If $f(x + 1) = 2x^2 + 7x + 5$, then one of the factors of $f(x)$ is ______.

(a) $2x + 3$ (b) $2x^2 + 3$

(c) $3x + 2$ (d) $2x + 1$

24. If $(x - p)$ and $(x - q)$ are the factors of $x^2 + px + q$, then the values of p and q are respectively ______.

(a) 1, −2 (b) 2, −3

(c) $\frac{-1}{3}, \frac{-2}{3}$ (d) −2, 1

PRACTICE QUESTIONS

25. Let $f\left(x-\frac{1}{x}\right)=x^2+\frac{1}{x^2}$, find the remainder when $f(x)$ is divided by $x-3$.

(a) $\frac{82}{9}$ (b) $\frac{8}{3}$

(c) 10 (d) 11

26. If $(x-2)^2$ is a factor of $f(x)=x^3+px+q$, then find the remainder when $f(x)$ is divided by $x-1$.

(a) 4 (b) −4

(c) −5 (d) 5

27. A quadratic polynomial in x leaves remainders 4, 4 and 0, respectively when divided by $(x-1)$, $(x-2)$ and $(x-3)$. Find the quadratic polynomial.

(a) $-2x^2+6x+3$ (b) $-2x^2+6x$

(c) $-2x^2+6x+5$ (d) $-2x^2+6x-5$

28. If $f(x+3)=x^2+x-6$, then one of the factors of $f(x)$ is ______.

(a) $x-3$ (b) $x-4$

(c) $x-5$ (d) $x-6$

29. If $(x-1)^2$ is a factor of $f(x)=x^3+bx+c$, then find the remainder when $f(x)$ is divided by $(x-2)$.

(a) 2 (b) −3

(c) 4 (d) −4

30. For what values of m and n, the expression $2x^2-(m+n)x+2n$ is exactly divisible by $(x-1)$ and $(x-2)$?

(a) $m=5$, $n=2$

(b) $m=3$, $n=4$

(c) $m=4$, $n=2$

(d) $m=2$, $n=4$

Level 2

31. The ratio of the remainders when the expression x^2+bx+c is divided by $(x-3)$ and $(x-2)$, respectively is 4 : 5. Find b and c, if $(x-1)$ is a factor of the given expression.

(a) $b=\frac{-11}{3}, c=\frac{14}{3}$

(b) $b=\frac{-14}{3}, c=\frac{11}{3}$

(c) $b=\frac{14}{3}, c=\frac{-11}{3}$

(d) None of these

32. If the polynomials $f(x)=x^2+9x+k$ and $g(x)=x^2+10x+l$ have a common factor, then $(k-l)^2$ is equal to ______.

(a) $9l-10k$ (b) $10l-9k$

(c) Both (a) and (b) (d) None of these

33. When $f(x)$ is divided by $(x-2)$, the quotient is $Q(x)$ and the remainder is zero. And when $f(x)$ is divided by $[Q(x)-1]$, the quotient is $(x-2)$ and the remainder is $R(x)$. Find the remainder $R(x)$.

(a) $x+2$ (b) $-x+2$

(c) $x-2$ (d) Cannot be determined

34. Find the values of m and n, if $(x-m)$ and $(x-n)$ are the factors of the expression x^2+mx-n.

(a) $m=-1$, $n=-2$

(b) $m=0$, $n=1$

(c) $m=\frac{-1}{2}, n=\frac{1}{2}$

(d) $m=-1$, $n=2$

35. Let $f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2}$, find the remainder when $f(x)$ is divided by $2x+1$.

(a) $\frac{-7}{4}$ (b) $\frac{9}{4}$

(c) $\frac{-9}{4}$ (d) $\frac{11}{4}$

36. A polynomial $f(x)$ leaves remainders 10 and 14, respectively when divided by $(x-3)$ and $(x-5)$. Find the remainder when $f(x)$ is divided by $(x-3)(x-5)$.

(a) $2x+6$ (b) $2x-4$

(c) $2x+4$ (d) $2x-6$

37. If $f(x+3)=x^2-7x+2$, then find the remainder when $f(x)$ is divided by $(x+1)$.

(a) 8 (b) −4

(c) 20 (d) 46

PRACTICE QUESTIONS

38. A polynomial $f(x)$ when divided by $(x - 5)$ and $(x - 7)$ leaves remainders 6 and 16, respectively. Find the remainder when $f(x)$ is divided by $(x - 5)(x - 7)$.

(a) $5x + 7$ (b) $5x - 7$
(c) $5x + 19$ (d) $5x - 19$

39. A polynomial $p(x)$ leaves remainders 75 and 15, respectively, when divided by $(x - 1)$ and $(x + 2)$.

Then the remainder when $f(x)$ is divided by $(x - 1)(x + 2)$ is ______.

(a) $5(4x + 11)$ (b) $5(4x - 11)$
(c) $5(3x + 11)$ (d) $5(3x - 11)$

40. The leading coefficient of a polynomial $f(x)$ of degree 3 is 2006. Suppose that $f(1) = 5$, $f(2) = 7$ and $f(c) = 9$. Then find $f(x)$.

(a) $2006\,(x - 1)(x - 2)(x - 3) + 2x + 3$
(b) $2006\,(x - 1)(x - 2)(x - 3) + 2x + 1$
(c) $2006\,(x - 1)(x - 2)(x - 3) + 2x - 1$
(d) $2006\,(x - 2)(x - 3)(x - 1) - (2x - 3)$

41. The ratio of the remainders when the expression $x^2 + ax + b$ is divided by $(x - 2)$ and $(x - 1)$, respectively is 4 : 3. Find a and b if $(x + 1)$ is a factor of the expression.

(a) 9, −10 (b) −9, 10
(c) 9, 10 (d) −9, −10

42. If $x^3 - ax^2 + bx - 6$ is exactly divisible by $x^2 - 5x + 6$, then $\frac{a}{b}$ is ______.

(a) $\frac{6}{11}$ (b) $\frac{-6}{11}$
(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

43. If $f(x) = x^2 + 5x + a$ and $g(x) = x^2 + 6x + b$ have a common factor, then which of the following is true?

(a) $(a - b)^2 + 5(a - b) + b = 0$
(b) $(a + b)^2 + 5(a + b) + a = 0$
(c) $(a + b)^2 + 6(a + b) + b = 0$
(d) $(a - b)^2 + 6(a - b) + b = 0$

44. If $ax^4 + bx^3 + cx^2 + dx$ is exactly divisible by $x^2 - 4$, then $\frac{a}{c}$ is ______.

(a) $\frac{1}{4}$ (b) $\frac{-1}{4}$
(c) $\frac{-1}{8}$ (d) $\frac{1}{8}$

45. If $x^2 + x + 1$ is a factor of $x^4 + ax^2 + b$, then the values of a and b, respectively are ______.

(a) 2, 4 (b) 2, 1
(c) 1, 1 (d) 1, 2

46. If $(x + 1)$ and $(x - 1)$ are the factors of $x^3 + ax^2 - bx - 2$, then find the other factor of the given polynomial.

The following are the steps involved in solving the problem given above. Arrange them in the sequential order.

(A) Put $x = 1$ and $x = -1$ in the given polynomial and obtain the equations in a and b.
(B) Substitute a and b in the given polynomial.
(C) Factorize the polynomial.
(D) Solve the equations in a and b.

(a) ADCB (b) ADBC
(c) ABCD (d) ABDC

47. The following are the steps involved in finding the value of a when $x - 2$ is a factor of $3x^2 - 7x + a$. Arrange them in sequential order.

(A) $12 - 14 + a = 0 \Rightarrow a = 2$
(B) By factor theorem, $f(2) = 0 \Rightarrow 3(2)^2 - 7(2) + a = 0$
(C) Let $f(x) = 3x^2 - 7x + a$

(a) CBA (b) BCA
(c) CAB (d) BAC

48. If $px^3 + qx^2 + rx + s$ is exactly divisible by $x^2 - 1$, then which of the following is/are necessarily true?

(A) $p = r$ (B) $q = s$
(C) $p = -r$ (D) $q = -s$

(a) Both (A) and (B) (b) Both (C) and (D)
(c) Both (A) and (D) (d) Both (B) and (C)

49. Which of the following is a factor of $x^3 + 3px^2 - 3pqx - q^3$? (where p and q are constants)

(a) $x + p$ (b) $x + q$

(c) $x - p$ (d) $x - q$

50. If $(x - k)$ is a common factor of $x^2 + 3x + a$ and $x^2 + 4x + b$, then find the value of k in terms of a and b.

(a) $a + b$ (b) $a - b$

(c) $2a + 3b$ (d) $2a - 3b$

Level 3

51. Find the remainder when x^{33} is divided by $x^2 - 3x - 4$.

(a) $\left(\frac{4^{33}-1}{5}\right)x + \left(\frac{4^{33}-4}{5}\right)$

(b) $\left(\frac{4^{33}+1}{5}\right)x + \left(\frac{4^{33}-4}{5}\right)$

(c) $\left(\frac{4^{33}-4}{5}\right)x + \left(\frac{4^{33}+1}{5}\right)$

(c) $\left(\frac{4^{33}+4}{5}\right)x + \left(\frac{4^{33}-1}{5}\right)$

52. If $6x^2 - 3x - 1$ is a factor of $ax^3 + bx - 1$ (where a, b are integers), then find the value of b.

(a) 1 (b) 3

(c) −5 (d) −7

53. If the polynomials $f(x) = x^2 + 6x + p$ and $g(x) = x^2 + 7x + q$ have a common factor, then which of the following is true?

(a) $p^2 + q^2 + 2pq + 6p - 7q = 0$

(b) $p^2 + q^2 - 2pq + 7p - 6q = 0$

(c) $p^2 + q^2 - 2pq + 6p - 7q = 0$

(d) $p^2 + q^2 + 2pq + 7p - 6q = 0$

54. A polynomial of degree 2 in x, when divided by $(x + 1)$, $(x + 2)$ and $(x + 3)$, leaves remainders 1, 4 and 3 respectively. Find the polynomial.

(a) $\frac{1}{2}(x^2 + 9x + 6)$

(b) $\frac{1}{2}(x^2 - 9x + 6)$

(c) $\frac{-1}{2}(x^2 - 9x + 6)$

(d) $\frac{-1}{2}(x^2 + 9x + 6)$

55. When a third degree polynomial $f(x)$ is divided by $(x - 3)$, the quotient is $Q(x)$ and the remainder is zero. Also when $f(x)$ is divided by $[Q(x) + x + 1]$, the quotient is $(x - 4)$ and remainder is $R(x)$. Find the remainder $R(x)$.

(a) $Q(x) + 3x + 4 + x^2$

(b) $Q(x) + 4x + 4 - x^2$

(c) $Q(x) + 3x + 4 - x^2$

(d) Cannot be determined

56. If the expression $x^2 + 3x - 3$, is divided by $(x - p)$, then it leaves remainder 1. Find the value of p.

(a) 1 (b) −3

(c) −4 (d) Either (a) or (c)

57. If $ax^3 - 5x^2 + x + p$ is divisible by $x^2 - 3x + 2$, then find the values of a and p.

(a) $a = 2$, $p = 2$ (b) $a = 2$, $p = 3$

(c) $a = 1$, $p = 3$ (d) $a = 1$, $p = 2$

58. Which of the following should be added to $9x^3 + 6x^2 + x + 2$, so that the sum is divisible by $(3x + 1)$?

(a) −4 (b) −3

(c) −2 (d) −1

59. If the expression $6x^2 + 13x + k$ is divisible by $2x + 3$, then which of the following is the factor of the expression?

(a) $3x + 1$ (b) $3x + 4$

(c) $3x + 2$ (d) $3x + 5$

60. Given $ax^2 + bx + c$ is a quadratic polynomial in x and leaves remainders 6, 11 and 18, respectively, when divided by $(x + 1)$, $(x + 2)$ and $(x + 3)$. Find the value of $a + b + c$.

(a) 1 (b) 2

(c) 3 (d) 4

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $a_1 + a_3 + a_5 + \cdots = a_0 + a_2 + a_4 + \cdots$
2. $f(-a)$
3. $n \in N$
4. -6
5. $a = 0$ or $a^2 + b + 1 = 0$
6. odd
7. 3
8. $a^2 + b^2$
9. -4
10. True
11. x^{15}
12. $x + 4$
13. -5
14. 4
15. False
16. 25
17. $\dfrac{5}{3}$
18. -10
19. 1
20. 1 and 3
21. $\dfrac{2}{5}$
22. $(a^2 + 3p)x + q$.
23. 41
24. -155
25. True
26. $9a + 6b + 4c = 0$
27. $(x + 4)$.
28. $x + 2$
29. even number
30. 4

Short Answer Type Questions

31. $m = 11$ and $n = -2$
32. $-6x$
34. $5(x - 1)$
35. $\dfrac{1}{2}(3^{555} - 1)x + \dfrac{3}{2}(1 - 3^{554})$
36. $a = c$
37. -8
38. $\left(x + \dfrac{b_1 - b_2}{a_1 - a_2}\right)$
39. -5
40. 42
41. $x + y$
42. 63
43. $5x - 8$
44. 0
45. 13

Essay Type Questions

46. $(x - 1)\ (x + 1)\ (x + 2)\ (x - 4)$.
47. $\left(\dfrac{3^{29} + 1}{4}\right)x + \left(\dfrac{3^{29} - 3}{4}\right)$
48. -3
49. 1, 1
50. $m = 0, n = \dfrac{-1}{3}$

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (d)	**9.** (a)	**10.** (c)
11. (b)	**12.** (b)	**13.** (a)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (a)	**18.** (c)	**19.** (d)	**20.** (b)
21. (c)	**22.** (a)	**23.** (a)	**24.** (a)	**25.** (d)	**26.** (d)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (c)

Level 2

31. (b)	**32.** (a)	**33.** (c)	**34.** (d)	**35.** (a)	**36.** (c)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (a)
41. (d)	**42.** (a)	**43.** (d)	**44.** (b)	**45.** (c)	**46.** (b)	**47.** (a)	**48.** (b)	**49.** (d)	**50.** (b)

Level 3

51. (b)	**52.** (c)	**53.** (b)	**54.** (d)	**55.** (c)	**56.** (d)	**57.** (a)	**58.** (c)	**59.** (c)	**60.** (b)

CONCEPT APPLICATION

Level 1

1. Use factor theorem.
2. Use remainder theorem.
3. Use remainder theorem.
4. Use remainder theorem.
5. The greatest possible value of n is the HCF of 1278 and 672.
6. Use remainder theorem.
7. Use factor theorem.
8. Use remainder theorem.
9. Use factor theorem.
10. $5x^{20} + 7x^{15} + x^9 = x^9(5x^{11} + 7x^6 + 1)$.
11. Since the coefficient of x^2 is zero, the sum of the roots is zero.
12. Use factor theorem.
13. Largest possible value of n is the HCF of 1215 and 945.
17. Use division algorithm.
18. Use division algorithm.
19. $(x + 1)$ and $(x - 1)$ are the factors of the given expression.
20. $P(1) = 6$, $P(-1) = 10$ and $P(0) = 5$.
21. Use division algorithm.
22. $f(0) = 7$, $f(2) = 9$ and $f(-3) = 49$.
23. Put $x = x - 1$ in $f(x + 1)$ to get $f(x)$.
    (i) Write $2x^2 + 7x + 5$ in terms of $x + 1$.
    (ii) Replace $x + 1$ by x.
    (iii) Apply remainder theorem.
24. (i) $x^2 + px + q = (x - p)(x - q)$.
    (ii) Compare the terms in LHS and RHS.
25. (i) $f\left(x - \dfrac{1}{x}\right) = \left(x - \dfrac{1}{x}\right)^2 + 2$.
    (ii) Replace $\left(x - \dfrac{1}{x}\right)$ with x.
    (iii) Use remainder theorem to obtain remainder.
26. (i) Since the coefficient of x^2 is 0, the sum of the roots is '0'.
    $\Rightarrow$ Third root is -4.
    (ii) Apply remainder theorem for $f(x) = (x - 2)^2(x + 4)$.
27. (i) Let $f(x) = ax^2 + bx + c$. $f(1) = 4$; $f(2) = 4$; $f(3) = 0$
    (ii) Solve for a, b, and c.
28. (i) Put $x = x - 3$ in $f(x + 3)$ to get $f(x)$.
    (ii) Apply factor theorem.
29. (i) Coefficient of x^2 is 0, therefore sum of roots is 0.
    $\therefore$ Third root $= -2$.
    (ii) Apply factor theorem.
    (iii) To obtain the remainder, use the remainder theorem.
30. (i) Take the given polynomial as $f(x)$.
    (ii) $f(1) = 0$, $f(2) = 0$.

Level 2

31. $\dfrac{f(3)}{f(2)} = \dfrac{4}{5}$ and $f(1) = 0$.
32. (i) Let the common factor be $x - a$ and find $f(a)$, and $g(a)$.
    (ii) Obtain the value of a in terms of k and l.
33. Dividend = Divisor × Quotient + Remainder.
34. (i) $x^2 + mx - n = (x - m)(x - n)$.
    (ii) Equate the corresponding terms.
35. (i) $f\left(x + \dfrac{1}{x}\right) = \left(x + \dfrac{1}{x}\right)^2 - 2$.
    (ii) Replace $x + \dfrac{1}{x}$ by x.
    (iii) Put $x = \dfrac{1}{2}$.
36. (i) $f(3) = 10$, $f(5) = 14$.
    (ii) Dividend = Divisor × Quotient + Remainder.

39. (i) $f(1) = 75, f(-2) = 15$.

(ii) Dividend = Divisor × Quotient + Remainder.

40. Verify from the options whether $f(1) = 5$, $f(2) = 7$ and $f(3) = 9$ by using remainder theorem.

41. $\dfrac{f(2)}{f(1)} = \dfrac{4}{0}$ and $f(-1) = 0$.

42. (i) $x^2 - 5x + 6 = (x - 2)(x - 3)$.

(ii) $f(2) = 0, f(3) = 0$.

43. (i) Let the common factor be $(x - a)$, then $f(a) = g(a)$, obtain value of 'a'.

(ii) Substitute value of 'a' in $f(x)$.

44. $f(2) = 0$ and $f(-2) = 0$.

45. $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$.

46. ADBC is the required sequential order.

47. CBA is the required sequential order.

48. Let $f(x) = px^3 + qx^2 + rx + s$

Given $x^2 - 1$ is a factor of $f(x)$.

$\therefore$ $(x + 1)$ and $(x - 1)$ are factors of $f(x)$

$\therefore f(-1) = 0$ and $f(1) = 0$

$-p + q - r + s = 0$

$\Rightarrow p + r = q + s$ (1)

$p + q + r + s = 0$ (2)

From Eqs. (1) and (2), we have

$p + r = 0$ and $q + s = 0$

$\Rightarrow p = -r$ and $q = -s$.

49. Let $f(x) = x^3 + 3px^2 - 3pqx - q^3$

From the options,

$f(q) = q^3 + 3pq^2 - 3pq^2 - q^3 = 0$

$x - q$ is a factor of $f(x)$.

50. Given, $x - k$ is a common factor of $x^2 + 3x + a$ and $x^2 + 4x + b$.

$\therefore$ $(k)^2 + 3k + a = 0$ and $(k)^2 + 4k + b = 0$

$\Rightarrow k^2 + 4k + b = k^2 + 3k + a$

$\Rightarrow k = a - b$.

Level 3

53. (i) Let the common factor be $(x - a)$, then make $f(a) = g(a)$, and get the value of 'a'.

(ii) Substitute value of 'a' in $f(x)$.

54. Let $f(x) = ax^2 + bx + c$, given $f(-1) = 1$, $f(-2) = 4$ and $f(-3) = 3$.

55. Dividend = Divisor × Quotient + Remainder.

56. Let $f(x) = x^2 + 3x - 3$

Given, $f(p) = 1$

$\Rightarrow p^2 + 3p - 3 = 1$

$\Rightarrow p^2 + 3p - 4 = 0$

$\Rightarrow (p + 4)(p - 1) = 0$

$\Rightarrow p = -4$ or 1.

57. Let $f(x) = ax^3 - 5x^2 + x + p$

Given, $f(x)$ is divisible $x^2 - 3x + 2$, i.e.,

$\Rightarrow f(x)$ is divisible by $(x - 1)$ and $(x - 2)$

$f(1) = 0$ and $f(2) = 0 \Rightarrow a + p - 4 = 0$

$\Rightarrow a + p = 4$ (1)

$8a + p = 18 = 0 \Rightarrow 8a + p = 18$ (2)

On solving Eqs. (1) and (2), we get $a = p = 2$.

58. Let k should be added to the given expression so that the sum is divisible by $(3x + 1)$.

Let $f(x) = 9x^3 + 6x^2 + x + 2 + k$

Given, $f\left(-\dfrac{1}{3}\right) = 0$

$$\Rightarrow 9\left(-\frac{1}{3}\right)^3 + 6\left(-\frac{1}{3}\right)^2 - \frac{1}{3} + 2 + k = 0$$

$$\Rightarrow -\frac{1}{3} + \frac{2}{3} - \frac{1}{3} + 2 + k = 0 \Rightarrow 2 + k = 0$$

$\Rightarrow k = -2$.

59. Let $f(x) = 6x^2 + 13x + k$

Given $2x + 3$ is a factor of $f(x)$

By factor theorem,

$$f\left(-\frac{3}{2}\right) = 0$$

$$\Rightarrow 6\left(-\frac{3}{2}\right)^2 + 13\left(-\frac{3}{2}\right) + k = 0$$

$$\Rightarrow \frac{27}{2} - \frac{39}{2} + k = 0 \Rightarrow k = 6$$

$\therefore f(x) = 6x^2 + 13x + 6$

$= (2x + 3)(3x + 2)$

$\therefore$ The other factor is $3x + 2$.

60. Let $f(x) = ax^2 + bx + c$.

Given, $f(-1) = 6$, $f(-2) = 11$ and $f(-3) = 18$

$\Rightarrow a - b + c = 6$ (1)

$\Rightarrow 4a - 2b + c = 11$ (2)

$\Rightarrow 9a - 3b + c = 18$ (3)

Eq. (2) − Eq. (1) $\Rightarrow 3a - b = 5$ (4)

Eq. (3) − Eq. (2) $\Rightarrow 5a - b = 7$ (5)

On solving Eqs. (4) and (5), we get

$a = 1$, $b = -2$

Substituting $a = 1$ and $b = -2$ in Eq. (1), we get

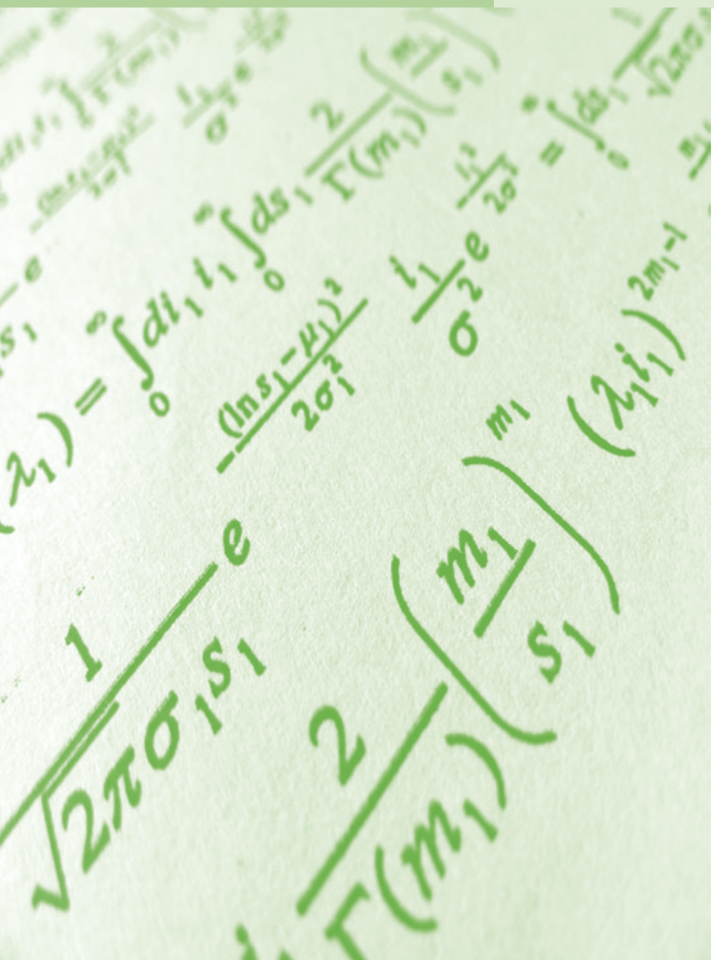
$\Rightarrow c = 3$

Now, $a + b + c = 2$.

Chapter

12

Statistics



REMEMBER

Before beginning this chapter, you should be able to:

- Define data, types of data, tabulation of data and statistical graphs
- Understand range, quartiles, deviation, etc.

KEY IDEAS

After completing this chapter, you would be able to:

- Explain types of data and statistical groups
- Study measures of central tendencies for grouped and ungrouped data—mean, median and mode
- Evaluate empirical relation between mean, median and mode
- Learn about standard deviation, coefficient of variation, quartiles, estimation of median and quartiles from ogives, estimation of mode from histogram

INTRODUCTION

The word 'statistics' is derived from the Latin word 'status' which means political state.

Political states had to collect information about their citizens to facilitate governance and plan for development. Then, in course of time, statistics came to mean a branch of mathematics which deals with collection, classification, presentation and analysis of numerical data.

In this chapter, we shall learn about the classification of data, i.e., grouped data and ungrouped data, measures of central tendency, and their properties.

Data

The word **'data'** means, information in the form of numerical figures or a set of given facts.

For example, the percentage of marks scored by 10 students of a class in a test are:

36, 80, 65, 75, 94, 48, 12, 64, 88 and 98.

The set of these figures is the data related to the marks scored by the 10 students in a class test.

Types of Data

Statistics is basically the study of numerical data. It includes methods of collection, classification, presentation, analysis of data and inferences from data. Data as can be qualitative or quantitative in nature. If one speaks of honesty, beauty, colour, etc., the data is qualitative, while height, weight, distance, marks, etc., are quantitative. Data can also be classified as raw data, and grouped data.

Raw Data

Data obtained from direct observation is called raw data.

The marks obtained by 10 students in a monthly test is an example of raw data or ungrouped data.

In fact, little can be inferred from this data. So, to make this data clearer and more meaningful, we group it into ordered intervals.

Grouped Data

To present the data in a more meaningful way, we condense the data into convenient number of classes or groups, generally not exceeding 10 and not less than 5. This helps us in perceiving at a glance, certain salient features of data.

Some Basic Definitions

Before getting into the details of tabular representation of data, let us review some basic definitions:

Observation Each numerical figure in a data is called an observation.

Frequency The number of times a particular observation occurs is called its frequency.

Tabulation or Presentation of Data

A systematical arrangement of the data in a tabular form is called 'tabulation' or 'presentation' of the data. This grouping results in a table called 'frequency table' that indicates the number of scores within each group.

Many conclusions about the characteristics of the data, the behaviour of variables, etc., can be drawn from this table.

The quantitative data that is to be analyzed statistically can be divided into three categories:

Individual Series Any raw data that is collected, forms an individual series.

Examples:

1. The weights of 5 students:

32, 40, 65, 48 and 54 (in kg).

2. Percentage of marks scored by 10 students in a test:

48, 59, 63, 72, 48, 72, 84, 98, 90 and 60.

Discrete Series A discrete series is formulated from raw data. Here, the frequency of the observations are taken into consideration.

Example:

Given below is the data showing the number of computers in 12 families of a locality:

1, 1, 2, 3, 2, 1, 4, 3, 2, 2, 1, 1.

Arranging the data in ascending order, we have

1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4.

To count, we can use tallymarks. We record tallymarks in bunches of five, the fifth one crossing the other four diagonally, i.e., $\cancel{||||}$.

Thus, we may prepare the following frequency table:

<table>
<tr><th>Number of Computers</th><th>Tally Marks</th><th>Number of Families (Frequency)</th></tr>
<tr><td>1</td><td>$\cancel{||||}$</td><td>5</td></tr>
<tr><td>2</td><td>||||</td><td>4</td></tr>
<tr><td>3</td><td>||</td><td>2</td></tr>
<tr><td>4</td><td>|</td><td>1</td></tr>
</table>

Continuous Series When the data contains large number of observations, we put them into different groups called 'class intervals', such as 1–10, 11–20, 21–30.

Here, 1–10 means data whose values lie between 1 and 10 including both 1 and 10.

This form is known as **'inclusive form'**. Also, 1 is called the **'lower limit'** and 10 is called the **'upper limit'**.

Example:

Given below are the marks (out of 50) obtained by 30 students in an examination:

43	19	25	32	48
17	29	9	15	50
7	24	20	37	44
22	2	50	27	25
18	42	16	1	33
25	35	45	35	28

Taking class intervals 1–10, 11–20, 21–30, 31–40 and 41–50, we prepare a frequency distribution table for the above data.

First, we write the marks in ascending order as:

1	2	7	9	15	16	17	18	19	20
22	24	25	25	25	27	28	29	32	33
35	35	37	42	43	44	45	48	50	50

Now, we can prepare the following frequency distribution table:

Class Interval	Tally Marks	Frequency
1–10	||||	4
11–20	~~||||~~ |	6
21–30	~~||||~~ |||	8
31–40	~~||||~~	5
41–50	~~||||~~ ||	7

Now, with this idea, let us review some more concepts about tabulation.

Class Interval

A group into which the raw data is condensed is called a class interval.

Each class is bounded by two figures which are called the class limits. The figure on the LHS is called lower limit and the figure on the RHS is called upper limit of the class. Thus, 0–10 is a class with lower limit being '0' and the upper limit being '10'.

Class Boundaries

In an exclusive form, the lower and upper limits are known as class boundaries or true lower limit and true upper limit of the class.

Thus, the boundaries of 15–25 in exclusive form are 15 and 25.

The boundaries in an inclusive form are obtained by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit.

Thus, the boundaries of 15–25 in the inclusive form are 14.5–25.5.

Class Size

The difference between the true upper limit and the true lower limit is called 'class size'.

Hence, in the above example, the class size = 25 − 15 = 10.

Class Mark or Mid-value

Class mark $= \frac{1}{2}$(upper limit + lower limit)

Thus, the class mark of 15–25 is, $\frac{1}{2}(25 + 15) = 20$.

Statistical Graphs

Information provided by a numerical frequency distribution is easily understood when represented by diagrams or graphs. Diagrams act as visual aids and leave a lasting impression on the mind. This enables the investigator to make quick conclusions about the distribution.

There are different types of graphs or diagrams to represent statistical data. Some of them are:

1. Bar chart or bar graph (for unclassified frequency distribution)
2. Histogram (for classified frequency distribution)
3. Frequency polygon (for classified frequency distribution)
4. Frequency curve (for classified frequency distribution)
5. Cumulative frequency curve (for classified frequency distribution).
   (i) Less than cumulative frequency curve.
   (ii) Greater than cumulative frequency curve.

Bar Graph

The important features of bar graphs are:

1. Bar graphs are used to represent unclassified frequency distributions.
2. Frequency of a value of a variable is represented by a bar (rectangle) whose length (i.e., height) is equal (or proportional) to the frequency.
3. The breadth of the bar is arbitrary and the breadth of all the bars are equal. The bars may or may not touch each other.

EXAMPLE 12.1

Represent the following frequency distribution by a bar graph:

Value of variable	2	4	6	8	10
Frequency	5	8	4	2	7

SOLUTION

Either of the following bar graphs [Figs. 12.1(a) or (b)] may be used to represent the above frequency distribution.

The first graph takes value of the variable along the X-axis and the frequency along the Y-axis, whereas the second one takes the frequency along the X-axis and the value of the variable on the Y-axis.

All the rectangles (i.e., bars) should be of same width and uniform spaces should be left between any two consecutive bars.

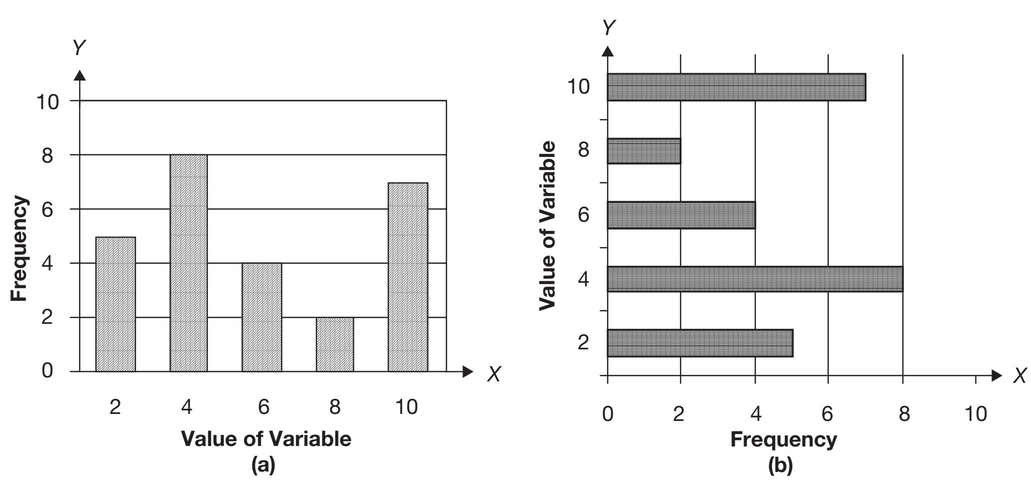


Figure 12.1

Histograms

Classified or grouped data is represented graphically by histograms. A histogram consists of rectangles each of which has its breadth proportional to the size of concerned class interval and its height proportional to the corresponding frequency. In a histogram, two consecutive rectangles have a common side.

Hence, in a histogram, we do the following:

1. We represent class boundaries along the *X*-axis.
2. Along the *Y*-axis, we represent class frequencies.
3. We construct rectangles with bases along the *X*-axis and heights along the *Y*-axis.

EXAMPLE 12.2

Construct a histogram for the following frequency distribution:

Class Interval	20–30	30–40	40–50	50–60	60–70
Frequency	5	8	3	7	4

SOLUTION

Here, the class intervals are continuous.

The following histogram is drawn according to the method described above:

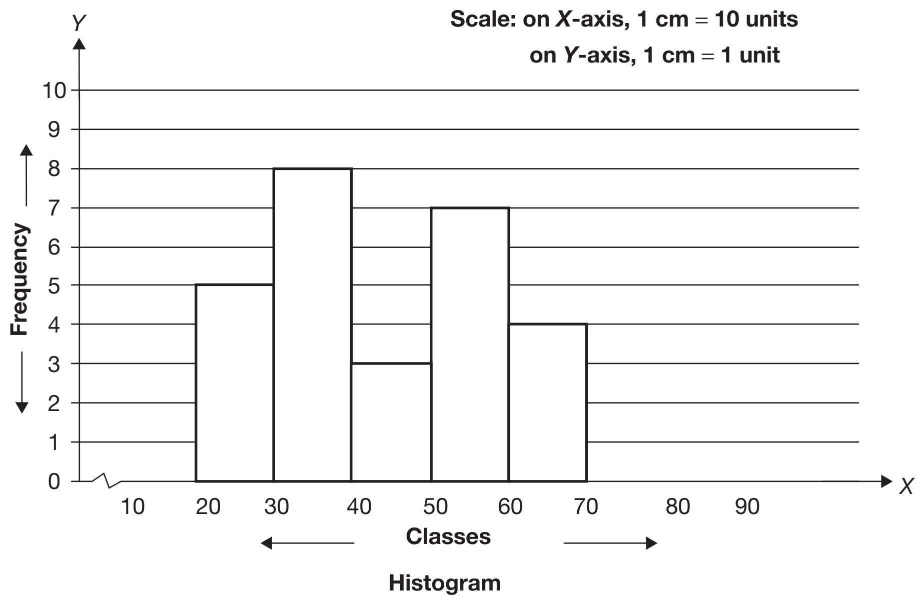


Figure 12.2

Remarks

The following points may be noted:

1. A link mark (—/\/—) made on the horizontal axis, between the vertical axis and first vertical rectangle, if there is a gap between 0 and the lower boundary of first class interval.
2. We may shade all rectangles. A heading for the histogram may also be given.

Important Observations

1. If the class intervals are discontinuous, the distribution has to be changed into continuous intervals, and then the histogram has to be drawn.
2. Bar graphs are used for unclassified frequency distributions, whereas histograms are used for classified frequency distribution. The breadths of rectangles in a bar graph are arbitary, while those in histogram are determined by class size.

Frequency Polygon

Frequency polygons are used to represent classified or grouped data graphically. It is a polygon whose vertices are the mid-points of the top sides of the rectangles, forming the histogram of the frequency distribution.

To draw a frequency polygon for a given frequency distribution, the mid-values of the class intervals are taken on *X*-axis and the corresponding frequencies on *Y*-axis and the points are plotted on a graph sheet.

These points are joined by straight line segments which form the frequency polygon.

EXAMPLE 12.3

Construct a frequency polygon for the following data:

Class Interval	12–17	18–23	24–29	30–35	36–41	Total
Frequency	10	7	12	8	13	50

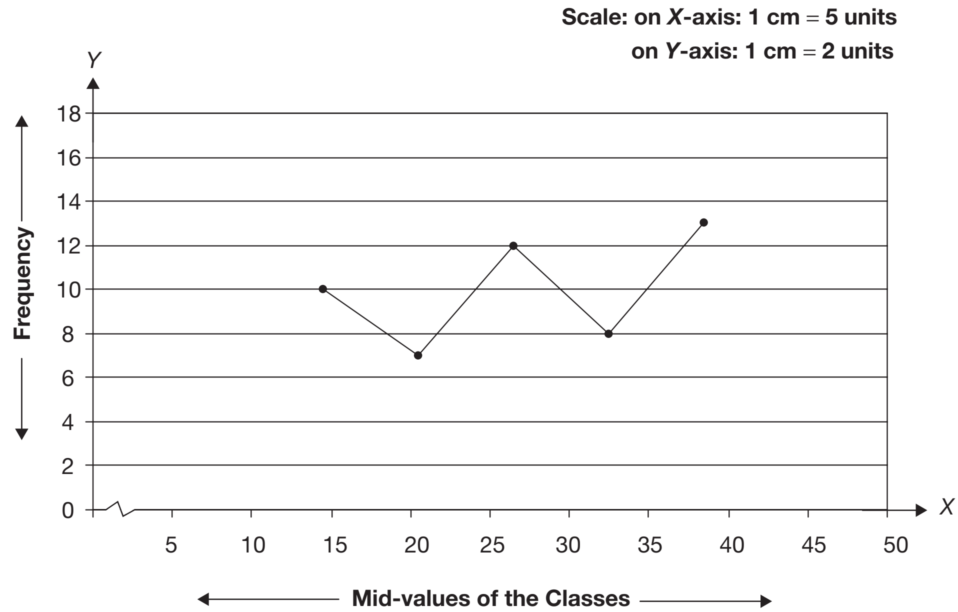
SOLUTION

Here, the class intervals are discontinuous.

Hence, first we convert the class intervals to continuous class intervals, and then find the mid-points of each class intervals. We do this by adding 0.5 to each upper limit and subtracting 0.5 from each lower limit.

Class Interval	Exclusive Class Interval	Mid-value of Class	Frequency
12–17	11.5–17·5	14.5	10
18–23	17.5–23.5	20.5	7
24–29	23.5–29.5	26.5	12
30–35	29.5–35.5	32.5	8
36–41	35.5–41.5	38.5	13

Now, taking the mid-values of class intervals on the *X*-axis and the corresponding frequencies on the *Y*-axis, we draw the frequency polygon as shown in the Fig. 12.3.



Frequency Polygon

Figure 12.3

Frequency Curve

Frequency curves are used to graphically represent classified or grouped data.

As the class-interval in a frequency distribution decreases, the points of the frequency polygon become closer, and closer, and then the frequency polygon tends to become a frequency curve. So, when the number of scores in the data is sufficiently large and the class-intervals become smaller (ultimately tending to zero), the limiting form of frequency polygon becomes frequency curve.

EXAMPLE 12.4

Draw a frequency curve for the following data:

Mid-values	5	10	15	20	25	30	35	40
Frequency	2	4	7	5	10	12	6	4

SOLUTION

For the given (i.e., a table) showing the mid-values of classes and frequencies is made.

Mid-value of Classes	Frequency
5	2
10	4
15	7
20	5
25	10
30	12
35	6
40	4

Now, taking the mid-values of the classes along the *X*-axis and the corresponding frequency along the *Y*-axis, we mark the points obtained from the above table in a graph sheet and join them with a smooth curve, which gives the following frequency curve:

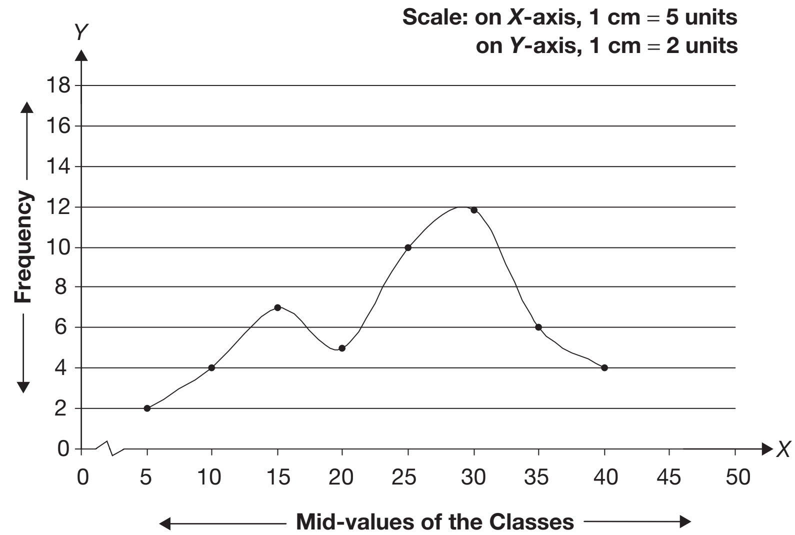


Figure 12.4

Cumulative Frequency Curves

The curves drawn for cumulative frequencies, less than or more than the true limits of the classes of a frequency distribution, are called 'cumulative frequency curves'. The curve drawn for the 'less than cumulative frequency distribution' is called the 'less than cumulative frequency curve' and the curve drawn for the greater than cumulative frequency distribution is called the 'greater than cumulative frequency curve'.

From these curves, we can find the total frequency above or below a particular value of the variable.

EXAMPLE 12.5

For the given distribution, draw the less than and greater than cumulative frequency curves.

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	2	4	5	7	17	12	6	4	3

SOLUTION

Less than cumulative frequency distribution:

Upper Boundaries of the Classes	Frequency	Less than Cumulative Frequency
20	2	2
30	4	6
40	5	11
50	7	18
60	17	35
70	12	47
80	6	53
90	4	57
100	3	60

Greater than cumulative frequency distribution:

Lower Boundaries of the Classes	Frequency	Greater than Cumulative Frequency
10	2	60
20	4	58
30	5	54
40	7	49
50	17	42
60	12	25
70	6	13
80	4	7
90	3	3

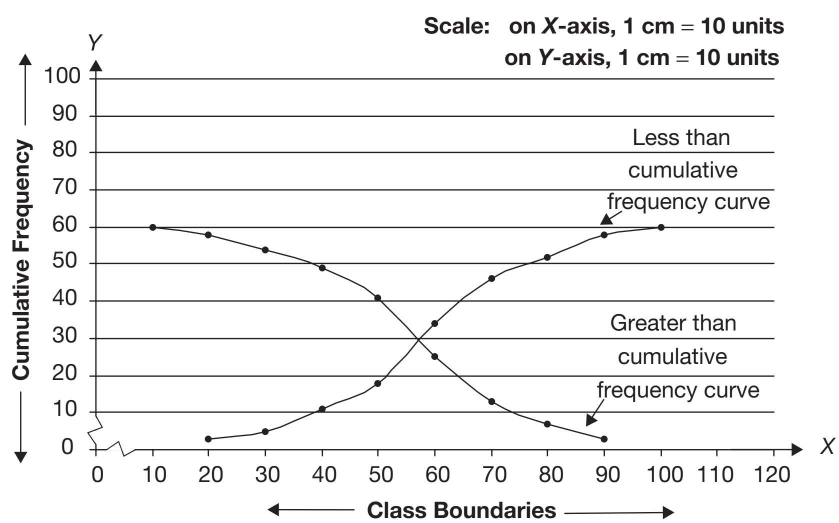


Figure 12.5

Measures of Central Tendencies for Ungrouped Data

Till now, we have observed that data collected in statistical enquiry or investigation happens in the form of raw data.

If data is very large, users cannot get much information. For this reason, data is grouped together to obtain some conclusions.

The measure of central tendency is a value which represents the total data, that is, it is the value in a data around which the values of all the other observations tend to concentrate.

The most commonly used measures of central tendency are the:

1. Arithmetic mean
2. Median
3. Mode

These measures give an idea about how the data is clustered or concentrated.

Arithmetic Mean or Mean (AM)

The arithmetic mean (or simply the mean) is the most commonly used measure of central tendency.

Arithmetic Mean for Raw Data The arithmetic mean of a statistical data is defined as the quotient obtained when the sum of all the observations or entries is divided by the total number of items.

If $x_1, x_2, \ldots, x_n$ are the n items, then:

$$\text{AM} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}, \text{ or briefly } \frac{\sum x}{n}.$$

AM is usually denoted by $\bar{x}$.

EXAMPLE 12.6

Find the mean of the first 10 natural numbers.

SOLUTION

Given data is 1, 2, 3, ..., 10

$$\therefore \text{ Arithmetic mean (AM)} = \frac{\text{Sum of observations}}{\text{Total number of observations}} = \frac{1+2+3+\cdots+10}{10} = \frac{55}{10} = 5.5.$$

Mean of Discrete Series Let $x_1, x_2, x_3, \ldots, x_n$ be n observations with respective frequencies $f_1, f_2, \ldots, f_n$.

This can be considered as a special case of raw data where the observation x_1 occurs f_1 times, x_2 occurs f_2 times, and so on.

$$\therefore \text{ The mean of the above data } = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n}.$$

It can also be represented by, $\bar{x} = \dfrac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$

Weighted Arithmetic Mean

When the variables $x_1, x_2, \ldots, x_n$ do not have same importance, and the weights $w_1, w_2, \ldots, w_n$ are given to each of the variables, the weighted arithmetic mean is given by, $\bar{X}_w = \dfrac{\sum x_iw_i}{\sum w_i}$.

EXAMPLE 12.7

The salaries of 100 workers of a factory are given below:

Salary (in ₹)	Number of Workers
6000	40
8000	25
10000	12

Salary (in ₹)	Number of Workers
12000	10
15000	8
20000	4
25000	1
Total	100

Find the mean salaries of the workers of the factory.

SOLUTION

The mean $\bar{x}$ is given by:

$$\bar{x} = \frac{(6000\times 40)+(8000\times 25)+(10000\times 12)+(12000\times 10) + (15000\times 8)+(20000\times 4)+(25000\times 1)}{40+25+12+10+8+4+1}$$

$\Rightarrow \bar{x} =$ ₹9050

$\therefore$ The mean salary of the workers is ₹9050.

Some Important Results About AM

1. The algebraic sum of deviations taken about the mean is zero. That is, $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$.
2. The value of the mean depends on all the observations.
3. The AM of two numbers a and b is $\frac{a+b}{2}$.
4. **Combined mean:** If $\bar{x}_1$ and $\bar{x}_2$ are the arithmetic means of two series with n_1 and n_2 observations, respectively, then the combined mean is:

$$\bar{x}_c = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

   The above result can be extended to any number of groups of data.

5. If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$, then the mean of $x_1 + a, x_2 + a, x_3 + a, \ldots, x_n + a$ is $\bar{x} + a$, for all values of a.
6. If $\bar{x}$ is the mean of $x_1, x_2, \ldots, x_n$, then the mean of $ax_1, ax_2, \ldots, ax_n$ is $a\bar{x}$ and that of $\frac{x_1}{a}, \frac{x_2}{a}, \ldots, \frac{x_n}{a}$ is $\frac{\bar{x}}{a}$.
7. The mean of the first n natural numbers is $\left(\frac{n+1}{2}\right)$.
8. The mean of the squares of the first n natural numbers $= \frac{(n+1)(2n+1)}{6}$.
9. The mean of the cubes of the first n natural numbers $= \frac{n(n+1)^2}{4}$.

EXAMPLE 12.8

If the average wage of 50 workers is ₹100 and the average wage of 30 of them is ₹120, then find the average wage of the remaining workers.

(a) ₹80 **(b)** ₹70 **(c)** ₹85 **(d)** ₹75

SOLUTION

We know that,

$$\text{Average } = \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$$

$$\therefore \frac{\text{Wage of 50 workers}}{50} = 100$$

$$\Rightarrow \text{Wage}_{50} = 50 \times 1000 = ₹5000$$

Similarly,

$$\frac{\text{Wage}_{30}}{30} = 120 \Rightarrow \text{Wage}_{30} = 30 \times 120 = ₹3600$$

Now,

$$\text{Wage of 20 workers} = \text{Wage}_{50} - \text{Wage}_{30}$$
$$= 5000 - 3000$$
$$= ₹1,400$$

$$\Rightarrow \text{Average wage}_{20} = \frac{1400}{20} = ₹70$$

$\therefore$ Hence, option (b) is the correct answer.

EXAMPLE 12.9

x	2	4	6	8
f	3	5	6	y

The mean of the above data is 5.5. Find the missing frequency (y) in the above distribution.

(a) 6 **(b)** 8 **(c)** 15 **(d)** 11

SOLUTION

Given AM = 5.5

$$\Sigma f \cdot x = 2 \times 3 + 4 \times 5 + 6 \times 6 + 8xy$$
$$= 6 + 20 + 36 + 8y$$
$$\Sigma f \cdot 2 = 62 + 8y \qquad (1)$$

Now $\Sigma f = 3 + 5 + 6 + y$

$$\Sigma f = 14 + y \qquad (2)$$

Using Eqs. (1) and (2)

$$\text{AM} = \frac{\sum f \cdot x}{\sum f} \Rightarrow 5.5 = \frac{62 + 8y}{14 + y}$$

$$\Rightarrow 5.5(14 + y) = 62 + 8y$$

$$\Rightarrow 77 + 5.5y = 62 + 8y$$

$$\Rightarrow 8y - 5.5y = 77 - 62$$
$$\Rightarrow 2.5y = 15$$
$$\Rightarrow y = 6$$

$\therefore$ Hence, option (a) is the correct answer.

Median

Another measure of central tendency of a given data is the median.

Definition

If the values x_i in the raw data are arranged either in the increasing or decreasing order of magnitude, then the middle-most value in this arrangement is called the median.

Thus, for the raw (i.e., ungrouped) data, the median is computed as follows:

1. The values of the observations are arranged in order of magnitude.
2. The middle-most value is taken as the median. Hence, depending on the number of observations (odd or even), we determine median as follows:
   - **(i)** When the number of observations(n) is odd, then the median is the value of $\left(\frac{n+1}{2}\right)$th observation.
   - **(ii)** If the number of observations(n) is even, then the median is the mean of $\left(\frac{n}{2}\right)$th observation and $\left(\frac{n}{2}+1\right)$th observation.

EXAMPLE 12.10

Find the median of the following data: 2, 7, 3, 15, 12, 17 and 5.

SOLUTION

Arranging the given numbers in the ascending order, we have 2, 3, 5, 7, 12, 15, 17.

Here, the middle term is 7.

$\therefore$ Median = 7.

EXAMPLE 12.11

Find the median of the data 5, 8, 4, 12, 16 and 10.

SOLUTION

Arranging the given data in ascending order, we have 4, 5, 8, 10, 12, 16.

As the given number of values is even, we have two middle values. Those are $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations. Those are 8 and 10.

$\therefore$ Median of the data = Average of 8 and 10.

$$= \frac{8+10}{2} = 9.$$

Some Important Facts About Median

1. The median does not take into consideration all the items.
2. The sum of absolute deviations taken about the median is the least.
3. The median can be calculated graphically, but not the mean.
4. The median is not effected by extreme values.
5. The sum of deviations taken about median is less than the sum of absolute deviations taken from any other observation in the data.

EXAMPLE 12.12

A sequence, a, ax, ax^2, …, ax^n, has odd number of terms. Find its median.

(a) ax^{n-1} **(b)** $ax^{(n/2)-1}$ **(c)** $ax^{n/2}$ **(d)** $ax^{(n/2)+1}$

SOLUTION

a, ax, ax^2, ax^3, …, ax^n

As there are odd number of terms, the median is:

$$\left(\frac{n+1+1}{2}\right)\text{th term is}\left(\frac{n+2}{2}\right)\text{th term}$$

$$\text{Median } = a(x^{((n+2)/2)-1}) = a \cdot x^{n/2}.$$

Mode

The third measure of central tendency of a data is the mode.

The most frequently found value in the data is called the mode.

This is the measure which can be identified in the simplest way.

EXAMPLE 12.13

Find the mode of 0, 5, 2, 7, 2, 1, 1, 3, 2, 4, 5, 7, 5, 1 and 2.

SOLUTION

Among the given observations, the most frequently found observation is 2. It occurs 4 times.

$\therefore$ Mode = 2.

Notes

1. For a given data, the mode may or may not exist. In a series of observations, if no item occurs more than once, then the mode is said to be ill-defined.
2. If the mode exists for a given data, it may or may not be unique.
3. Data having unique mode is uni-model, while data having two modes is bi-model.

Properties of Mode

1. It can be graphically calculated.
2. It is not effected by extreme values.
3. It can be used for open-ended distribution and qualitative data.

Empirical Relationship Among Mean, Median, and Mode

For a moderately symmetric data, the above three measures of central tendency can be related by the formula, Mode = 3 × Median − 2 × Mean.

EXAMPLE 12.14

Find the mode when median is 12 and mean is 16 of a data.

SOLUTION

$$\begin{aligned}\text{Mode} &= 3 \times \text{Median} - 2 \times \text{Mean}\\ &= (3 \times 12) - (2 \times 16) = 36 - 32 = 4.\end{aligned}$$

Observations

1. For a symmetric distribution,
$$\text{Mean} = \text{Median} = \text{Mode}.$$
2. Given any two of the mean, median and mode, the third can be calculated.
3. This formula is to be applied in the absence of sufficient data.

Measure of Central Tendencies for Grouped Data

We studied the measure of central tendencies of ungrouped or raw data. Now, we study the measures of central tendencies (i,e., mean, median and mode) for grouped data.

Mean of Grouped Data

If the frequency distribution of 'n' observations of a variable x has k classes, x_i is the mid-value, and f_i is the frequency of ith class, then the mean $\overline{x}$ of grouped data is defined as:

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_kx_k}{f_1 + f_2 + \cdots + f_k} = \frac{\sum_{i=1}^{k} f_ix_i}{\sum_{i=1}^{k} f_i}$$

(or) simply, $\overline{x} = \dfrac{\sum f_ix_i}{N}$; where $N = \sum_{i=1}^{k} f_i$.

In grouped data, it is assumed that the frequency of each class is concentrated at its mid-value.

EXAMPLE 12.15

Calculate the arithmetic mean (AM) of the following data:

Percentage of marks	0–20	20–40	40–60	60–80	80–100
Number of students	2	12	13	15	8

SOLUTION

Let us write the tabular form as given below:

Percentage of Marks	Number of Students (f_i)	Midpoints of Classes (x_i)	$f_i x_i$
0–20	2	10	20
20–40	12	30	360
40–60	13	50	650
60–80	15	70	1050
80–100	8	90	720
	$\Sigma f_i = N = 50$		$\Sigma f_i x_i = 2800$

$$\therefore \text{ Mean } = \bar{x} = \frac{\sum fx}{N} = \frac{2800}{50} = 56.$$

Short-cut Method for Finding the Mean of Group Data (Deviation Method)

Sometimes, when the frequencies are large in number, the calculation of mean using the given formula is cumbersome. This can be simplified if the class interval of each class of grouped data is the same. Under the assumption of equal class interval, we get the following formula for the mean of grouped data:

$$\bar{x} = A + \frac{1}{N}\left(\sum_{i=1}^{k} f_i u_i\right) \times c$$

Where, A = Assumed value from among mid-values

C = Length of class interval

K = Number of classes of the frequency distribution

N = Sum of frequencies $= \sum_{i=1}^{k} f_i$

$u_i = \dfrac{x_i - A}{C}$, $i = 1, 2, 3, \ldots, k$ and

x_i = mid-value of the ith class

u_i is called the deviation or difference of the mid-value of the ith class from the assumed value, divided by the class interval.

Using this method, the previous example can be worked out as follows:

Percentage of Marks	Number of Students (f_i)	Mid-values (x_i)	Deviation $u_i = \dfrac{x_i - A}{C}$	$f_i u_i$
0–20	2	10	−3	−6
20–40	12	30	−2	−24
40–60	13	50	−1	−13
60–80	15	70 (A)	0	0
80–100	8	90	1	8
	$N = 50$			$\Sigma f_i u_i = -35$

Here, $A = 70$; $N = 50$; $C = 20$; $\Sigma f_i u_i = -35$

$$\therefore \text{ AM } = A + \frac{1}{N}(\Sigma f_i u_i) \times c = 70 + \frac{1}{50}(-35) \times 20 = 70 - 14 = 56.$$

Median of Grouped Data

Before finding out how to obtain the median of grouped data, first we review what a median class is.

If 'n' is the number of observations, then from the cumulative frequency distribution, the class in which $\left(\frac{n}{2}\right)$th observation lies is called the median class.

Formula for calculating median:

$$\text{Median } (M) = L + \frac{\frac{n}{2} - F}{f}(C)$$

Where, L = Lower boundary of the median class, i.e., class in which $\left(\frac{n}{2}\right)$th observation lies.

n = Sum of frequencies

F = Cumulative frequency of the class just preceding the median class

f = Frequency of median class

C = Length of class interval

EXAMPLE 12.16

Following is the data showing weights of 40 students in a class. Find its median.

Weight	45	46	47	48	49	50	51	52	53
Number of students	6	2	3	4	7	4	7	4	3

SOLUTION

To find the median, we prepare less than cumulative frequency table as given below:

Weight in kgs	Number of Students	Cumulative Frequency (*F*)
45	6	6
46	2	8
47	3	11
48	4	15
49	7	22
50	4	26
51	7	33
52	4	37
53	3	40

Here, $n = 40$ which is even.

$\therefore$ Median = Value of $\frac{40}{2}$, or the 20th observation.

From the column of cumulative frequency, the value of 20th observation is 49.

$\therefore$ Median = 49 kg.

Note In the above example, we do not have any class interval. As there is no class interval, we cannot use the formula.

EXAMPLE 12.17

Find the median of the following data:

Class interval	0–10	10–20	20–30	30–40	40–50
Frequency	7	6	5	8	9

SOLUTION

To find the median, we prepare the following table:

Class Interval	Frequency	Cumulative Frequency
0–10	7	7
10–20	6	13 (F)
20–30	5 (f)	18
30–40	8	26
40–50	9	35
Total	35	

$$\text{Median } = L + \frac{\left(\frac{n}{2} - F\right)}{f} \times C$$

Here, $n = 35 \Rightarrow \frac{n}{2} = \frac{35}{2} = 17.5$

This value appears in the class 20–30.

L = Lower boundary of median class = (20−30) = 20

$F = 13$; $f = 5$ and $C = 10$ (class length)

$$\therefore \text{ Median } = 20 + \frac{\left(\frac{35}{2} - 13\right)}{5} \times 10 = 20 + \frac{9}{2 \times 5} \times 10 = 29.$$

Mode of Grouped Data

The formula for determining the mode of grouped data is $L_1 + \dfrac{\Delta_1 C}{\Delta_1 + \Delta_2}$.

Where, L_1 = Lower boundary of model class (class with highest frequency).

$\Delta_1 = f - f_1$ and $\Delta_2 = f - f_2$, where f is the frequency of model class

f_1 = Frequency of the previous class of the model class

f_2 = Frequency of the next class of the model class.

Rewriting the formula:

$$\text{Mode } = L_1 + \frac{(f - f_1)C}{(f - f_1) + (f - f_2)}$$

$$\text{Mode } = L_1 + \frac{(f - f_1)C}{2f - (f_1 + f_2)}$$

EXAMPLE 12.18

The following information gives the monthly salaries of 100 employees. Find the mode of the data.

Salary (₹)	2000–3499	3500–4999	5000–6499	6500–7999	8000–9499
Number of Persons	35	37	21	12	5

SOLUTION

Here, the given classes are not continuous.

Hence, we first rewrite it as follows:

Salary	Adjusted Salary (₹)	Number of Persons
2000–3499	1999.5–3499.5	35 (f_1)
3500–4999	3499.5–4999.5	37 (f)
5000–6499	4999.5–6499.5	21 (f_2)
6500–7999	6499.5–7999.5	12
8000–9499	7999.5–9499.5	5

From the above table, it can be known that the maximum frequency occurs in the class interval 3500–4999.

$\therefore f = 37; f_1 = 35; f_2 = 29;$

$L_1 = 3499 \cdot 5;\ C = 1500$

$$\therefore \text{Mode } = L_1 + \frac{(f - f_1)C}{(f - f_1) + (f - f_2)} = 3499.5 + \frac{1500(2)}{2+8} = 3499.5 + 300 = 3799.5.$$

EXAMPLE 12.19

Mode for the following distribution is 22 and $10 > y > x$. Find y.

Class interval	0–10	10–20	20–30	30–40	40–50	Total
Frequency	5	8	10	x	y	30

(a) 2 **(b)** 5 **(c)** 3 **(d)** 4

HINTS

(i) 10 is the model class.

(ii) Using mode $= L + \dfrac{\Delta_1}{\Delta_1 + \Delta_2} \times C$, we can get the value of x.

Range

The difference between the maximum and the minimum values of the given observations is called the range of the data.

Given $x_1, x_2, \ldots, x_n$ (n individual observations)

Range = (Maximum value) − (Minimum value).

EXAMPLE 12.20

Find the range of {2, 7, 6, 4, 3, 8, 5, 12}.

HINTS

By arranging the given data in the ascending order.

We have, {2, 3, 4, 5, 6, 7, 8, 12}

$\therefore$ Range = (Maximum value) − (Minimum value) = 12 − 2 = 10.

Note The range of the class interval is the difference of the actual limits of the class.

Calculation of Variance and Standard Deviation for Raw Data

Standard deviation (SD) is referred to as the root mean squared deviation about the mean. It is denoted by the symbol σ and read as sigma.

Variance is denoted by σ^2, which is the square of the standard deviation.

$\therefore$ Varience = $(\text{SD})^2$, or SD $= \sqrt{\text{Varience}}$

$$\text{SD } (\sigma) = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

where, x_1, x_2, ..., x_n are n observations with mean as $\bar{x}$.

Alternatively, the above formula can also be written as

$$\text{SD } (\sigma) = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}.$$

EXAMPLE 12.21

Calculate variance and standard deviation of the following data:

10, 12, 8, 14, 16.

SOLUTION

$$\text{AM } (\bar{x}) = \frac{10 + 12 + 8 + 14 + 16}{5} = \frac{60}{5} = 12$$

$$\text{Varience } = \frac{(10 - 12)^2 + (12 - 12)^2 + (8 - 12)^2 + (14 - 12)^2 + (16 - 12)^2}{5}$$

$$= \frac{4 + 0 + 16 + 4 + 16}{5} = \frac{40}{5} = 8$$

$$\text{SD } (\sigma) = \sqrt{\text{Variance}} = \sqrt{8}.$$

Calculation of Variance and SD for a Grouped Data

1. $N = \Sigma f$ = The sum of the frequencies
2. $\text{AM } (\bar{x}) = \dfrac{\sum fx}{N}$

3. D = Deviation from the AM $(x - \bar{x})$

4. Standard deviation (SD) $= \sigma = \sqrt{\dfrac{\sum fD^2}{N}}$

EXAMPLE 12.22

Calculate SD for the given data:

f	1	2	3	4
x	5	10	15	20

SOLUTION

f	x	fx	$D = x - \bar{x}$	D^2	fD^2
1	5	5	−10	100	100
2	10	20	−5	25	50
3	15	45	0	0	0
4	20	80	5	25	100
$\Sigma f = 10$		$\Sigma fx = 150$			$\Sigma fD^2 = 250$

$$\text{AM } (\bar{x}) = \frac{\sum fx}{N} = \frac{150}{10} = 15$$

$$\text{SD } = \sigma = \sqrt{\frac{\sum fD^2}{N}} = \sqrt{\frac{250}{10}} = 5.$$

EXAMPLE 12.23

Find the SD for the given data:

CI	0–10	10–20	20–30	30–40
Frequency	4	3	2	1

CI	f	x Mid-point	fx	$D = x - \bar{x}$	D^2	fD^2
0–10	4	5	20	–10	100	400
10–20	3	15	45	0	0	0
20–30	2	25	50	10	100	200
30–40	1	35	35	20	400	400
	$\Sigma f = 10$		$\Sigma fx = 150$			$\Sigma fD^2 = 1000$

$$\text{AM } = \bar{x} = \frac{\sum fx}{\sum f} = \frac{150}{10} = 15$$

$$\text{SD } = \sigma = \sqrt{\frac{\sum fD^2}{N}} = \sqrt{\frac{1000}{10}} = \sqrt{100} = 10.$$

EXAMPLE 12.24

If the standard deviation of $2x_i + 3$ is 8, then the variance of $\frac{3x_i}{2}$.

(a) 24 **(b)** 36 **(c)** 6 **(d)** 18

SOLUTION

SD $(2x_i) + 3 = 8$

SD of $2x_i = 8$ ($\because$ SD dose not alter when term is decreased by fixed constant.)

SD of $x_i = 4$

Variance of $x_i = \text{SD}^2 = 16$

$$\text{Variance of } \frac{3x_i}{2} = \left(\frac{9}{4}\right) \times \text{Variance of } x_i \; (\because \text{Var}(ax + b) = a^2 \text{ Var}(x))$$

$$= \left(\frac{9}{4}\right)(16) = 36$$

$\therefore$ Hence, option (b) is the correct answer.

Coefficient of Variation (CV)

For any given data, let SD be the standard deviation and AM be the arithmetic mean, then coefficient of variation (CV) of the data is defined as:

$$\text{CV} = \frac{\text{SD}}{\text{AM}} \times 100$$

This is a relative measure and helps in measuring the consistency. Smaller the CV, greater is the consistency.

EXAMPLE 12.25

In a series of observations, find the coefficient of variation, given SD = 12.5 and AM = 50.

SOLUTION

$$\text{CV} = \frac{\text{SD}}{\text{AM}} \times 100 = \frac{12.5}{50} \times 100 = 25.$$

$\therefore$ Coefficient of variation = 25.

Quartiles

In a given data, the observations that divide the given set of observations into four equal parts are called quartiles.

First Quartile or Lower Quartile

When the given observations are arranged in ascending order, the observation which lies mid-way between the lower extreme and the median is called the first quartile, or the lower quartile, and is denoted as Q_1.

Third Quartile or Upper Quartile

Of the data when the given observations are arranged in ascending order, the observation that lies in mid-way between the median and the upper extreme observation is called the third quartile, or the upper quartile, and is denoted by Q_3.

We can find Q_1 and Q_3 for an ungrouped data containing n observations as follows:

We arrange the given n observations or items in ascending order, then

Lower or first quartile, Q_1 is $\left(\frac{n}{4}\right)$th item or observation if n is even and $\left(\frac{n+1}{4}\right)$th item or observation when n is odd.

EXAMPLE 12.26

Find Q_1 for 8, 12, 7, 5, 16, 10, 21 and 19.

SOLUTION

Arranging the given observations in ascending order.

We have, 5, 7, 8, 10, 12, 16,19, 21

Here, $n = 8$ (n is even)

$\therefore$ First quartile, $Q_1 = \left(\frac{n}{4}\right)$th item $= \left(\frac{8}{4}\right)$th item = 2nd item of the data, i.e., 7.

$\therefore\ Q_1 = 7.$

EXAMPLE 12.27

Find Q_1 of the observations 21, 12, 9, 6, 18, 16 and 5.

SOLUTION

Arranging the observations in ascending order, we have 5, 6, 9, 12, 16, 18, 21.

Here, $n = 7$ (odd)

$\therefore Q_1 = \left(\frac{n+1}{4}\right)$th item, i.e., $\left(\frac{7+1}{4}\right)$th item = 2nd item.

$\therefore\ Q_1 = 6.$

EXAMPLE 12.28

The marks of 10 students in a class are 38, 24, 16, 40, 25, 27, 17, 32, 22, and 26. Find Q_1.

SOLUTION

The given observations, when arranged in ascending order, we have 16, 17, 22, 24, 25, 26, 27, 32, 38, 40.

Here, $n = 10$ (even)

$\therefore Q_1 = \left(\frac{n}{4}\right)$th item $= \left(2\frac{1}{2}\right)$th item of the data

$\therefore\ Q_1 = 2\text{nd item } + \frac{1}{2}(3\text{rd} - 2\text{nd})\text{ item } = 17 + \frac{1}{2}(22 - 17) = 17 + \frac{5}{2} = 19.5$

$\therefore\ Q_1 = 19.5$

Third quartile: Q_3 is $\left(\frac{3n}{4}\right)$th item, when n is even and $3\left(\frac{n+1}{4}\right)$th item when n is odd.

EXAMPLE 12.29

Find Q_3 for 7, 16, 19, 10, 21 and 12.

SOLUTION

By arranging the data in ascending order, we have 7, 10, 12, 16, 19, 21.

Here, $n = 6$ (even)

$\therefore Q_3 = 3\left(\frac{n}{4}\right)$th item $= 4\frac{1}{2}$ th item

$\Rightarrow Q_3 = \left[4\text{th} + \frac{1}{2}(5\text{th} - 4\text{th})\right]$ item $= 16 + \frac{1}{2}(19 - 16)$

$Q_3 = 17.5$.

Semi-inter Quartile Range or Quartile Deviation (QD)

Quartile deviation is given by the formula, QD $= \frac{Q_3 - Q_1}{2}$.

EXAMPLE 12.30

Find semi-inter quartile range of the following data:

X	2	5	6	8	9	10	12
Frequency	1	8	12	16	11	9	3

SOLUTION

X	Frequency	Cumulative Frequency
2	1	1
5	8	9
6	12	21
8	16	37
9	11	48
10	9	57
12	3	60
	$N = 60$	

Here, $N = 60$

$$\therefore Q_1 = \left(\frac{N}{4}\right)\text{th item} = \left(\frac{60}{4}\right)\text{th item} = 15\text{th item}$$

$\therefore$ $Q_1 = 6$ (as the 15th item lies in 21 in the cumulative frequency)

$$Q_3 = 3\left(\frac{N}{4}\right)\text{th item} = 45\text{th item}$$

$\therefore$ $Q_3 = 9$ (as 45th item lies in 48 in the cumulative frequency)

$$\text{Semi-inter quartile range (QD)} = \frac{Q_3 - Q_1}{2} = \frac{9-6}{2} = 1.5.$$

Note For an individual data, the second quartile Q_2 coincides with median.

$\Rightarrow Q_2 =$ Median of the data.

EXAMPLE 12.31

The heights of 31 students in a class are given below:

Height (in cm)	126	127	128	129	130	131	132
Number of Students	7	3	4	2	5	6	4

1. Find the median of the above frequency distribution.
2. Find the semi-interquartile range of the above frequency distribution.

HINTS

1. Find the less than cumulative frequency, then find the median by using formulae.
2. Find the less than cumulative frequency, then find the inter-quartile range by using formulae

Estimation of Median and Quartiles from Ogive

1. Prepare the cumulative frequency table with the given data.
2. Draw ogive.
3. Let, total number of observations = Sum of all frequencies = N.
4. Mark the points A, B and C on Y-axis, corresponding to $\frac{N}{4}$, $\frac{N}{2}$ and $\frac{3N}{4}$.
5. Mark three points (P, Q, R) on ogive corresponding to $\frac{3N}{4}$, $\frac{N}{2}$ and $\frac{N}{4}$.
6. Draw vertical lines from the points R, Q and P to meet X-axis Q_1, M and Q_3.
7. Then, the abscissas of Q_1, M and Q_3 gives lower quartile, median and upper-quartile.

EXAMPLE 12.32

The following table shows the distribution of the weights of a group of students:

Weight in Kg	30–35	35–40	40–45	45–50	50–55	55–60	60–65
No. of students	5	6	7	5	4	3	2

SOLUTION

Class Interval	Number of Students (Frequency)	Cumulative Frequency
30–35	5	5
35–40	6	11
40–45	7	18
45–50	5	23
50–55	4	27
55–60	3	30
60–65	2	32
	$N = 32$	

$N = 32$

$$\Rightarrow \frac{N}{4} = 8, \frac{N}{2} = 16 \text{ and } \frac{3N}{4} = 24$$

From the graph:

Lower quartile $(Q_1) = 38$

Upper quartile $(Q_3) = 52$

Median $(M) = 44$

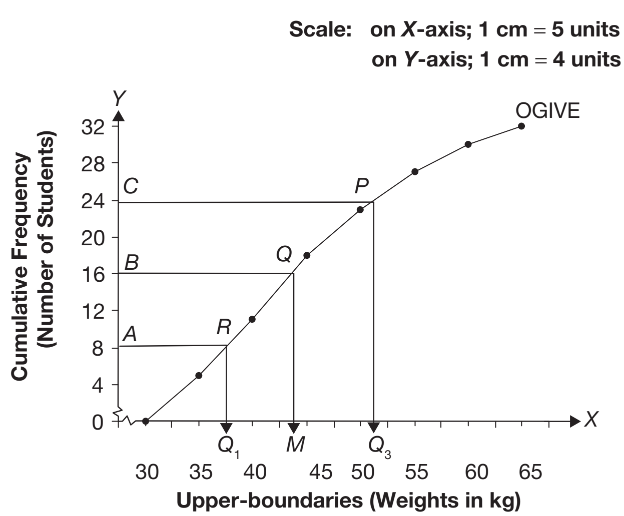


Figure 12.6

Estimation of Mode from Histogram

1. Draw a histogram to represent the given data.
2. From the upper corners of the highest rectangle, draw segments to meet the opposite corners of adjacent rectangles, diagonally as shown in the given example. Mark the intersecting point as *P*.
3. Draw *PM* perpendicular to *X*-axis, to meet *X*-axis at *M*.
4. Abscissa of *M* gives the mode of the data.

EXAMPLE 12.33

Estimate mode of the following data from the histogram:

CI	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	10	16	17	20	15	13	12

From the graph, mode (M) = 34

SOLUTION

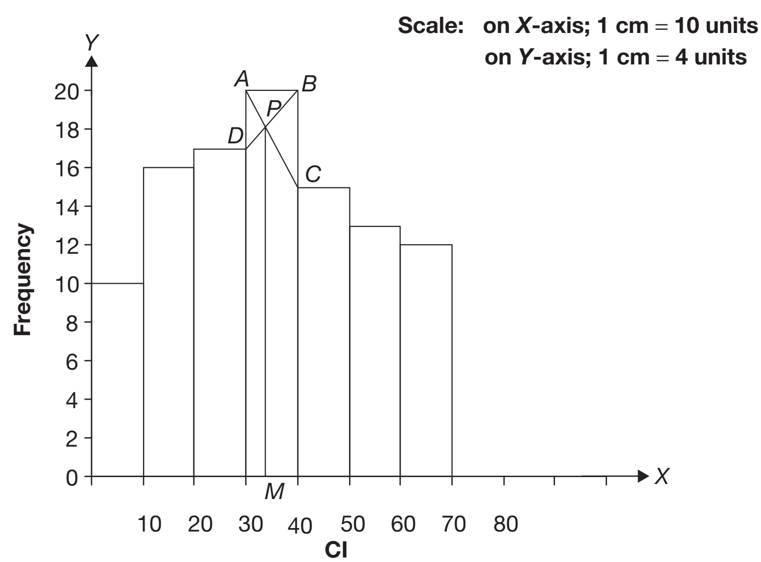


Figure 12.7

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. The class mark of a class is 25, and if the upper limit of that class is 40, then its lower limit is _______.
2. Consider the data: 2, 3, 2, 4, 5, 6, 4, 2, 3, 3, 7, 8, 2, 2. The frequency of 2 is _______.
3. 1–5, 6–10, 11–15, …, are the classes of a distribution, the upper boundary of the class 1–5 is _______.
4. 0–10, 10–20, 20–30, …, are the classes, the lower boundary of the class 20–30 is _______.
5. The mid-value of 20–30 is _______.
6. If 1–5, 6–10, 11–15, …, are the classes of a frequency distribution, then the size of the class is _______.
7. A class interval of a data has 15 as the lower limit and 25 as the size. Then, the class mark is _______.
8. In a histogram, the _______ of all rectangles are equal. (width/length/area)
9. The sum of 12 observations is 600, then their mean is _______.
10. If the lower boundary of the class is 25 and the size of the class is 9, then the upper boundary of the same class _______.
11. If 1–5, 6–10, 11–15, 16–20, …, are the classes of a frequency distribution, then the lower boundary of the class 11–15 is _______.
12. Arithmetic mean of first n natural numbers is _______.
13. The width of a rectangle in a histrogram represents frequency of the class. (True/False)
14. If 16 observations are arranged in ascending order, then the median is

$$\frac{(\text{8th observation } + \text{9th observation})}{2}. \text{(True/False)}$$

15. The mean of x, y, z is y, then $x + z = 2y$. (True/False)
16. Range of the scores 25, 33, 44, 26, 17 is _______.
17. Upper quartile of the data 4, 6, 7, 8, 9 is _______.
18. 2(Median − Mean) = Mode − Mean. (True/False)
19. Lower quartile of the data 5, 7, 8, 9, 10 is _______.
20. Consider the data: 2, x, 3, 4, 5, 2, 4, 6, 4, where $x > 2$. The mode of the data is _______.
21. Find the mean and the median of the data 10, 15, 17, 19, 20, and 21.
22. Find the semi-inter quartile range of the data: 32, 33, 38, 39, 36, 37, 40, 41, 47, 34, and 49.
23. Find the mean of first 726 natural numbers.
24. Find the range of the data: 14, 16, 20, 12, 13, 4, 5, 7, 29, 32, and 6.
25. Find the mean of the observations 425, 430, 435, 440, 445, …, 495. (Difference between any two given consecutive observation is equal)
26. The mean of 10 observations is 15.5. By an error, one observation is registered as 13 instead of 34. Find the actual mean.
27. Observations of the certain data are $\frac{x}{8}, \frac{x}{4}, \frac{x}{2}, x, \frac{x}{16}$ where $x > 0$. If median of the given data is 8, then find the mean of the given data.
28. The mean of 12 observations is 14. By an error, one observation is registered as 24 instead of −24. Find the actual mean.
29. The mean weight of 20 students is 25 kg and the mean weight of another 10 students is 40 kg. Find the mean weight of the 30 students.
30. Find the variance and standard deviation of the scores 7, 8, 9, 10 and 16.

Short Answer Type Questions

31. Tabulate the given data by taking

    Class intervals: 1–10, 11–20, 21–30, 31–40

    Data: 9, 10, 8, 6, 7, 4, 3, 2, 16, 28, 22, 36, 24, 18, 27, 35, 19, 29, 23, 34.
32. If the mean and the median of a unimodel data are 34.5 and 32.5, then find the mode of the data.
33. The heights of 100 students in primary classes is classified as follows. Find the median.

Height (in cm)	Number of Students
81	22
82	14
83	26
84	23
85	15

34. The weight (in kg) of 25 children of 9th class is given. Find the mean weight of the children.

Weight (in kg)	Number of Children
40	3
41	4
42	6
43	2
44	5
45	5

35. If the mean of the following data is 5.3, then find the missing frequency y of the following distribution:

x	f
4	11
8	2
6	3
7	y

36. The mean of the data is 15. If each observation is divided by 5 and 2 is added to each result, then find the mean of the observations so obtained.

37. Draw the histogram for the following distribution:

Marks	Number of Students
0–10	3
10–20	4
20–30	8
30–40	9
40–50	6

38. Find the mode of the following data:

Class Interval	Frequency
1–5	3
6–10	4
11–15	10
16–20	6
21–25	7

39. A six-faced balanced dice is rolled 20 times, and the frequency distribution of the integers obtained is given below. Find the inter quartile range.

Integer	Frequency
1	3
2	4
3	2
4	5
5	4
6	2

40. Construct a less than cumulative frequency curve and a greater than cumulative frequency curve and answer the following:

Daily Wages (in ₹)	Number of Persons
20–30	5
30–40	12
40–50	17
50–60	36
60–70	20
70–80	10
80–90	8
90–100	2

(i) Find the number of persons who received ₹60 and more than ₹60.

(ii) Find the number of persons who received ₹90 and less than ₹90.

41. Draw the frequency polygon for the following distribution:

Class Interval	Frequency
0–5	8
5–10	12
10–15	20
15–20	16
20–25	4

42. Find the median of the following data:

Class Interval	Frequency
0–20	8
20–40	10
40–60	12
60–80	9
80–100	9

43. Construct a greater than cumulative frequency curve.

Class Interval	Frequency
5–9	1
10–14	5
15–19	10
20–24	19
25–29	25
30–34	21
35–39	15
40–44	3
45–49	1

44. Draw a histogram of the following data on a graph paper and estimate the mode.

Percentage of Marks	Number of Students
0–20	10
20–40	12
40–60	16
60–80	14
80–100	8

45. Find the coefficient of variation of the following dice create series.

Scores	Frequency
1	0
2	4
3	3
4	2
5	1

Essay Type Questions

46. If the mean of the following table is 30, then find the missing frequencies.

Class Interval	Frequency
0–15	10
15–30	a
30–45	b
45–60	8
Total	60

47. Calculate the AM of the following data using short-cut method.

Marks	Number of Students
0–10	3
10–20	4
20–30	6
30–40	8
40–50	9

48. For the following frequency distribution, construct a less than cumulative frequency curve. And also find Q_1, Q_2, Q_3 by using graph.

Class Interval	Frequency
0–9	4
10–19	3
20–29	5
30–39	6
40–49	1
50–59	2
60–69	1

49. Find the standard deviation of the following discrete series.

Scores	Frequency
1	0
2	4
3	3
4	2
5	1

50. Find the variance and SD for the given frequency distribution.

Class Interval	Frequency
1–5	4
6–10	1
11–15	2
16–20	3

CONCEPT APPLICATION

Level 1

1. If the arithmetic mean of the first n natural numbers is 15, then n is _______.
   (a) 15 (b) 30
   (c) 14 (d) 29

2. If the arithmetic mean of 7, 8, x, 11, 14 is x, then x is _______.
   (a) 9 (b) 9.5
   (c) 10 (d) 10.5

3. Find the mode of the data 5, 3, 4, 3, 5, 3, 6, 4, 5.
   (a) 5 (b) 4
   (c) 3 (d) Both (a) and (c)

4. The median of the data 5, 6, 7, 8, 9, 10 is _______.
   (a) 7 (b) 8
   (c) 7.5 (d) 8.5

5. If a mode exceeds a mean by 12, then the mode exceeds the median by _______.
   (a) 4 (b) 8
   (c) 6 (d) 10

6. If the less than cumulative frequency of a class is 50 and that of the previous class is 30, then the frequency of that class is _______.
   (a) 10 (b) 20
   (c) 40 (d) 30

7. If the median of the data, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 is a, then find the median of the data x_3, x_4, x_5, x_6. (where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8$)
   (a) a (b) $\frac{a}{2}$
   (c) $\frac{a}{4}$ (d) $\frac{a}{3}$

8. The mode of the data 6, 4, 3, 6, 4, 3, 4, 6, 5 and x can be:
   (a) Only 5 (b) Both 4 and 6
   (c) Both 3 and 6 (d) 3, 4 or 6

9. If the greater than cumulative frequency of a class is 60 and that of the next class is 40, then find the frequency of that class.
   (a) 10 (b) 20
   (c) 50 (d) 30

10. If the difference between the mode and median is 2, then the difference between the median and mean is _______ (in the given order).
   (a) 2 (b) 4
   (c) 1 (d) 0

11. In a series of observations, SD is 7 and mean is 28. Find the coefficient of variation.
   (a) 4 (b) $\frac{1}{4}$
   (c) 25 (d) 12.5

12. If the SD of x_1, x_2, x_3, …, x_n is 5, then find SD of $x_1 + 5$, $x_2 + 5$, $x_3 + 5$, …, $x_n + 5$.
   (a) 0 (b) 10
   (c) 5 (d) 2

13. In a series of observations, coefficient of variation is 16 and mean is 25. Find the variance.
   (a) 4 (b) 8
   (c) 12 (d) 16

14. If the SD of y_1, y_2, y_3, …, y_n is 6, then variance of $(y_1 - 3)$, $(y_2 - 3)$, $(y_3 - 3)$, …, $(y_n - 3)$ is _______.
   (a) 6 (b) 36
   (c) 3 (d) 27

15. Lower quartile, upper quartile and interquartile range are Q_1, Q_3 and Q. If the average of Q, Q_1 and Q_3 is 40 and semi-interquartile range is 6, then find the lower quartile.
   (a) 24 (b) 36
   (c) 48 (d) 60

Direction for questions 16 and 17:
These questions are based on the following data.

The weights of 20 students in a class are given below.

Weight (In kg)	Number of Students
31	6
32	3
33	5
34	2
35	4

16. Find the median of the above frequency distribution.

(a) 32.5 (b) 33
(c) 33.5 (d) 32

17. The interquartile range of the above frequency distribution is _______.

(a) 4 (b) 3
(c) 2 (d) 1

18. If the average of a, b, c and d is the average of b and c, then which of the following is necessarily true?

(a) $(a + d) = (b + c)$
(b) $(a + b) = (c + d)$
(c) $(a - d) = (b - c)$
(d) $\frac{(a+b)}{4} = \frac{(b+c)}{2}$

19. Find the interquartile range of the data 3, 6, 5, 4, 2, 1 and 7.

(a) 4 (b) 3
(c) 2 (d) 1

20. If the mean of the lower quartile and upper quartile is 10 and the semi-interquartile range is 5, then the lower quartile and the upper quartile are _______ and _______.

(a) 2, 12 (b) 3, 13
(c) 4, 14 (d) 5, 15

21. The lower quartile of the data 5, 3, 4, 6, 7, 11, 9 is _______.

(a) 4 (b) 3
(c) 5 (d) 6

22. Find the arithmetic mean of the first 567 natural numbers.

(a) 284 (b) 283.5
(c) 283 (d) 285

23. If $a < b < c < d$ and a, b, c, d are non-zero integers, the mean and median of a, b, c, d is 0, then which of the following is correct?

(a) $b = -c$
(b) $a = -d$
(c) Both (a) and (b)
(d) None of these

24. The mean of 16 observations is 16. If one observation 16 is deleted and three observations 5, 5 and 6 are included, then find the mean of the final observations.

(a) 16 (b) 15.5
(c) 13.5 (d) None of these

25. If $L = 44.5$, $N = 50$, $F = 15$, $f = 5$ and $C = 20$, then find the median from of given data.

(a) 84.5 (b) 74.5
(c) 64.5 (d) 54.5

26. If $L = 39.5$, $\Delta_1 = 6$, $\Delta_2 = 9$ and $c = 10$, then find the mode of the data.

(a) 45.5 (b) 43.5
(c) 46.5 (d) 44.5

27. The average weight of 55 students is 55 kg, and the average weight of another 45 students is 45 kg. Find the average weight of all the students.

(a) 48 kg (b) 50 kg
(c) 50.5 kg (d) 52.25 kg

28. If the mean of 26, 19, 15, 24, and x is x, then find the median of the data.

(a) 23 (b) 22
(c) 20 (d) 21

Level 2

29. The mean and median of the data a, b and c are 50 and 35, where $a < b < c$. If $c - a = 55$, then find $(b - a)$.

(a) 8 (b) 7
(c) 3 (d) 5

30. If $a < b < 2a$, and the mean and the median of a, b and $2a$ are 15 and 12, then find a.

(a) 7 (b) 11
(c) 10 (d) 8

31. The variance of $6x_i + 3$ is 30, find the standard deviation of x_i.

(a) $\dfrac{5}{\sqrt{6}}$ (b) $\sqrt{\dfrac{5}{6}}$
(c) 30 (d) $\sqrt{30}$

32. The frequency distribution of the marks obtained by 28 students in a test carrying 40 marks is given below:

Marks	Number of Students
0–10	6
10–20	x
20–30	y
30–40	6

If the mean of the above data is 20, then find the difference between x and y.

(a) 3 (b) 2
(c) 1 (d) 0

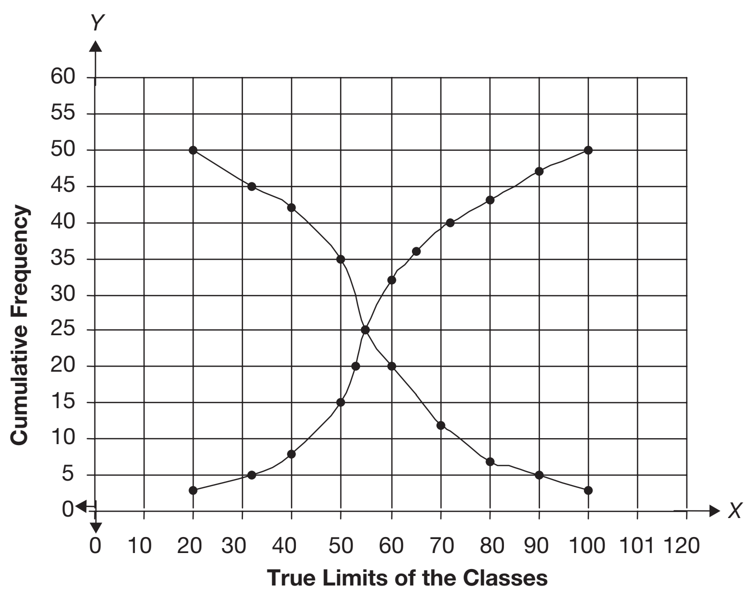


Figure 12.8

Direction for questions 37 and 38:
These question are based on the following data above (figure)

The given figure represents the percentage of marks on the X-axis and the number of students on Y-axis.

33. Find the number of students who scored less than or equal to 50% of marks.

(a) 35 (b) 15
(c) 20 (d) 30

34. Find the number of students who scored greater than or equal to 90% of marks.

(a) 47 (b) 45
(c) 5 (d) 10

35. Find the variance of the scores 2, 4, 6, 8 and 10.

(a) 2 (b) 4
(c) 6 (d) 8

36. If $A = 55.5$, $N = 100$, $C = 20$, and $\Sigma f_i d_i = 60$, then find the mean from the given data.

(a) 67.5 (b) 57.5

(c) 77.5 (d) 47.5

37. Mode for the following distribution is 17.5 and x is less than 6. Find x.

Class Interval	Frequency
0–5	5
5–10	2
10–15	3
15–20	6
20–25	x

(a) 3 (b) 2

(c) 4 (d) 5

Direction for questions 38 and 39:
These questions are based on the following data: consider the following distribution table.

Class Interval	Frequency
0–6	2
6–12	4
12–18	6

38. Find the coefficient of variation for the given distribution.

(a) $\frac{200\sqrt{6}}{11}$ (b) $\frac{200\sqrt{3}}{11}$

(c) $\frac{500}{11}$ (d) $\frac{200\sqrt{5}}{11}$

39. Find the variance for the given distribution:

(a) 24 (b) 12

(c) 20 (d) 25

Directions for questions 40 to 44:
Select the correct alternative from the given choices.

40. Find the mean of the quartiles Q_1, Q_2 and Q_3 of the data 5, 9, 8, 12, 7, 13, 10, 14.

(a) 9 (b) 10

(c) 9.5 (d) 11.5

41. Which of the following cannot be determined?

(A) Range of the factors of 64

(B) Range of the first 10 positive integers

(a) A

(b) B

(c) Both (A) and (B)

(d) None of these

42. Find the mean of the following data.

Range of first n natural numbers, range of negative integers from $-n$ to -1 (where $-n < -1$), range of first n positive even integers and range of first n positive odd integers.

(a) $\frac{3}{2}(n-1)$ (b) $\frac{3n-2}{2}$

(c) $\frac{3}{2}(n-2)$ (d) $\frac{4n-3}{2}$

43. The following are the steps involved in finding the mean of the data.

x	f
10	1
8	3
6	5
4	7
2	9

Arrange them in sequential order.

(A) $\therefore$ Mean $= \frac{\sum fx}{\sum f} = \frac{110}{25}$

(B) $\Sigma fx = 10 + 24 + 30 + 28 + 18$

$\Sigma f = 1 + 3 + 5 + 7 + 9$

(C) $\therefore$ Mean = 4.4

(D) $\Sigma fx = 110$ and $\Sigma f = 25$

(a) ABDC (b) ACBD

(c) BDAC (d) BCAD

44. The mean weight of a group of 9 students is 19 kg. If a boy of weight 29 kg is joined in the group, then find the mean weight of 10 students.

The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) The mean weight of 10 students $= \frac{200}{10}$ kg

(B) The total weight of 9 students $= 9 \times 19$ kg $= 171$ kg

(C) The total weight of 10 students $= (171 + 29)$ kg $= 200$ kg

(D) $\therefore$ The mean weight $= 20$ kg

(a) BCAD (b) BDAC

(c) BDCA (d) BCDA

Level 3

45. The arithmetic mean of the following data is 7. Find $(a + b)$.

x	f
4	a
6	4
7	b
9	5

(a) 4 (b) 2

(c) 3 (d) Cannot be determined

Direction for questions 46 and 47:
The questions are based on the following data.

The performance of four students in annual report is given below.

Name of Student	Mean Score $(\overline{x})$	SD (σ)
Dheeraja	75	11.25
Nishitha	65	5.98
Sindhuja	48	8.88
Akshitha	44	5.28

46. Who is more consistent than the others?

(a) Dheeraja (b) Nishitha

(c) Sindhuja (d) Akshitha

47. Who is less consistent than the others?

(a) Dheeraja (b) Nishitha

(c) Sindhuja (d) Akshitha

48. If the mean of the squares of first n natural numbers is 105, then find the median of the first n natural numbers.

(a) 8 (b) 9

(c) 10 (d) 11

49. Range of the scores 18, 13, 14, 42, 22, 26 and x is 44 $(x > 0)$. Find the sum of the digits of x.

(a) 16 (b) 14

(c) 12 (d) 18

50. Find the arithmetic mean of the series 1, 3, 5, …, $(2n - 1)$.

(a) $\frac{2n-1}{n}$ (b) $\frac{2n+1}{n}$

(c) n (d) $n + 2$

51. The arithmetic mean of the squares of first n natural numbers is _______.

(a) $\frac{(n+1)(2n+1)}{6}$ (b) $\frac{n+1}{6}$

(c) $\frac{n^2-1}{6}$ (d) $\frac{n-1}{6}$

52. If X, M, Z are denoting mean, median and mode of a data and $X : M = 9 : 8$, then find the ratio $M : Z$.

(a) 8 : 9 (b) 4 : 3

(c) 7 : 6 (d) 5 : 4

53. The arithmetic mean of the series 1, 3, 3^2, …, 3^{n-1} is _______.

(a) $\frac{3^n}{2n}$ (b) $\frac{3^n-1}{2n}$

(c) $\frac{3^{n-1}}{n+1}$ (d) $\frac{3^n+1}{2n}$

54. The mean of the data x, $x + a$, $x + 2a$, $x + 3a$, …, $(2n + 1$ terms$)$ is _______.

(a) $x + (n - 1)a$ (b) $x + (n + 1)a$

(c) $x + (n + 2)a$ (d) $x + an$

PRACTICE QUESTIONS

55. The mean height of 25 boys in a class is 150 cm, and the mean height of 35 girls in the same class is 145 cm. The combined mean height of 60 students in the class is ______ (approximately).

(a) 143 cm (b) 146 cm

(c) 147 cm (d) 145 cm

56. The sum of 15 observations of a data is $(434 + x)$. If the mean of the data is x, then find x.

(a) 25 (b) 27

(c) 31 (d) 33

57. The mean weight of 9 students is 25 kg. If one more student is joined in the group the mean is unaltered, then the weight of the 10th student is ______ (in kg).

(a) 25 (b) 24

(c) 26 (d) 23

58. Observations of some data are $\frac{x}{5}, x, \frac{x}{3}, \frac{2x}{3}, \frac{x}{4}, \frac{2x}{5}$ and $\frac{3x}{4}$ where $x > 0$. If the median of the data is 4, then find the value of 'x'.

(a) 5 (b) 7

(c) 8 (d) 10

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 10
2. 5
3. 5.5
4. 20
5. 25
6. 5
7. 27.5
8. width
9. 50
10. 34
11. 10.5
12. $\frac{(n+1)}{2}$
13. False
14. True
15. True
16. 27
17. 8.5
18. False
19. 6
20. 4
21. Mean = 17

    Median = 18
22. 3.5
23. 363.5
24. 28.
25. 460
26. 17.6
27. 12.4
28. 10
29. 30 kg
30. Variance = 10, SD $= \sqrt{10}$

Short Answer Type Questions

31.

<table>
<tr><th>Class Interval</th><th>Tally Marks</th><th>Frequency</th></tr>
<tr><td>1–10</td><td>𝍸 ||||</td><td>8</td></tr>
<tr><td>11–20</td><td>|||</td><td>3</td></tr>
<tr><td>21–30</td><td>𝍸 |</td><td>6</td></tr>
<tr><td>31–40</td><td>|||</td><td>3</td></tr>
</table>

32. 28.5
33. 83 cm
34. 42.68 kg
35. 4
36. 5
38. 13.5
39. 3
40. (i) 40 (ii) 108
42. 50
44. 53
45. $33\frac{1}{3}$

Essay Type Questions

46. 18, 24
47. 30.33
49. 1
50. Variance = 41

    SD $= \sqrt{41}$

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (a)	**8.** (d)	**9.** (b)	**10.** (c)
11. (c)	**12.** (c)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (d)
21. (a)	**22.** (a)	**23.** (c)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (d)		

Level 2

29. (d)	**30.** (b)	**31.** (b)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (d)	**36.** (a)	**37.** (a)	**38.** (d)
39. (c)	**40.** (c)	**41.** (d)	**42.** (a)	**43.** (c)	**44.** (a)				

Level 3

45. (d)	**46.** (b)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (c)	**51.** (a)	**52.** (b)	**53.** (b)	**54.** (d)
55. (c)	**56.** (c)	**57.** (a)	**58.** (d)						

CONCEPT APPLICATION

Level 1

1. Arithmatic mean of first n natural numbers is $\frac{n+1}{2}$.

2. Arithmatic mean $= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$.

3. An observation which has more frequency in the data is called the mode of the data.

4. If the number of observations is even, then the median of the data is the average of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

5. Use the 'empirical formula'.

6. Frequency of a particular class = Cumulative frequency of that class − Cumulative frequency of the previous class.

7. If the number of observations is even, then the median of the data is the average of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

8. An observation which has more frequency in the data is called the mode of the data.

9. Frequency of a particular Class = (cumulative frequency of that class) − (Cumulative frequency of the next class).

10. Use the empirical formula.

11. Coefficient of variation $= \frac{\text{Standard deviation}}{\text{Mean}} \times 100$.

12. SD does not alter when each term is increased by fixed constant.

13. Coefficient of variation $= \frac{\text{Standard deviation}}{\text{Mean}} \times 100$.

14. SD does not alter when each term is decreased by fixed constant.

15. Semi-interquartile range $= \frac{Q_3 - Q_1}{2}$.

16. Find the less than cumulative frequency, then find the median by using formulae.

17. Find the less than cumulative frequency, then find inter-quartile range by using formulae.

18. Average $= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$.

19. $Q = Q_3 - Q_1$.

20. Semi-interquartile range $= \frac{Q_3 - Q_1}{2}$.

21. Write the data in the ascending order.

    Lower quartile is $\left(\frac{n+1}{4}\right)$ observation.

22. Arithmetic mean of first 'n' natural numbers is $\frac{n+1}{2}$.

23. Average $= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$.

24. Mean $= \frac{\text{Sum of the observations}}{\text{Number of the observations}}$.

25. Median $= L + \left[\frac{\left(\frac{N}{2} - F\right)C}{f}\right]$.

26. Mode $= L_1 + \left[\frac{(f - f_1)C}{2f - (f_1 - f_2)}\right]$.

27. Average $= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$.

28. Arithmatic mean $= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$.

Level 2

29. (i) Since, a, b, c are in ascending order, b is the median.

    (ii) AM $= \frac{\text{Sum of all observations}}{\text{Total number of observations}}$.

    (iii) Using the above relation, write the relation between c and a, and find the value of a and c. Finally, find the values of b and a.

HINTS AND EXPLANATION

30. (i) Since, $a < b < 2a$, Median $= b = 12$.

(ii) Mean $= \dfrac{\text{Sum of all observations}}{\text{Total number of observations}}$.

31. (i) If the variance of $(ax_i + b)$ is k, then SD of $(ax_i + b) = \sqrt{k}$.

(ii) SD of $x_i = \dfrac{\sqrt{k}}{a}$.

32. (i) Find the mid-values (x_i) of class intervals.

(ii) Find $\dfrac{\sum fx}{\sum f}$.

(iii) Equate $\dfrac{\sum fx}{\sum f}$ with mean, and find the values of x and y.

33. In the given graph, one curve represents the less than cumulative frequency, another curve represents the greater than cumulative frequency.

34. Apply the greater than cumulative frequency concept.

35. (i) SD of AP with common difference 'd' is $d\sqrt{\dfrac{n^2-1}{12}}$.

(ii) Variance $= (\text{SD})^2$.

36. (i) Use arithmetic mean formula for grouped data.

(ii) Mean of grouped data

$= A + \dfrac{\Sigma f_i d_i}{N} \times C$.

38. Find mode of the given data in terms of x and form an equation to find x.

39. (i) First calculate the mean.

(ii) Find deviations about the mean $(x_i - \overline{x})$.

$D = x_i - \overline{x}$, $N =$ Sum of all the frequencies, SD $= \sqrt{\dfrac{\Sigma f D^2}{N}}$.

40. The increasing order of given the observations:

5, 7, 8, 9, 10, 12, 13, 14

$Q_1 = \left(\dfrac{n}{4}\right)$th observation

$Q_1 =$ 2nd observation

$\therefore\ Q_1 = 7$

$Q_2 =$ Mean of $\left(\dfrac{n}{2}\right)$th, $\left(\dfrac{n}{2}+1\right)$th observation

$=$ Mean of 4th, 5th items $= \dfrac{9+10}{2} = 9.5$

$Q_3 = \left(\dfrac{3n}{4}\right)$th $= \left(3 \times \dfrac{8}{4}\right)$th 6th observation

$Q_3 = 12$

Mean of Q_1, Q_2, $Q_3 = \dfrac{7+9.5+12}{3} = \dfrac{28.5}{3} = 9.5$.

41. (a) Range of the factors of 64 is $64 - 1 = 63$.

(b) Range of the first ten-positive integers is 9.

42. Range of the first 'n' natural numbers $= n - 1$.

Range of the last n negative integers $= -1 - (-n) = n - 1$.

Range of the first 'n' positive even integers $= 2n - 2$.

Range of first 'n' positive odd integers $= (2n - 1) - (1) = 2n - 2$.

$\therefore$ Mean of the given data is

$\dfrac{(n-1)+(n-1)+(2n-2)+(2n-2)}{4}$

$= \dfrac{6n-6}{4} = \dfrac{3n-3}{2}$

$= \dfrac{3}{2}(n-1)$.

43. BDAC is the required sequential order.

44. BCAD is the required sequential order.

Level 3

45. Use the arithmetic mean formulae for descreate series.

46. (i) Coefficient of variation $= \dfrac{\text{SD}}{\text{Mean}} \times 100$.

(ii) Using the above, find CV of all the four members.

(iii) The member whose CV is least is more consistent.

47. (i) Find coefficient of variation then decide.

(ii) The one with highest CV is less consistent.

48. The mean of the squares of n natural numbers $\dfrac{(n+1)(2n+1)}{6}$.

49. Range is the maximum value − minimum value.

50. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

$\therefore$ AM $= \dfrac{n^2}{n} = n.$

51. $\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

AM $= \dfrac{n(n+1)(2n+1)}{6n} = \dfrac{(n+1)(2n+1)}{6}$

52. Mode = 3 × Median − 2 × Mean

$Z = 3M - 2X$

Given, $X : M = 9 : 8$

$\Rightarrow \dfrac{X}{M} = \dfrac{9}{8}$

$X = \dfrac{9M}{8}$

$\therefore Z = 3M - 2 \times \dfrac{9M}{8} = 3M - \dfrac{9M}{4}$

$Z = \dfrac{3M}{4} \quad \therefore \dfrac{M}{Z} = \dfrac{4}{3}$

$\Rightarrow M : Z = 4 : 3.$

53. $\bar{x} = \dfrac{1 + 3 + 3^2 + \cdots + 3^{n-1}}{n}$

$= \dfrac{1\left(\dfrac{3^n - 1}{3 - 1}\right)}{n} = \dfrac{3^n - 1}{2n}$

54. Sum of the given observations $S_n = x + x + a + x + 2a + \ldots$ for $(2n + 1)$ terms

$\therefore$ The total number of terms in the given series is $(2n + 1)$ and first term $= x$ and common difference $= a$

$= \dfrac{2n+1}{2}[2 \cdot x + (2n + 1 - 1)a]$

$= \dfrac{2n+1}{2}[2x + 2an] = (2n + 1)(x + an)$

AM $= \dfrac{S_n}{2n+1} = \dfrac{(2n+1)(x+an)}{2n+1}$

AM $= x + an.$

55. Given $n_1 = 25$, $n_2 = 35$

$\bar{x}_1 = 150$ and $\bar{x}_2 = 145$

Combined mean, $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$\bar{x} = \dfrac{25 \times 150 + 35 \times 145}{25 + 35} = \dfrac{3750 + 5075}{60}$

$= \dfrac{8825}{60} = 147$ (approximately).

56. Given sum of all observations $= 434 + x$ and $n = 15$, $\bar{x} = x$.

$\therefore \quad \dfrac{434 + x}{15} = x$

$434 + x = 15x$

$x = \dfrac{434}{14} \Rightarrow x = 31$.

57. The sum of the weights of the 9 students = 25 × 9 = 225 kg.

If one more student is joined in the group, then total number of students is 10, and the mean is 25.

The sum of the weights of the 10 students is 25 × 10 = 250 kg.

The weight of the 10th student is 250 − 225 = 25 kg.

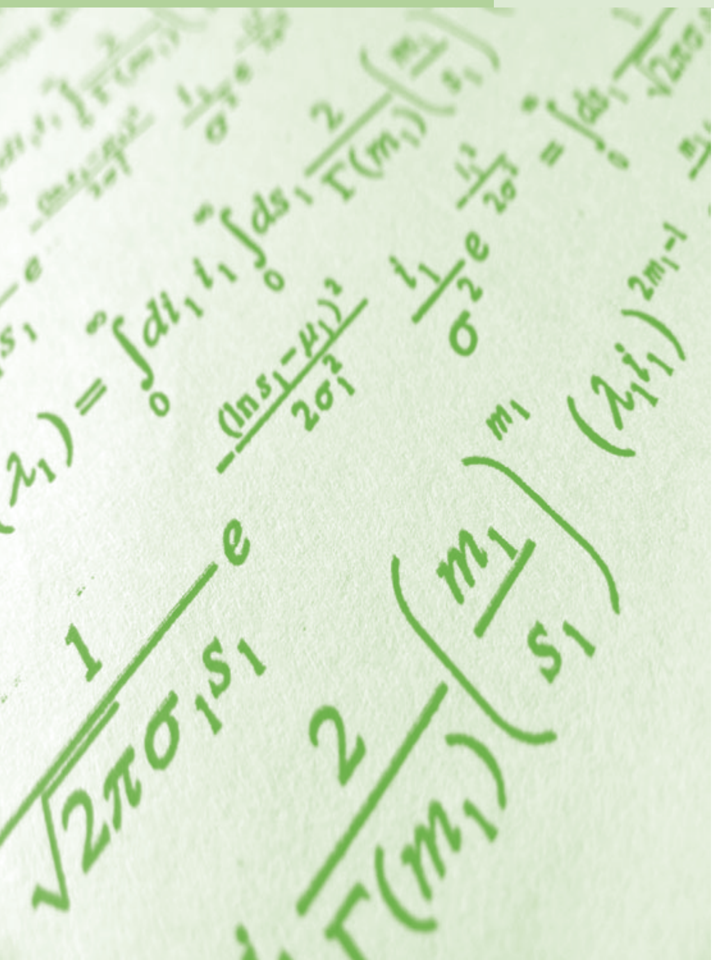
58. $\dfrac{x}{5}\ \dfrac{x}{4}\ \dfrac{x}{3}\ \dfrac{2x}{5}\ \dfrac{2x}{3}\ \dfrac{2x}{4}\ x \Rightarrow$ Median $= \dfrac{2x}{5}$

$\Rightarrow$ Median $= \dfrac{2x}{5} = 4$ (given)

$\Rightarrow x = 10.$

Chapter 13

Geometry



REMEMBER

Before beginning this chapter, you should be able to:

- Know types and properties of quadrilaterals
- Recall Pythagoras theorem and its converse
- Construct polygons, triangles, quadrilaterals, circles

KEY IDEAS

After completing this chapter, you would be able to:

- Learn about symmetry and similarity of geometrical figures in general
- Explain criteria for similarity of triangles, right angle theorem, results on areas of similar triangle, Pythagorean, Apollonius, Thales, vertical angle bisector theorems
- Understand properties of chords or arcs and prove theorems based on tangents
- Learn construction of geometrical figures
- Study equation of locus

INTRODUCTION

In this chapter, we shall learn about symmetry, similarity of geometrical figures in general, triangles in particular. We shall understand similarity through size transformation. Further we shall learn about concurrent lines, geometric centres in triangles, basic concepts of circles and related theorems. We shall also focus on some constructions related to polygons and circles. Finally, we shall discuss the concept of locus.

SYMMETRY

Let us examine the following figures (Fig. 13.1) drawn on a rectangular piece of paper:

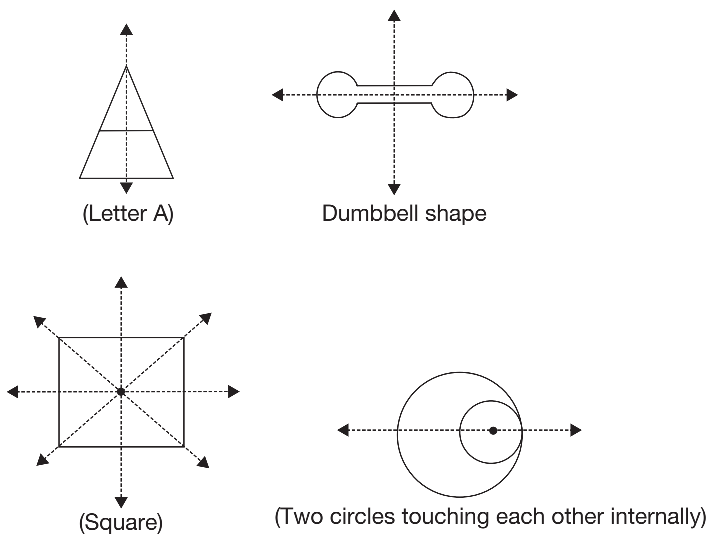


Figure 13.1

What do you infer? We can observe that when these figures are folded about the dotted lines, the two parts on either side of the dotted lines coincide. This property of geometrical figures is called symmetry.

In this chapter, we shall discuss two basic types of symmetry—**line symmetry and point symmetry**. Then we shall see how to obtain the image of a point, a line segment and an angle about a line.

Line Symmetry

Trace a geometrical figure on a rectangular piece of paper as shown in adjacent figure.

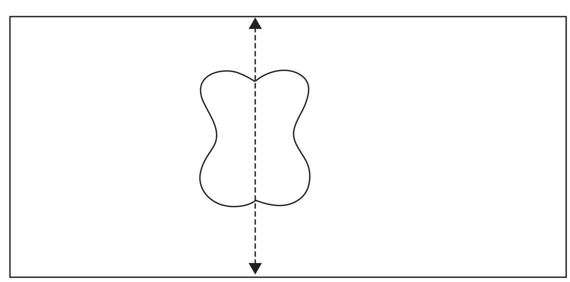


Figure 13.2

Now fold the paper along the dotted line. You will find that the two parts of the figure on either sides of the line coincide. Thus the line divides the figure into two identical parts. In this case, we say that the figure is symmetrical about the dotted line or Line Symmetric. Also, the dotted line is called the Line of Symmetry or The Axis of Symmetry.

So, a geometrical figure is said to be line symmetric or symmetrical about a line if there exists at least one line in the figure such that the parts of the figure on either sides of the line coincide, when it is folded about the line.

Example: Consider the following figure (Fig. 13.3).

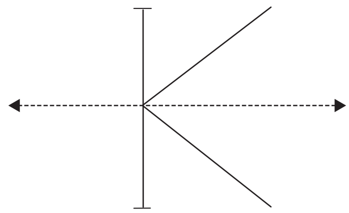


Figure 13.3

The above figure is symmetrical about the dotted line. Also, there is only one line of symmetry for the figure.

Example: Observe the figure given below (Fig. 13.4).

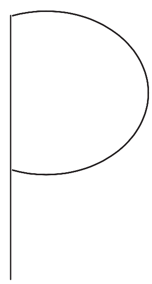


Figure 13.4

There is no line in the figure about which the figure is symmetric.

Example: Consider a rectangle as shown in the Fig. 13.5.

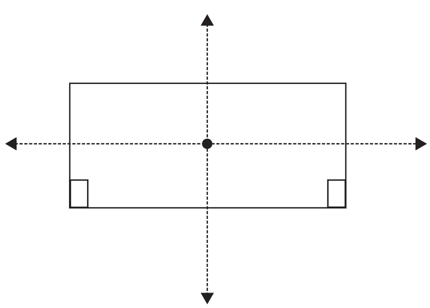


Figure 13.5

The rectangle is symmetrical about the two dotted lines. So, a rectangle has two lines of symmetry.

Example: Consider a figure in which 4 equilateral triangles are placed, one on each side of a square as shown below.

The figure is symmetrical about the dotted lines. It has 4 lines of symmetry.

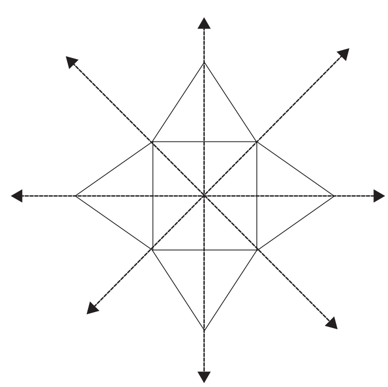


Figure 13.6

Example: A circle has an infinite number of lines of symmetry, some of which are shown in the Fig. 13.7.

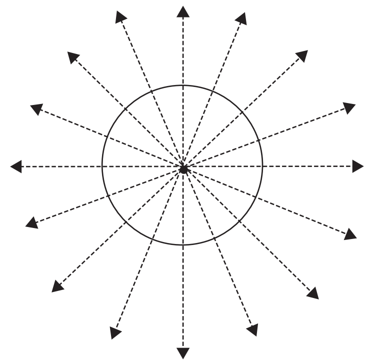


Figure 13.7

From the above illustrations, we observe the following:

1. A geometrical figure may not have a line of symmetry, i.e., a geometric figure may not be line symmetric.
2. A geometrical figure may have more than one line of symmetry, i.e., a geometrical figure may be symmetrical about more than one line.

Point Symmetry

Trace the letter N on a rectangular piece of paper as shown in Fig. 13.8. Let P be the mid-point of the inclined line in the figure. Now draw a line segment through the point P touching the two vertical strokes of N. We find that the point P divides the line segment into two equal parts. Thus, every line segment drawn through the point P and touching the vertical strokes of N is bisected at the point P. Also, when we rotate the letter N about the point P through an angle of 180°, we find that it coincides exactly with the initial position. This property of geometrical figures is called *Point Symmetry*. In this case, we say that the point P is the point of symmetry of the figure.

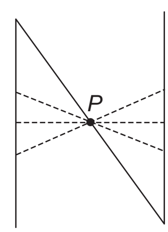


Figure 13.8

So, a geometrical figure is said to have symmetry about a point P if every line segment through the point P touching the boundary of the figure is bisected at the point P.

(Or)

A geometrical figure is said to have point symmetry if the figure does not change when rotated through an angle of 180°, about the point P.

Here, point P is called the centre of symmetry.

Illustrations

Example: Consider a rectangle $ABCD$. Let P be the point of intersection of the diagonals AC and BD. Then, the rectangle $ABCD$ has the point symmetry about the point P.

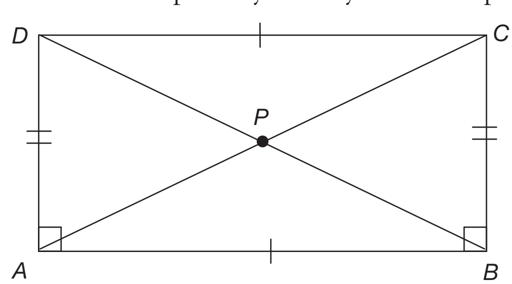


Figure 13.9

Example: The letter '*S*' has a point of symmetry about the turning point in *S*.

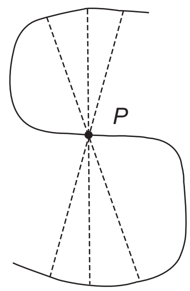


Figure 13.10

Example: A circle has a point of symmetry. This point is the centre of the circle.

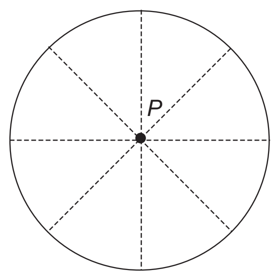


Figure 13.11

Image of a Point About a Line

Draw a line l and mark a point A on a rectangular piece of paper as shown in the Fig. 13.12.

Draw AM perpendicular to l and produce it to B such that $AM = MB$. Then the point B is said to be the image of the point A about the line l.

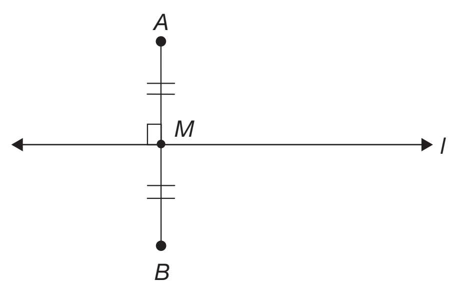


Figure 13.12

Image of a Line Segment about a Line

Draw a line l and a line segment PQ on a rectangular piece of paper as shown in the Fig. 13.13.

Draw perpendiculars PL and QM from the points P and Q respectively to the line $l$and produce them to P' and Q' respectively, such that $PL = LP'$ and $QM = MQ'$. Join the points P', Q'. Then the line segment $P'Q'$ is called the image of the line segment PQ about the line l.

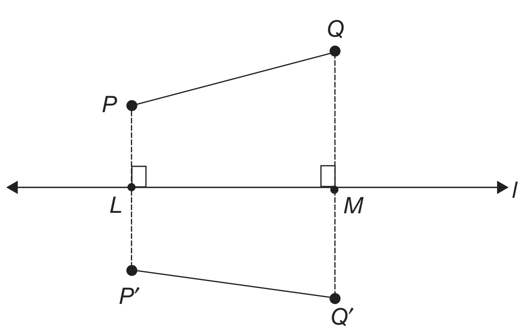


Figure 13.13

Image of an Angle About a Line

Draw a line l and an angle PQR on a piece of paper as shown below:

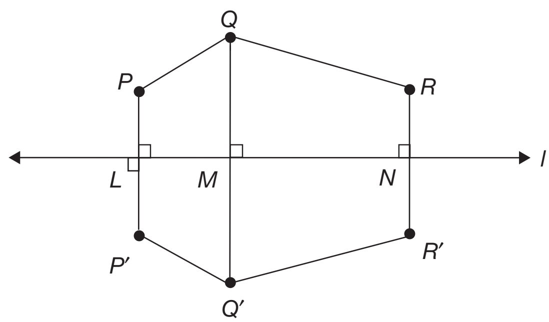


Figure 13.14

Draw perpendiculars PL, QM and RN from the points P, Q and R respectively to the line l and produce them to P', Q' and R' respectively, such that $PL = LP'$, $QM = MQ'$ and $RN = NR'$. Join the points P', Q' and Q', R'. Then, the angle $P'Q'R'$ is called the image of the angle PQR about the line l.

EXAMPLE 13.1

Determine the line of symmetry of a triangle ABC in which $\angle A = 40°$, $\angle B = 70°$ and $\angle C = 70°$.

SOLUTION

The line which bisects $\angle A$ is the line of symmetry of the given triangle ABC.

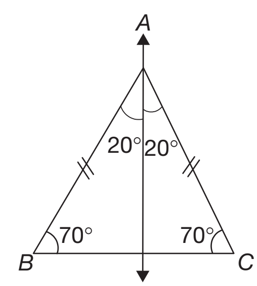


Figure 13.15

EXAMPLE 13.2

Determine the point of symmetry of a regular hexagon.

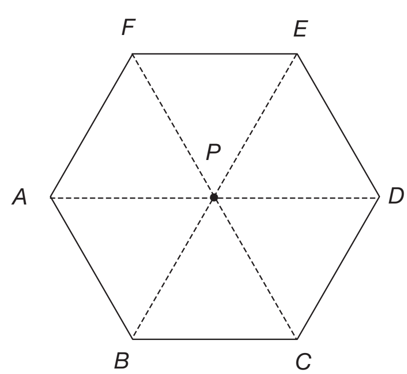


Figure 13.16

SOLUTION

The point of intersection of the diagonals of regular hexagon is the required point of symmetry.

EXAMPLE 13.3

Complete adjacent figure so that X-axis and Y-axis are the lines of symmetry of the completed figure.

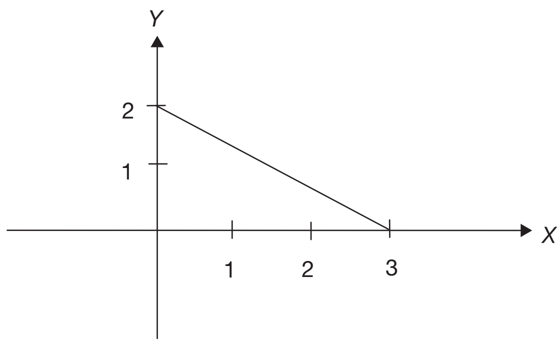


Figure 13.17

SOLUTION

Required figure is a rhombus of side $\sqrt{13}$ units.

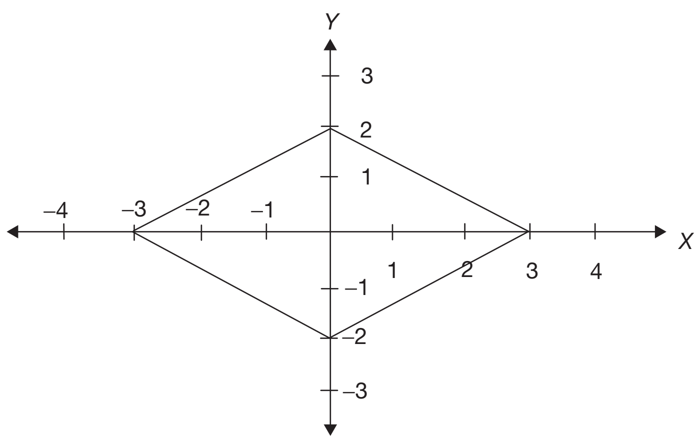
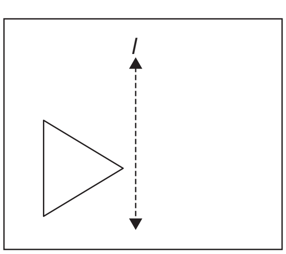


Figure 13.18

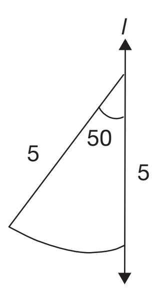
EXAMPLE 13.4

Determine the images of the following figure about the given line:

(a)

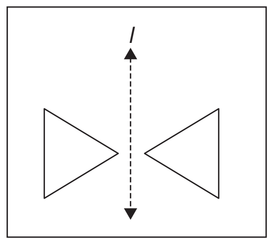


(b)

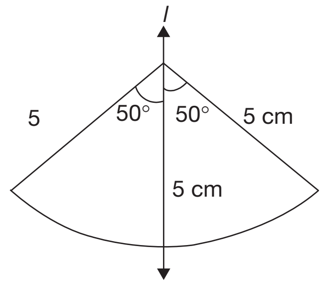


SOLUTION

(a)



(b)



SIMILARITY

Similarity of Geometrical Figures

Two geometrical figures of same shape, but not necessarily of same size are said to be similar.

Examples:

1. Any two circles are similar.
2. Any two squares are similar.
3. Any two equilateral triangles are similar.

Two circles C_1 and C_2 are similar, since their shapes are the same.

C_1 and C_2 may have an equal area.

While considering the circles C_1 and C_3 they may not be of equal area, but their shapes are the same.

C_1 and C_2 are of the same shape and of same size.

C_1 and C_3 are of the same shape but of different sizes.

In both the cases, they are of the same shape.

Hence C_1, C_2 and C_3 are similar circles.

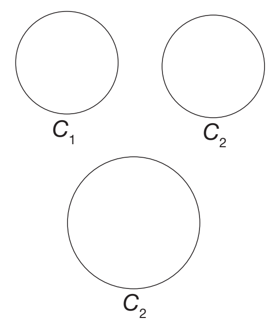


Figure 13.19

Similarity as a Size Transformation

Enlargement

Consider triangle ABC.

Construct triangle PQR whose sides are twice the corresponding sides of triangle ABC, as shown in the following figure.

Construction: First draw the triangle ABC. Take a point D outside the triangle. Join DA, DB and DC.

Now, extend DA, DB and DC to the points P, Q and R respectively, such that $DP = 2DA$, $DQ = 2DB$ and $DR = 2DC$.

PQR is the enlarged image of ABC.

On verification, we find that $PQ = 2AB$

$QR = 2BC$ and $RP = 2AC$.

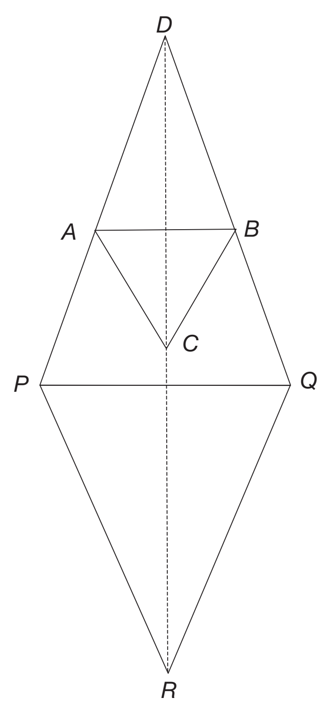


Figure 13.20

It can be defined that triangle ABC has been enlarged by a scale factor of 2 about the centre of the enlargement D to give image PQR.

Reduction

Consider square $ABCD$. One can draw a square $PQRS$ whose side is half the length of the side of $ABCD$.

Construction: Draw square $ABCD$.

Take a point E outside the square.

Join EA, EB, EC and ED. Mark the points P, Q, R and S on $\overline{EA}$, $\overline{EB}$, $\overline{EC}$, and $\overline{ED}$, such that

$$EP = \frac{1}{2}EA,\ EQ = \frac{1}{2}EB,\ ER = \frac{1}{2}EC \text{ and } ES = \frac{1}{2}ED$$

Square $PQRS$ is the reduced image of square $ABCD$.

We find that,

$$PQ = \frac{1}{2}AB,\ QR = \frac{1}{2}BC,$$

$$RS = \frac{1}{2}CD \text{ and } SP = \frac{1}{2}AD$$

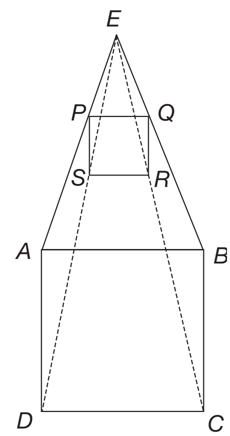


Figure 13.21

$ABCD$ has been reduced by a scale factor $\frac{1}{2}$ about the centre of reduction E to give the image $PQRS$.

Size transformation is the process which a geometrical figure is enlarged or reduced by a scale k, such that the image formed is similar to the given figure.

Properties of Size Transformation

1. The shape of the given figure remains the same.
2. If k is the scale factor of a given size transformation and $k > 1 \Rightarrow$ the image is enlarged and $k < 1 \Rightarrow$ the image is reduced.

   And $k = 1 \Rightarrow$ the image is identical to the original figure.
3. Each side of the given geometrical figure = k (The corresponding side of the given figure).
4. Area of the image = k_2 (Area of the given geometrical figure).
5. Volume of the image, if it is a 3-dimensional figure, is equal to k^3 (Volume of the original figure).

Model

The model of a plane figure and the given figure are similar to each other. If the model of a plane figure is drawn to the scale 1 : x, then scale factor, $k = \frac{1}{x}$.

1. Length of the model = k (Length of the original figure)
2. Area of the model = k^2 (Area of the original figure)
3. Volume of the model = k^3 (Volume of the original figure)

Map

If the map of a plane figure, is drawn to the scale 1 : x, then, scale factor, $k = \frac{1}{x}$.

1. Length in the map = k (Original length)
2. Area in the map = k^2 (Original area)

The discussion on similarity can be extended further as follows:

Two polygons are said to be similar to each other if

1. their corresponding angles are equal and
2. the lengths of their corresponding sides are proportional.

Note '~' is the symbol used for 'is similar to'.

If quadrilateral $ABCD$ is similar to quadrilateral $PQRS$, we denote this as ▱ $ABCD$ ~ ▱ $PQRS$. The relation 'is similar to' satisfies the following properties.

1. It is reflexive as every figure is similar to itself.
2. It is symmetric as A is similar to B, then B is also similar to A.
3. It is transitive as, if A is similar to B and B is similar to C, then A is similar to C.

∴ The relation is 'similar to' is an equivalence relation.

Criteria for Similarity of Triangles

Similar Triangles

In two triangles, if either the corresponding angles are equal or the ratio of corresponding sides are equal, then the two triangles are similar to each other.

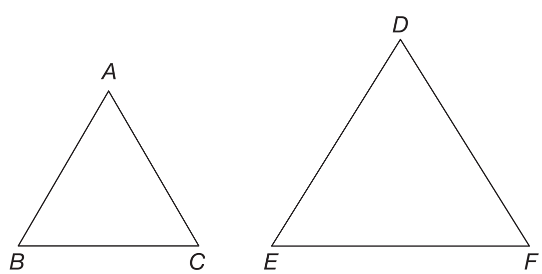


Figure 13.22

In ΔABC and ΔDEF, if,

1. $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ or
2. $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$,

   then $\Delta ABC \sim \Delta DEF$ or ΔABC is similar to ΔDEF.

Three Similarity Axioms for Triangles

1. **AA–Axiom or AAA–Axiom:** In two triangles, if corresponding angles are equal then the triangles are similar to each other.

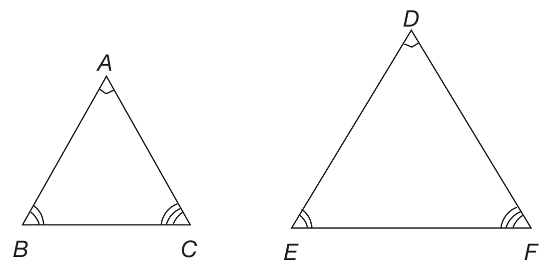


Figure 13.23

In ΔABC and ΔDEF, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

$\therefore \Delta ABC \sim \Delta DEF$.

Note In two triangles, if two pairs of corresponding angles are equal then the triangles are similar to each other, because the third pair corresponding angles will also be equal.

2. **SSS–Axiom:** In two triangles, if the corresponding sides are proportional, then the two triangles are similar to each other.

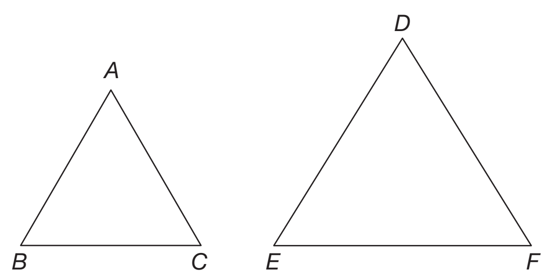


Figure 13.24

In ΔABC and ΔDEF, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

$\Rightarrow \Delta ABC \sim \Delta DEF$.

3. **SAS–Axiom:** In two triangles, if one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

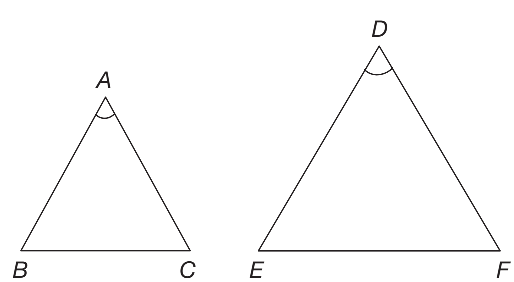


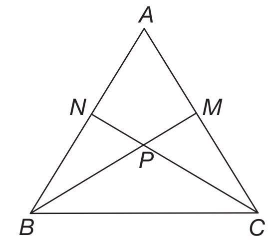
Figure 13.25

In ΔABC and ΔDEF,

If $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ and $\angle A = \angle D$, then $\Delta ABC \sim \Delta DEF$.

EXAMPLE 13.5

In the given figure (not to scale), $AM : MC = 3 : 4$, $BP : PM = 3 : 2$ and $BN = 12$ cm. Find AN.



(a) 10 cm **(b)** 12 cm **(c)** 14 cm **(d)** 16 cm

SOLUTION

Given, $AM : MC = 3 : 4$

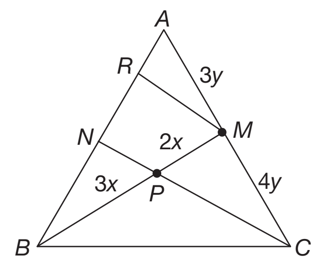
$BP : PM = 3 : 2$ and $BN = 12$ cm

Draw MR parallel to CN which meets AB at the point R,

Consider ΔBMR,

$PN \parallel MR$ (Construction)

By BPT,



$$\frac{BN}{NR} = \frac{BP}{PM} \Rightarrow \frac{12}{NR} = \frac{3}{2}$$

$NR = 8$ cm

Consider ΔANC, $RM \parallel NC$ (By construction)

By BPT,

$$\frac{AR}{RN} = \frac{AM}{MC} \Rightarrow \frac{AR}{8} = \frac{3}{4} \Rightarrow AR = 6 \text{ cm}$$

$\therefore AN = AR + RN = 6 + 8 = 14$ cm.

Right Angle Theorem

In a right angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, then the two right triangles formed on either side of the perpendicular are similar to each other and similar to the given triangle.

In the figure, ΔABC is right angled at A. $\overline{AD}$ is the perpendicular drawn from A to BC.

Let $\angle ABD = \theta$

Then, $\angle BAD = 90° - \theta$

$\angle DAC = \angle BAC - \angle BAD = 90° - (90° - \theta)$

$\Rightarrow \angle DAC = \theta$

$\therefore \angle DCA = 90° - \theta$

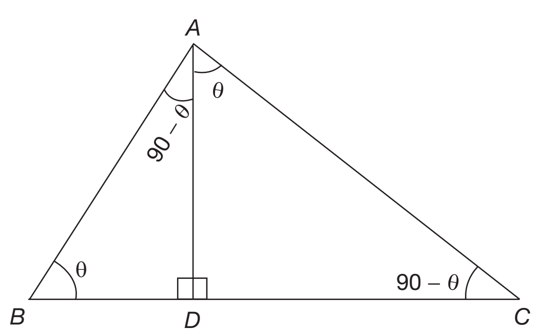


Figure 13.26

In triangles BAD, DAC, and ABC, three corresponding angles are equal. Therefore, the triangle ABC is similar to triangle DAC or triangle DBA.

In triangles DBA and DAC,

$$\because \frac{BD}{AD} = \frac{AD}{DC}$$

$\Rightarrow AD^2 = BD \times DC$

AD is the mean proportional of BD and DC.

$\Rightarrow AD^2 = BD \cdot DC.$

Results on Areas of Similar Triangles The ratio of areas of the two similar triangles is equal to the ratio of the squares of any two corresponding sides of the triangles.

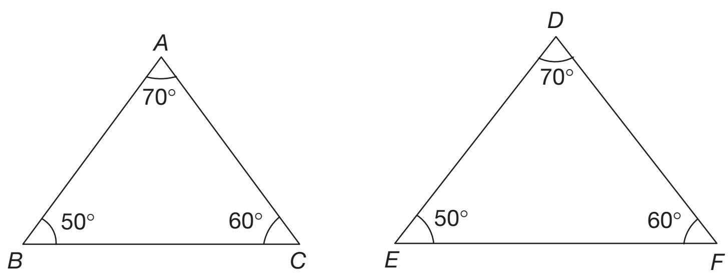


Figure 13.27

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2}.$$

1. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes.

   In the following figures, $\Delta ABC \sim \Delta DEF$ and AX, DY are the altitudes.

   Then, $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \dfrac{AX^2}{DY^2}.$

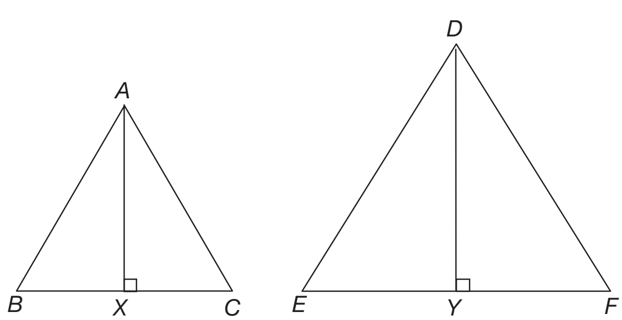


Figure 13.28

2. The ratio of areas of two similar triangles is equal to the ratio of the squares on their corresponding medians (see Fig. 13.29).

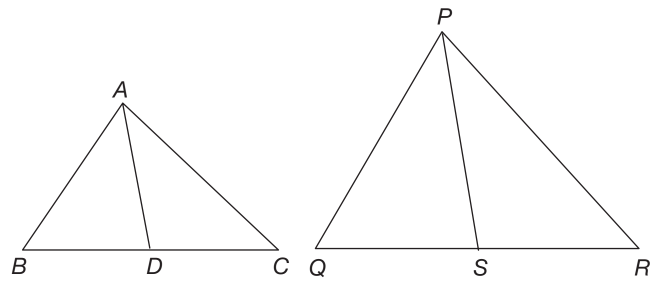


Figure 13.29

In the above figure, $\Delta ABC \sim \Delta PQR$ and AD and PS are medians.

Then, $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AD^2}{PS^2}$.

3. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding angle bisector segments.

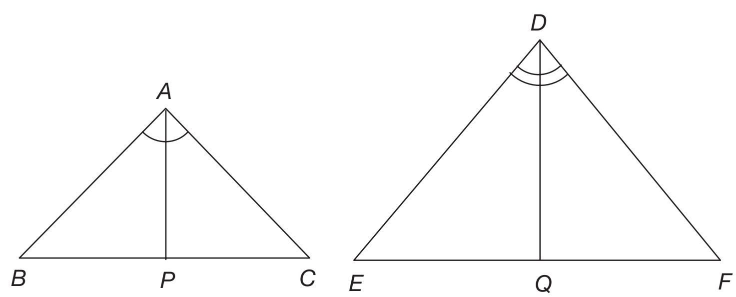


Figure 13.30

In the figure, $\Delta ABC \sim \Delta DEF$ and AP, DQ are bisectors of $\angle A$ and $\angle D$ respectively, then

$\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \dfrac{AP^2}{DQ^2}$.

Pythagoras Theorem

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

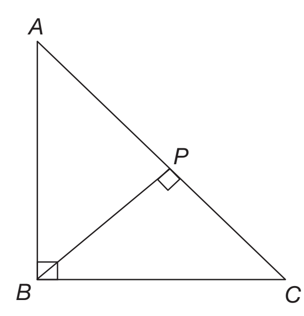


Figure 13.31

To prove: $AC^2 = AB^2 + BC^2$

Construction: Draw BP perpendicular to AC.

Proof: In triangles APB and ABC,

$\angle APB = \angle ABC$ (right angles)

$\angle A = \angle A$ (common)

$\therefore$ Triangle APB is similar to triangle ABC.

$\Rightarrow \dfrac{AP}{AB} = \dfrac{AB}{AC}$

$\Rightarrow AB^2 = (AP)(AC)$

Similarly, $BC^2 = (PC)(AC)$

$\therefore AB^2 + BC^2 = (AP)(AC) + (PC)(AC)$

$AB^2 + BC^2 = (AC)(AP + PC)$

$AB^2 + BC^2 = (AC)(AC)$

$\Rightarrow AC^2 = AB^2 + BC^2$

Hence proved.

The results in an obtuse triangle and an acute triangle are as follows:

In ΔABC, $\angle ABC$ is obtuse and AD is drawn perpendicular to BC, then

$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$

ΔABC is an acute angled triangle, acute angle at B and AD is drawn perpendicular to BC, then $AC^2 = AB^2 + BC^2 - 2BC \cdot BD$.

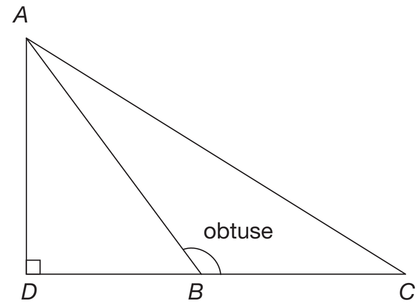


Figure 13.32

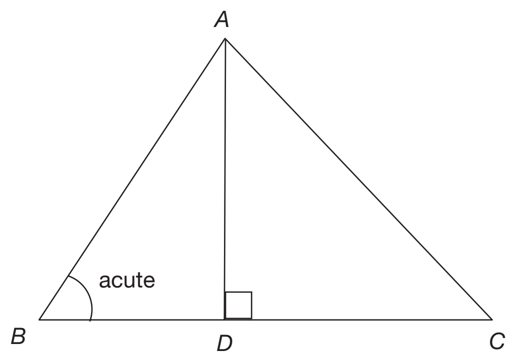


Figure 13.33

Appolonius Theorem

In a triangle, the sum of the squares of two sides of a triangle is equal to twice the sum of the square of the median which bisects the third side and the square of half the third side.

Given: In ΔABC, AD is the median.

RTP: $AB^2 + AC^2 = 2(AD^2 + DC^2)$ or $2(AD^2 + BD^2)$

Construction: Draw AE perpendicular to BC.

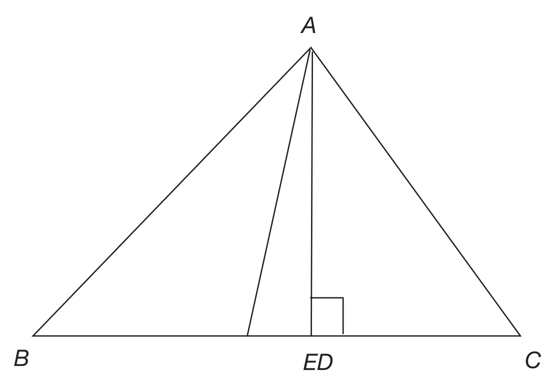


Figure 13.34

Case 1: If $\angle ADB = \angle ADC = 90°$

According to Pythagoras theorem,

In ΔABD, $AB^2 = BD^2 + AD^2$ (1)

In ΔADC, $AC^2 = CD^2 + AD^2$ (2)

Adding Eqs. (1) and (2), we have

$AB^2 + AC^2 = BD^2 + CD^2 + 2AD^2 = 2BD^2 + 2AD^2$. [$\because CD = BD$] or $2[CD^2 + AD^2]$

Case 2: If $\angle ADC$ is acute and $\angle ADB$ is obtuse.

In triangle ADB,

$AB^2 = AD^2 + BD^2 + 2 \times BD \times DE$

In triangle ADC,

$AC^2 = AD^2 + DC^2 - 2 \times CD \times DE$

But $BD = CD$

$\therefore AB^2 + AC^2 = 2AD^2 + 2BD^2$ or $2(AD^2 + CD^2)$

Hence proved.

Basic Proportionality Theorem In a triangle, if a line is drawn parallel to one side of a triangle, then it divides the other two sides in the same ratio.

Given: In ΔABC, $\overline{DE} \parallel \overline{BC}$

RTP: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

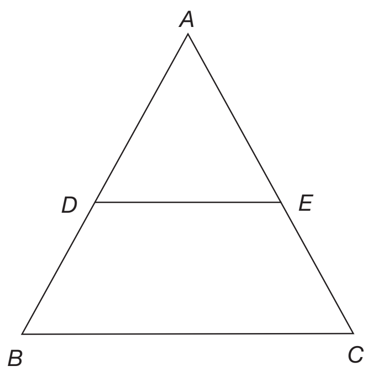


Figure 13.35

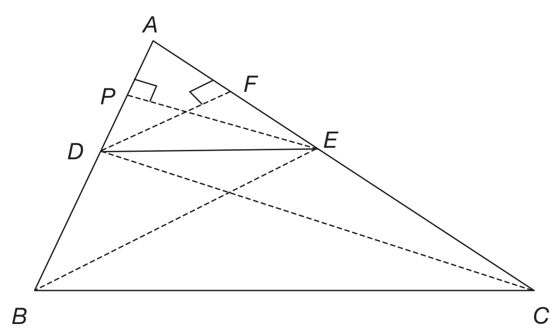


Figure 13.36

Construction: Draw $EP \perp AB$ and $DF \perp AC$. Join $\overline{DC}$ and $\overline{BE}$.

Proof: $\dfrac{\text{Area of triangle } ADE}{\text{Area of triangle } BDE} = \dfrac{\frac{1}{2} \times AD \times PE}{\frac{1}{2} \times BD \times PE} = \dfrac{AD}{BD}$

and $\dfrac{\text{Area of triangle } ADE}{\text{Area of triangle } CDE} = \dfrac{\frac{1}{2} \times AE \times DF}{\frac{1}{2} \times EC \times DF} = \dfrac{AE}{EC}$

But, area of triangles BDE and CDE are equal. (Two triangles lying on the same base and between the same parallel lines are equal in area).

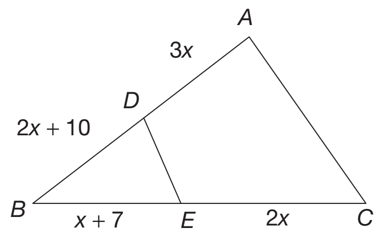
$\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$

Note From the above result we can prove that $\frac{AB}{AD} = \frac{AC}{AE}$ and $\frac{AB}{BD} = \frac{AC}{CE}$.

EXAMPLE 13.6

In the given figure, $\overline{DE} \parallel \overline{AC}$. Find the value of x.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4



SOLUTION

In ΔABC,

$DE \parallel AC$

$\therefore \frac{BD}{DA} = \frac{BE}{EC}$ (Basic proportionality theorem)

$\Rightarrow \frac{2x+10}{3x} = \frac{x+7}{2x}$

$\Rightarrow 2x(2x+10) = 3x(x+7)$

$\Rightarrow 4x^2 + 20x = 3x^2 + 21x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x - 1 = 0$

$\Rightarrow x = 1$

$\therefore$ Hence, option (a) is the correct answer.

Converse of Basic Proportionality Theorem

If a line divides two sides of a triangle in the same ratio, then that line is parallel to the third side.

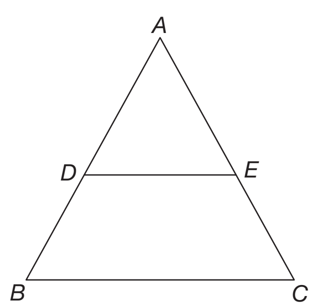


Figure 13.37

In the figure given, $\frac{AD}{DB} = \frac{AE}{EC} \Rightarrow \overline{DE} \parallel \overline{BC}$.

Vertical Angle Bisector Theorem

The bisector of the vertical angle of a triangle divides the base in the ratio of the other two sides.

Given: In ΔABC, AD is the bisector of $\angle A$.

RTP : $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction: Draw CP parallel to AD to meet BA produced at P.

Proof: $\angle DAC = \angle ACP$ (alternate angles and $\overline{AD} \parallel \overline{CP}$)

$\angle BAD = \angle APC$ (corresponding angles)

But $\angle BAD = \angle DAC$ (given)

$\therefore\ \angle ACP = \angle APC$

In triangle APC,

$AC = AP$ (sides opposite to equal angles are equal)

In triangle BCP,

$\dfrac{BD}{DC} = \dfrac{BA}{AP}$ (by basic proportionality theorem)

$\Rightarrow \dfrac{BD}{DC} = \dfrac{BA}{AC}$ $\quad (\because AP = AC)$

Hence proved.

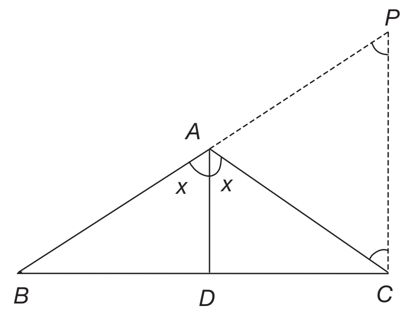


Figure 13.38

Converse of Vertical Angle Bisector Theorem

If a line that passes through a vertex of a triangle, divides the base in the ratio of the other two sides, then it bisects the angle.

In the adjacent figure, AD divides BC in the ratio $\dfrac{BD}{DC}$ and if $\dfrac{BD}{DC} = \dfrac{AB}{AC}$, then AD is the bisector of $\angle A$.

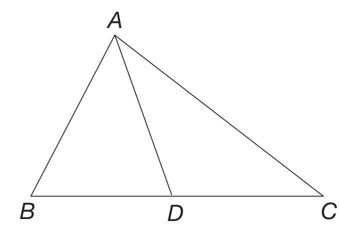
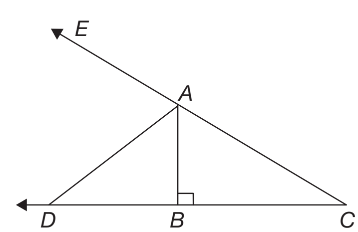


Figure 13.39

EXAMPLE 13.7

In the figure above (not to scale), $\overline{AB} \perp \overline{CD}$ and AD is the bisector of $\angle BAE$. $AB = 3$ cm and $AC = 5$ cm. Find CD.

(a) 6 cm **(b)** 8 cm **(c)** 10 cm **(d)** None of these



SOLUTION

Let BD be x cm.

Given $AB = 3$ cm, $AC = 5$ cm and $\angle ABC = 90°$

$\therefore$ BC = 4 cm

And also given, AD is the bisector of $\angle BAE$.

$\therefore$ By vertical angle bisector theorem.

$\dfrac{AB}{AC} = \dfrac{BD}{CD} \Rightarrow \dfrac{3}{5} = \dfrac{x}{4+x}$

$\Rightarrow 12 + 3x = 5x \Rightarrow x = 6$ cm.

$\therefore\ CD = 4 + 6 = 10$ cm.

Concurrency—Geometric Centres of a Triangle

Let us recall that if three or more lines pass through a fixed point, then those lines are said to be concurrent and that fixed point is called the point of concurrence. In this context, we recall different concurrent lines and their points of concurrence associated with a triangle, also called geometric centres of a triangle.

Circum-centre

The locus of the point equidistant from the end points of the line segment is the perpendicular bisector of the line segment. The three perpendicular bisectors of the three sides of a triangle are concurrent and the point of their concurrence is called the circum-centre of the triangle and is usually denoted by S. The circum-centre is equidistant from all the vertices of the triangle. The circum-centre of the triangle is the locus of the point in the plane of the triangle, equidistant from the vertices of the triangle.

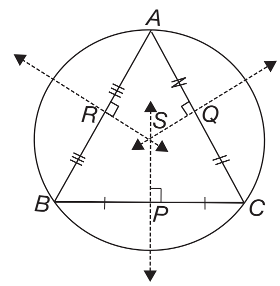


Figure 13.40

In-centre

The angle bisectors of the triangle are concurrent and the point of concurrence is called the in-centre and is usually denoted by I. I is equidistant from the sides of the triangle. The in-centre of the triangle is the locus of the point, in the plane of the triangle, equidistant from the sides of the triangle.

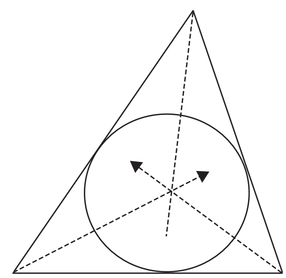


Figure 13.41

Ortho-centre

The altitudes of the triangle are concurrent and the point of concurrence of the altitudes of a triangle is called ortho-centre and is usually denoted by O.

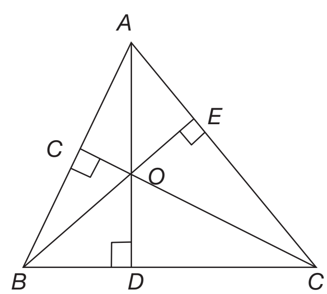


Figure 13.42

Centroid

The medians of a triangle are concurrent and the point of concurrence of the medians of a triangle is called the centroid and it is usually denoted by G. The centroid divides each of the medians in the ratio 2 : 1, starting from vertex, i.e., in the figure given below, $AG : GD = BG : GE = CG : GF = 2 : 1$.

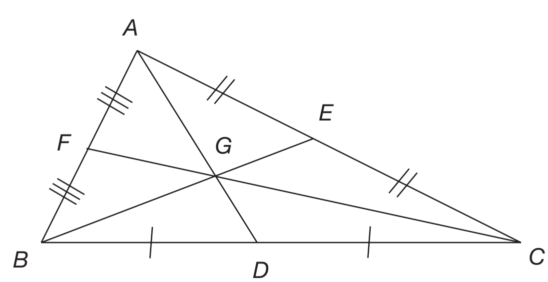


Figure 13.43

Some Important Points

1. In an equilateral triangle, the centroid, the ortho-centre, the circum-centre and the in-centre all coincide.
2. In an isosceles triangle, the centroid, the ortho-centre, the circum-centre and the in-centre all lie on the median to the base.
3. In a rightangled triangle the length of the median drawn to the hypotenuse is equal to half of the hypotenuse. The median is also equal to the circum-radius. The mid-point of the hypotenuse is the circum-centre.
4. In an obtuseangled triangle, the circum-centre and ortho-centre lie outside the triangle and for an acute angled triangle the circum-centre and the ortho-centre lie inside the triangle.
5. For all triangles, the centroid and the in-centre lie inside the triangle.

CIRCLES

A circle is a set of points in a plane which are at a fixed distance from a fixed point.

The fixed point is the centre of the circle and the fixed distance is the radius of the circle.

In the Fig. 13.44, O is the centre of the circle and OC is a radius of the circle.

AB is a diameter of the circle. OA and OB are also the radii of the circle.

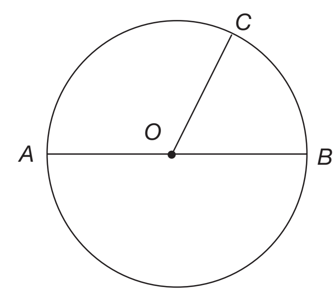


Figure 13.44

The diameter is twice the radius.

The centre of the circle is generally denoted by O, diameter by d and radius by r.

$d = 2r$.

The perimeter of the circular line is called the circumference of the circle.

The circumference of the circle is π times the diameter.

In the Fig. 13.45 with centre O; A, B and C are three points in the plane in which the circle lies. The points O and A are in the interior of the circle. The point B is located on the circumference of the circle.

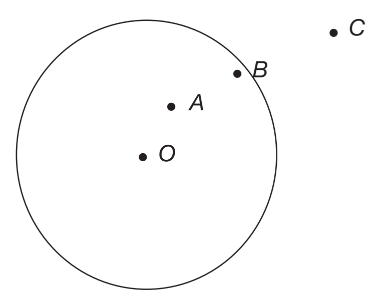


Figure 13.45

Hence, B belongs to the circle.

C is located in the exterior of the circle.

If $OB = r$, B is a point on the circumference of the circle.

As $OA < r$, A is a point in the interior of the circle.

As $OC > r$, C is a point in the exterior of the circle.

Chord

The line segment joining any two points on the circumference of a circle is a chord of the circle.

In the Fig. 13.46, $\overline{PQ}$ and $\overline{AB}$ are the chords.

AB passes through centre *O*, hence, it is a diameter of the circle. A diameter is the longest chord of the circle. It divides the circle into two equal parts.

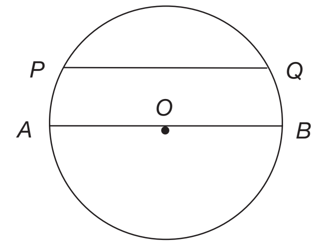


Figure 13.46

Theorem 1

One and only one circle exists through three non-collinear points.

Given: *P*, *Q* and *R* are three non-collinear points.

RTP: One and only one circle passes through the points *P*, *Q* and *R*.

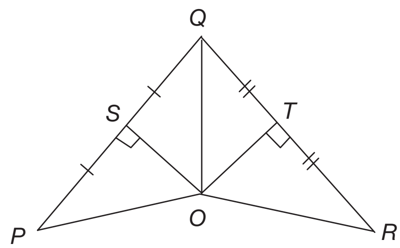


Figure 13.47

Construction:

1. Join *PQ* and *RQ* and draw the perpendicular bisectors of *PQ* and *RQ*.

   Let them meet at the point *O*.

   Join $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$.

   In triangles *OPS* and *OQS*, $PS = SQ$

   $\angle OSP = \angle OSQ$ (right angles)

   $OS = OS$ (common)

   $\therefore\ \Delta OSP \cong OSQ$. (SAS Congruence Property)

   $\therefore\ OQ = OP$

   Similarly, it can be proved that $OQ = OR$.

   $\therefore\ OP = OQ = OR$. Therefore, the circle with *O* as the centre and passing through *P*, also passes through *Q* and *R*.

2. The perpendicular bisectors of *PQ* and *QR* intersect at only one point and that point is *O*. A circle passing through *P*, *Q* and *R* has to have this point as the centre. Thus, there can only be one circle passing through *P*, *Q* and *R*.

Properties of Chords and Related Theorems

Theorem 2

The perpendicular bisector of a chord of a circle passes through the centre of the circle.

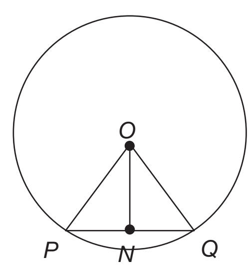


Figure 13.48

Given: *PQ* is a chord of a circle with centre *O*. *N* is the mid-point of chord *PQ*.

RTP: *ON* is perpendicular to *PQ*.

Construction: Join OP and OQ.

Proof: In triangles OPN and OQN,

$OP = OQ$ (radius of the circle)

$PN = QN$ (given)

ON is common.

By SSS Congruence Property,

$\Delta OPN \cong \Delta OQN$.

$\therefore \angle ONP = \angle ONQ$

But,

$\angle ONP + \angle ONQ = 180°$ (straight line)

$\Rightarrow \angle ONP = \angle ONQ = 90°$

$\therefore$ ON is perpendicular to chord PQ.

Note The converse of the above theorem is also true, i.e., the diameter which is perpendicular to a chord of a circle bisects the chord.

Theorem 3

Two equal chords of a circle are equidistant from the centre of the circle.

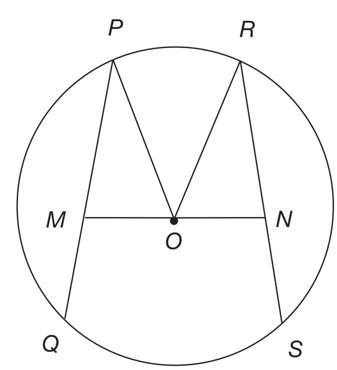


Figure 13.49

Given: In a circle with centre O, chord PQ = chord RS.

$OM \perp PQ$ and $ON \perp RS$.

RTP: $OM = ON$

Construction: Join OP and OR.

Proof: Since PQ is a chord of the circle and OM is perpendicular to PQ, OM bisects PQ (theorem 1).

Similarly, ON bisects RS.

$PQ = RS$ (given)

$\frac{1}{2}PQ = \frac{1}{2}RS$

$\Rightarrow PM = RN$

$\angle OMP = \angle ONR$ (right angles)

$OP = OR$ (radii of the circle)

In triangles OMP and ONR,

$OP = OR$,

$PM = RN$ and $\angle OMP = \angle ONR$

$\therefore \Delta OMP \cong \Delta ONR$ (SAS axiom)

$\therefore OM = ON$

Hence proved.

The converse of the above theorem, is also true, i.e., two chords which are equidistant from the centre of a circle are equal in length.

Note Longer chords are closer to the centre and shorter chords are farther from the centre.

In the given figure, AB and CD are two chords of the circle with centre at O and $AB > CD$. OP is perpendicular to chord AB and OQ is perpendicular to chord CD. By observation, we can say that $OQ > OP$.

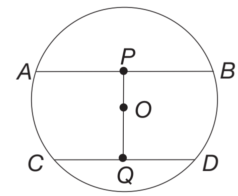


Figure 13.50

EXAMPLE 13.8

AB and CD are two equal and parallel chords of lengths 24 cm each, in a circle of radius 13 cm. What is the distance between the chords?

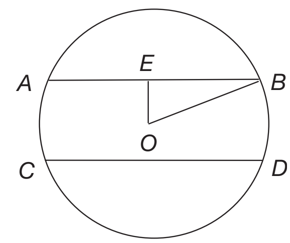


Figure 13.51

SOLUTION

In the given circle with centre O, AB and CD are the two chords each of length 24 cm and are equidistant from the centre of the circle. Therefore the distance between AB and $CD = 5 + 5 = 10$ cm.

Angles Subtended by Equal Chords at the Centre

Theorem 4

Equal chords subtend equal angles at the centre of the circle.

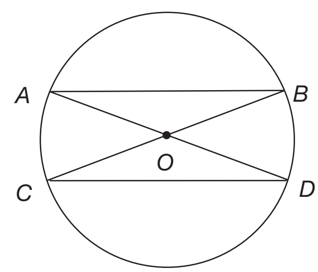


Figure 13.52

Given: AB and CD are equal chords of a circle with centre O. Join OA, OB, OC and OD.

RTP: $\angle AOB = \angle COD$.

Proof: In triangles ABO and CDO,

$OA = OC$, (radii of the same circle)

$OB = OD$

$AB = CD$ (given)

By SSS (side-side-side congruence property), triangles ABO and CDO are congruent. Hence, the corresponding angles are equal.

$\angle AOB = \angle COD$

Hence proved.

The converse of the theorem is also true, i.e., chords of a circle subtending equal angles at the centre are equal.

Angles Subtended by an Arc

Property 1

Angles subtended by an arc at any point on the rest of the circle are equal.

In the given circle, AXB is an arc of the circle. The angles subtended by the arc AXB at C and D are equal, i.e., $\angle ADB = \angle ACB$.

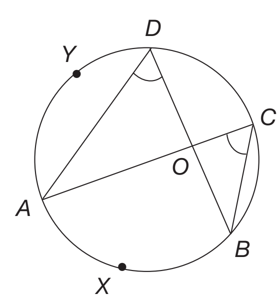


Figure 13.53

Property 2

Angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining part of the circle.

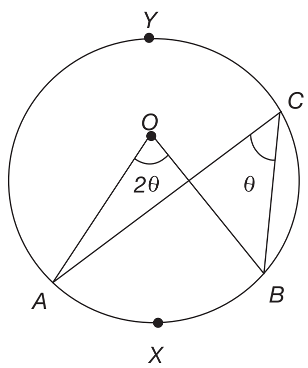


Figure 13.54

In the above figure, AXB is an arc of the circle. $\angle ACB$ is subtended by the arc AXB at the point C (a point on the remaining part of the circle), i.e., arc AYB. If O is the centre of the circle, $\angle AOB = 2\angle ACB$.

EXAMPLE 13.9

In the following figure, AB is an arc of the circle. C and D are the points on the circle. If $\angle ACB = 30°$, find $\angle ADB$.

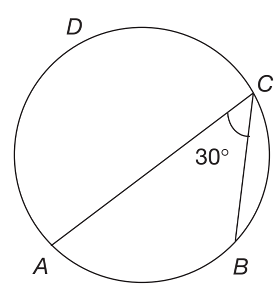


Figure 13.55

SOLUTION

Angles made by an arc in the same segment are equal. Angles made by the arc in the segment $ADCB$ are equal.

$\therefore\ \angle ADB = \angle ACB = 30°$.

EXAMPLE 13.10

In the following figure, O is the centre of the circle.

AB is an arc of the circle, such that $\angle AOB = 80°$. Find $\angle ACB$.

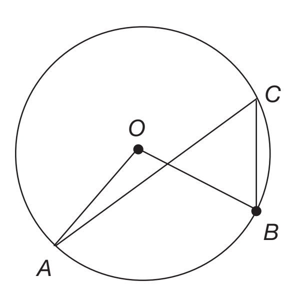


Figure 13.56

SOLUTION

The angle made by an arc at the centre of a circle is twice the angle made by the arc at any point on the remaining part of the circle.

$\angle AOB = 2\angle ACB$

$\Rightarrow 2\angle ACB = 80°$

$\Rightarrow \angle ACB = 40°$.

EXAMPLE 13.11

In the following figure, O is the centre of the circle. AB and CD are equal chords. If $\angle AOB = 100°$, find $\angle CED$.

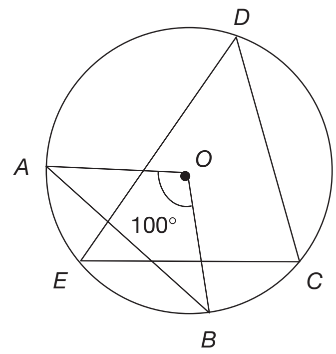


Figure 13.57

SOLUTION

Equal chords subtend equal angles at the centre of the circle.

$\angle AOB = 100°$

$\Rightarrow \angle DOC = 100°$

(Angle subtended by an arc at the centre of the circle is twice the angle subtended by it anywhere in the remaining part of the circle)

$\therefore \angle DEC = \frac{1}{2}(100°) = 50°$.

Cyclic Quadrilateral

If all the four vertices of a quadrilateral lie on a circle, it is called a cyclic quadrilateral.

In the given figure, the four vertices, A, B, C and D of the quadrilateral $ABCD$ lie on the circle. Hence $ABCD$ is a cyclic quadrilateral.

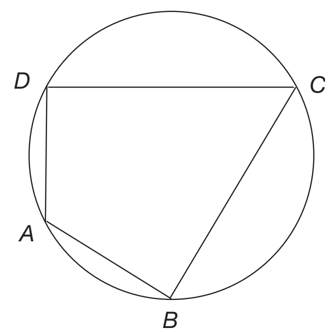


Figure 13.58

Theorem 5

The opposite angles of a cyclic quadrilateral are supplementary.

Given: $ABCD$ is a cyclic quadrilateral.

RTP:

$\angle A + \angle C = 180°$ or $\angle B + \angle D = 180°$

Construction: Join OA and OC, where O is the centre of the circle.

$$\angle AOC = 2\angle ADC$$

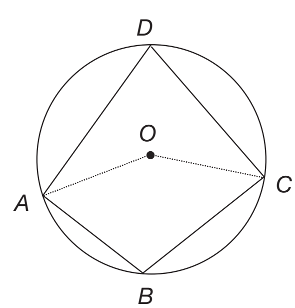


Figure 13.59

Angle subtended by arc ABC is double the angle subtended by arc ABC at any point on the remaining part of the circle and D is such a point.

Similarly,

Reflex $\angle AOC = 2\angle ABC$

$\angle AOC +$ Reflex $\angle AOC = 360°$

$2\angle ADC + 2\angle ABC = 360°$

$\Rightarrow \angle ADC + \angle ABC = 180°$

$\Rightarrow \angle B + \angle D = 180°$

Similarly, it can be proved that $\angle A + \angle C = 180°$.

Theorem 6

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

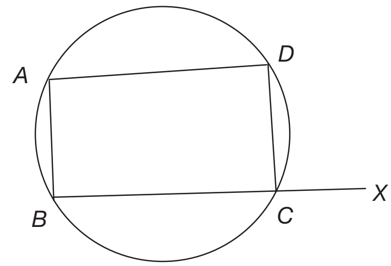


Figure 13.60

Given: $ABCD$ is a cyclic quadrilateral.

Construction: Extend BC to X.

RTP: $\angle DCX = \angle BAD$.

Proof:

$\angle BAD + \angle BCD = 180°$ (1)

(The opposite angles of a cyclic quadrilateral are supplementary).

$\angle BCD + \angle DCX = 180°$ (2)

(Angle of a straight line)

From Eqs. (1) and (2), we get

$\angle BAD + \angle BCD = \angle BCD + \angle DCX$

$\Rightarrow \angle DCX = \angle BAD$.

Hence proved.

We are already familiar with what a circle is and studied some of its properties. Now, we shall study the properties of tangents and chords.

Tangents

When a line and a circle are drawn in the same plane we have the following cases:

1. The line and the circle may not intersect at all, as shown in the Fig. 13.61(a). This means the line and the circle do not meet.
2. The line may intersect the circle at two points as shown in Fig. 13.61(b).
3. The line may touch the circle at only one point as shown in Fig. 13.61(c).

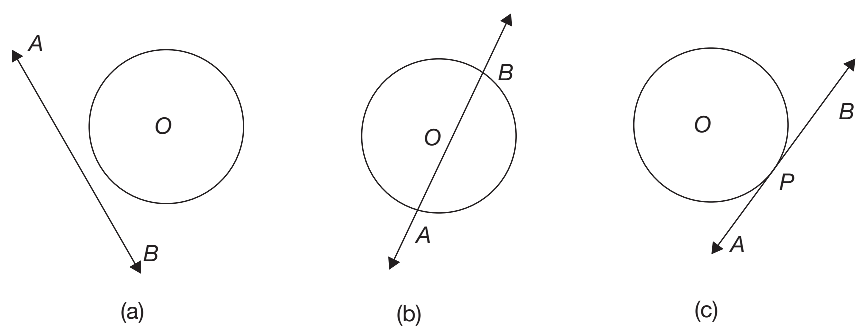


Figure 13.61

(i) If a line meets a circle at two points, then the line is called a secant of the circle.
(ii) When a line touches the circle at only one point or a line meets a circle at only one point, then the line is called a tangent to the circle at that point and that point is called point of contact or the point of tangency.
(iii) At a point on a circle, only one tangent can be drawn.
(iv) From any given external point, two tangents can be drawn to a circle.
(v) From any point inside a circle, no tangent can be drawn to the circle.

Theorem 7

The tangent at any point on a circle is perpendicular to the radius through the point of contact.

Given: PQ is a tangent to a circle with centre O. Point of contact of the tangent and the circle is A.

RTP: OA is perpendicular to PQ.

Construction: Take another point B on PQ and join OB.

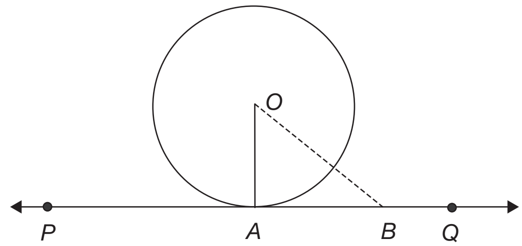


Figure 13.62

Proof: Since B is a point other than A, B may lie inside or outside the circle.

Case 1: If point B lies inside the circle but on the line PQ, then PQ cannot be a tangent to the circle. Hence B does not lie inside the circle.

Case 2: If B lies outside the circle, $OB > OA$, i.e., among all the segment joining a point on the line to the point O, OA is the shortest.

But, among all the line segments joining point O to a point on PQ, the shortest is the perpendicular line.
$\therefore$ OA is perpendicular to PQ.

Converse of the Theorem

A line drawn through the end point of a radius of a circle and perpendicular to it is a tangent to the circle.

Theorem 8

Two tangents drawn to a circle from an external print are equal in length.

Given PQ and PT are two tangents drawn from point P to circle with centre O.

RTP: $PQ = PT$

Construction: Join OP, OQ and OT.

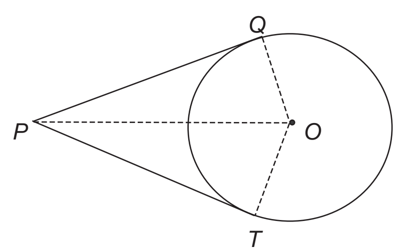


Figure 13.63

Proof: In triangles OPQ and OPT,

$OP = OP$ (Common)

$\angle OQP = \angle OTP = 90°$

$OQ = OT$ (Radius)

The tangent is perpendicular to the radius at the point of tangency. By the RHS congruence axiom, $\Delta OPQ \cong \Delta OPT$

$\Rightarrow PQ = PT$.

Note We also note that $\angle OPQ = \angle OPT$, i.e., the centre of a circle lies on the bisector of the angle formed by the two tangents.

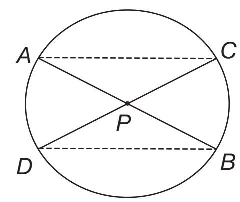
Chords

A line joining any two points on a circle is a chord of the circle. If AB is a chord of a circle and P is a point on it, P is said to divide AB internally into two segments AP and PB. Similarly, if Q is a point on the line AB, outside the circle, Q is said to divide AB externally into two segments AQ and QB.

Theorem 9

If two chords of a circle intersect each other, then the products of the lengths of their segments are equal.

Case 1: Let the two chords intersect internally.



Given: AB and CD are two chords intersecting at point P in the circle.

RTP: $(PA)\ (PB) = (PC)\ (PD)$

Construction: Join AC and BD.

Proof: In triangles APC and PDB,

$\angle APC = \angle DPB$ (Vertically opposite angels)

$\angle CAP = \angle CDB$ (Angles made by arc BC in the same segment)

ΔAPC is similar to ΔDPB. (If two angles of one triangle are equal to the corresponding angles of another triangle, the two triangles are similar or the AA Similarity Property)

$$\therefore\ \frac{PA}{PD} = \frac{PC}{PB} \Rightarrow (PA)(PB) = (PC)(PD)$$

Case 2: Let the two chords intersect externally.

Given: Two chords BA and CD intersect at point P which lies outside the circle.

RTP: $(PA)(PB) = (PC)(PD)$

Construction: Join AC and BD.

Proof: In triangles PAC and PDB,

$\angle PAC = \angle PDB$ and $\angle PCA = \angle PBD$ (An external angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.

$$\frac{PA}{PD} = \frac{PC}{PB} \text{ (The } AA \text{ Similarity Property)}$$

$\Rightarrow (PA)(PB) = (PC)(PD)$.

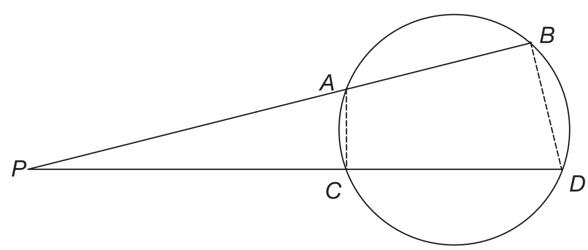


Figure 13.64

Notes

1. The converse of the above theorem is also true, i.e., if two line segments AB and CD intersect at P and $(PA)(PB) = (PC)(PD)$, then the four points are concyclic.
2. If one of the secants (say PCD) is rotated around P so that it becomes a tangent, i.e., points C and D say at T, coincide. We get the following result:

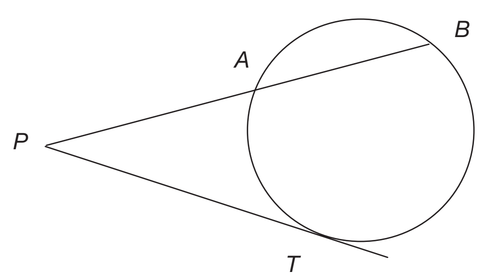


Figure 13.65

If PAB is a secant to a circle intersecting the circle at A and B and PT is the tangent drawn from P to the circle, then $PA \cdot PB = PT^2$

P is any point out side the circle with centre O and PAB is these cant drawn from P and PT is the tangent. Then $(PA)(PB) = PT^2$.

Alternate Segment and Its Angles

AB is a chord in a circle with centre O. A tangent is drawn to the circle at A. Chord AB makes two angles with the tangents $\angle BAY$ and $\angle BAX$. Chord AB divides the circle into two segments ACB and ADB. The segments ACB and ADB are called alternate segments to angles $\angle BAY$ and $\angle BAX$ respectively.

Angles made by the tangent with the chord are $\angle BAY$ and $\angle BAX$. ACB and ADB are alternate segments to those angles respectively.

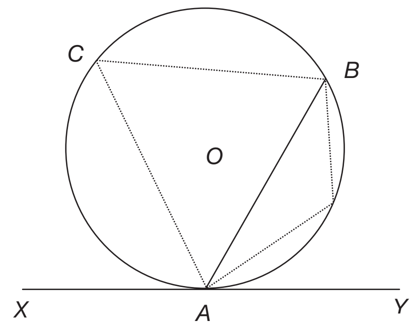


Figure 13.66

Theorem 10—Alternate Segment Theorem

If a line touches the circle at a point and if a chord is drawn from the point of contact then the angles formed between the chord and the tangent are equal to the angles in the alternate segments.

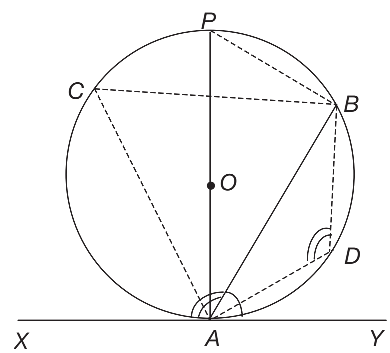


Figure 13.67

Given: XY is a tangent to the given circle with centre O at the point A, which lies in between X and Y. AB is a chord. C and D are points on the circle either side of line AB.

RTP: $\angle BAY = \angle ACB$ and $\angle BAX = \angle ADB$.

Construction: Draw the diameter AOP and join PB.

Proof: $\angle ACB = \angle APB$ (Angles in the same segment)

$\angle ABP = 90°$ (Angle in a semi-circle)

In the triangle ABP,

$\angle APB + \angle BAP = 90°$ (1)

$\angle PAY = 90°$ (the radius makes a right angle with the tangent at the point of tangency).

$\Rightarrow \angle BAP + \angle BAY = 90°$ (2)

From the Eqs. (1) and (2), we get

$\angle APB = \angle BAY$

$\Rightarrow \angle ACB = \angle BAY$. ($\because \angle APB = \angle ACB$)

Similarly, it can be proved that

$\angle BAX = \angle ADB$.

Converse of Alternate Segment Theorem

A line is drawn through the end point of a chord of a circle such that the angle formed between the line and the chord is equal to the angle subtended by the chord in the alternate segment. Then, the line is tangent to the circle at the point.

Common Tangents to Circles

When two circles are drawn on the same plane with radii r_1 and r_2, with their centres d units apart, then we have the following possibilities.

1. The two circles are concentric, then $d = 0$. The points C_1 and C_2 coincide.

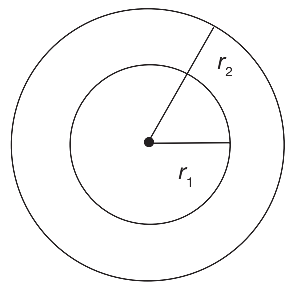


Figure 13.68

2. The two circles are such that one lies in side the other, then $|r_1 - r_2| > d$.

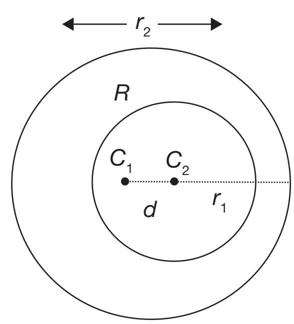


Figure 13.69

3. The two circles may touch each other internally, then $d = |r_1 - r_2|$.

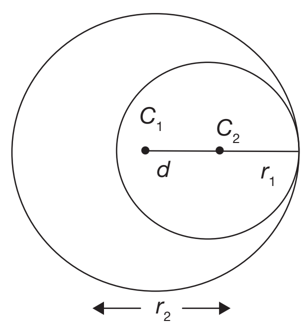


Figure 13.70

4. The two circles intersect at two points, in which case, $|r_1 - r_2| < d < r_1 + r_2$.

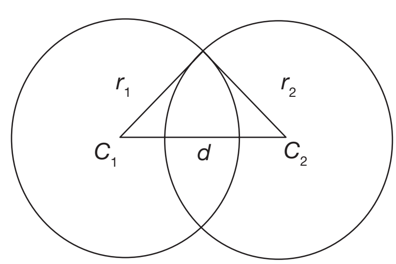


Figure 13.71

5. The two circles may touch each other externally, then $d = r_1 + r_2$.

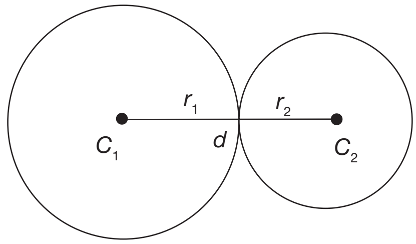


Figure 13.72

6. The two circles do not meet each other, then $d > r_1 + r_2$.

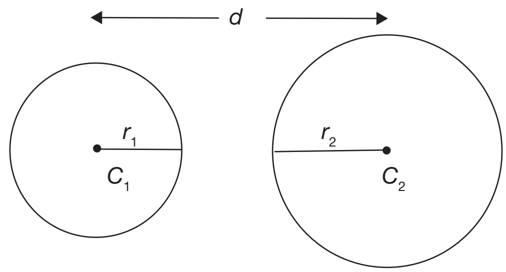


Figure 13.73

Common Tangent

If the same line is tangent to two circles drawn on the same plane, then the line is called a common tangent to the circles. The distance between the point of contacts is called the length of the common tangent.

In the figure, PQ is a common tangent to the circles, C_1 and C_2. The length of PQ is the length of the common tangent.

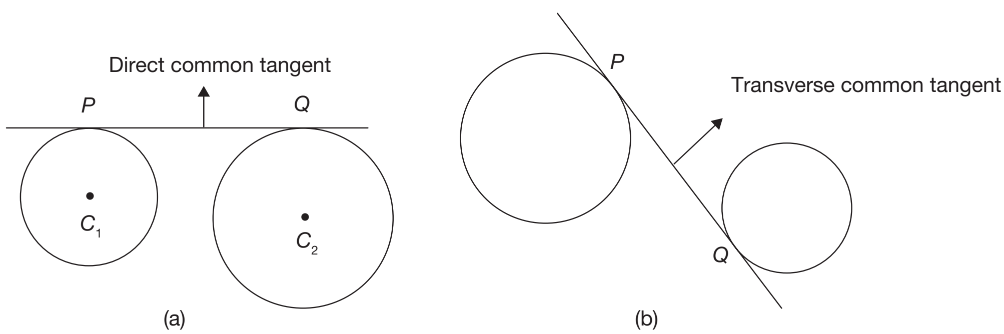


Figure 13.74

In Fig. 13.74(a), we observe that both the circles lie on the same side of PQ. In this case, PQ is a ***direct*** common tangent and in Fig. 13.74(b), we notice that the two circles lie on either side of PQ. Here PQ is a ***transverse*** common tangent.

1. The number of common tangents to the circles, one lying inside the other is zero.

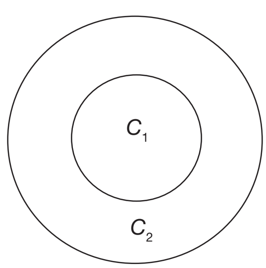


Figure 13.75

2. The number of common tangents to two circles touching internally is one.

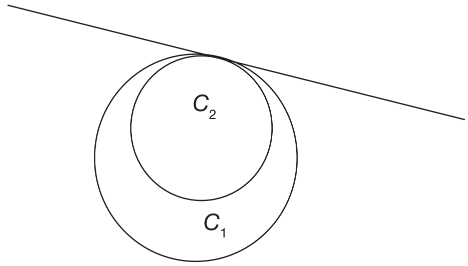


Figure 13.76

3. The number of common tangents to two intersecting circles is two, i.e., two direct common tangents.

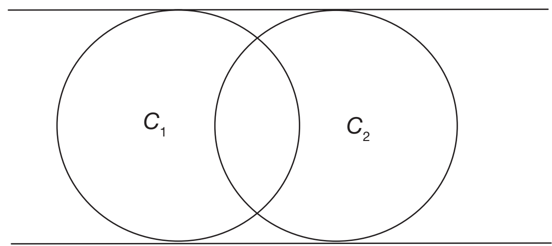


Figure 13.77

4. The number of common tangents to two circles touching externally is three, i.e., two direct common tangents and one transverse common tangent.

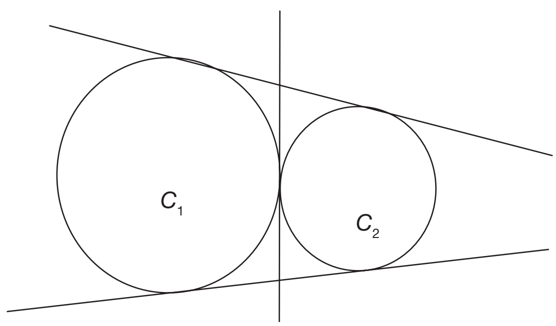


Figure 13.78

5. The number of common tangents to non-intersecting circles is four, i.e., 2 direct common tangents and 2 transverse common tangents.

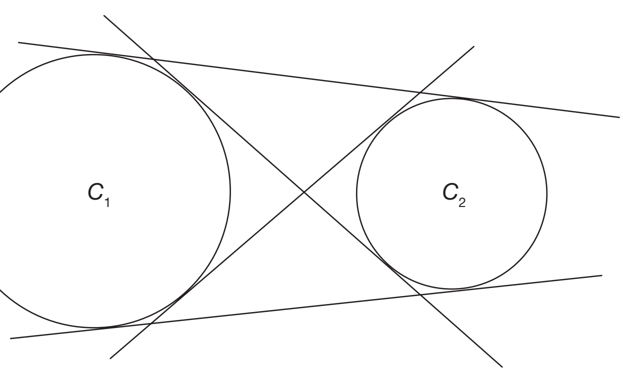


Figure 13.79

Properties of Common Tangents

1. When two circles touch each other internally or externally, then the line joining the centres is perpendicular to the tangent drawn at the point of contact of the two circles.

Case 1: Two circles with centres C_1 and C_2 touch each other internally at P. C_1C_2P is the line drawn through the centres and XY is the common tangent drawn at P which is common tangent to both the circles.
$\therefore$ C_1C_2 is perpendicular to XY.

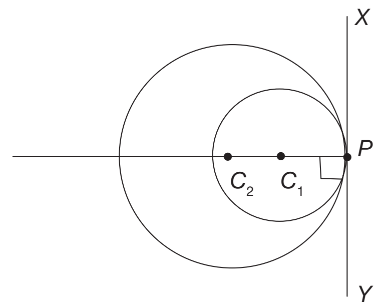


Figure 13.80

Case 2: The given two circles with centres C_1 and C_2 touch each other externally at P.

C_1PC_2 is the line joining the centres of the circles and XY is the common tangent to the two circles drawn at P.

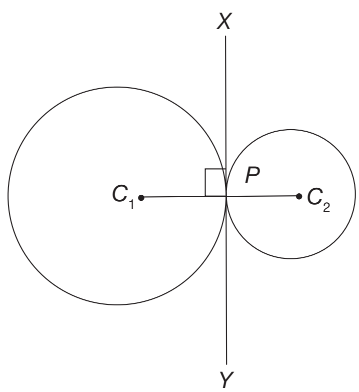


Figure 13.81

$\therefore$ C_1C_2 is perpendicular to XY.

2. The direct common tangents to two circles of equal radii are parallel to each other.

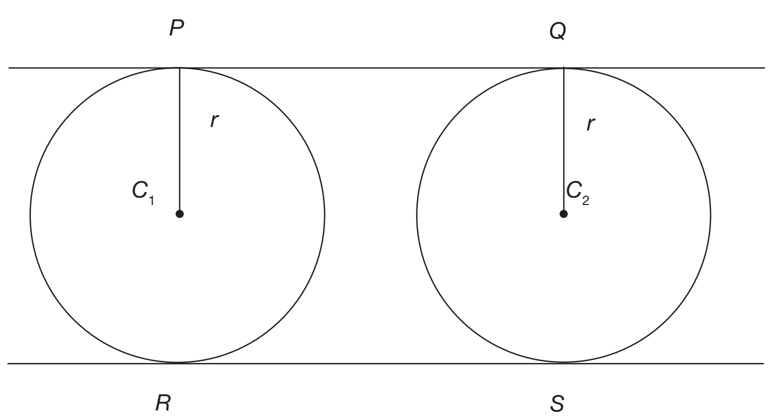


Figure 13.82

Let two circles of equal radii 'r' have centres C_1 and C_2 and PQ and RS be the direct common tangents drawn to the circles. Then PQ is parallel to RS.

EXAMPLE 13.12

In the figure, find the value of x.

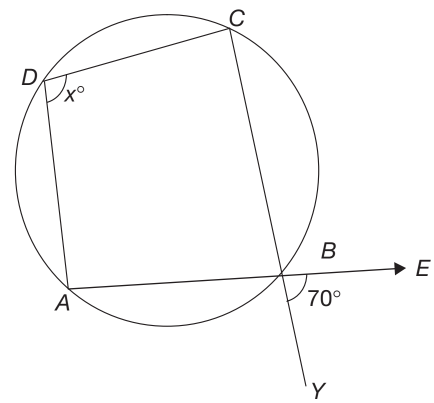


Figure 13.83

SOLUTION

In the figure, $\angle CBE$ is an exterior angle which is equal to the opposite interior angle at the opposite vertex, $\angle ADC$.

$\therefore \angle CBE = \angle ADC$ (1)

$\angle CBE + \angle EBY = 180°$ ($\because$ linear pair)

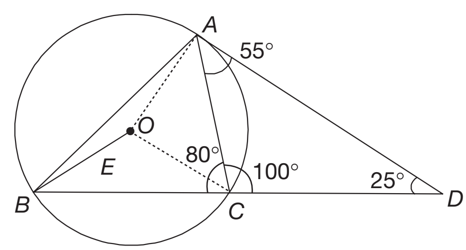
$\therefore \angle CBE = 180° - 70° = 110°$

$x° = \angle ADC = \angle CBE = 110°$.

EXAMPLE 13.13

In the given figure, O is the centre of the circle and AD is a tangent to the circle at A. If $\angle CAD = 55°$ and $\angle ADC = 25°$, then find $\angle ABO$.

(a) 10° **(b)** 15° **(c)** 20° **(d)** 25°



SOLUTION

(i) Join OA and OC.

(ii) $\angle ACD = 180° - 80° = 100°$.

(iii) $\angle ACB$ and $\angle ACD$ are supplementary.

$\Rightarrow \angle ACB + \angle ACD = 180°$

$\Rightarrow \angle ACB = 80°$

(iv) $\angle AOB = 2\angle ACB$. $\Rightarrow \angle AOB = 2 \times 80° = 160°$

(v) Now, In ΔAOB,

$\angle O + \angle A + \angle B = 180°$

$160° + x + x = 180°$

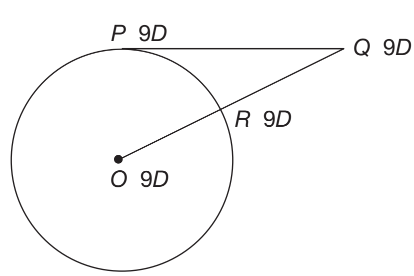
$2x = 20°$

$x = 10°$

$\therefore$ Hence, option (a) is the correct answer.

EXAMPLE 13.14

In the given figure (not to scale), PQ is a tangent segment, drawn to the circle with centre O at P and $QR = RO$. If $PQ = 3\sqrt{3}$ cm, and ORQ is a line segment, then find the radius of the circle.



(a) $\sqrt{3}$ cm **(b)** 3 cm **(c)** $2\sqrt{3}$ cm **(d)** 2 cm

SOLUTION

Let radius of the circle be x.

$\therefore OR = RQ = x$ (given $OR = RQ$)

Join OP

$\therefore\ \overline{OP} \perp \overline{PQ}$

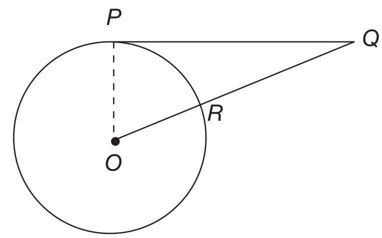
$OP^2 + PQ^2 = OQ^2$

$\Rightarrow x^2 + (3\sqrt{3})^2 = (2x)^2$

$\Rightarrow 3x^2 = 27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

But x is positive.

$\therefore\ x = 3$ cm.



Constructions Related to Circles

Construction 1: To construct a segment of a circle, on a given chord and containing a given angle.

Example: Construct a segment of a circle on a chord of length 8.5 cm, containing an angle of $65°(\theta)$

Step 1: Draw a line segment BC of the given length, 8.5 cm.

Step 2: Draw $\overrightarrow{BX}$ and $\overrightarrow{CY}$ such that $\angle CBX = \angle BCY = \dfrac{180 - 2\theta}{2} = 25°$.

Step 3: Mark the intersection of $\overrightarrow{BX}$ and $\overrightarrow{CY}$ as O.

Step 4: Taking O as centre and OB or OC as radius, draw BAC

Step 5: In ΔBOC, $\angle BOC = 130°$

$\Rightarrow \angle BAC = (1/2)\ \angle BOC = (1/2)(130°) = 65°$

$\angle BAC = 65°$.

The segment bounded by $\overline{BAC}$ and $\overline{BC}$ is the required segment.

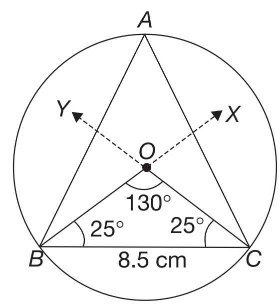


Figure 13.84

Construction 2: Construct an equilateral triangle inscribed in a circle of radius 3.5 cm.

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ such that $\angle AOB = \angle BOC = 120°$.

$(\therefore\ \angle COA = 120°)$

Step 3: Join AB, BC and CA which is the required equilateral ΔABC in the given circle.

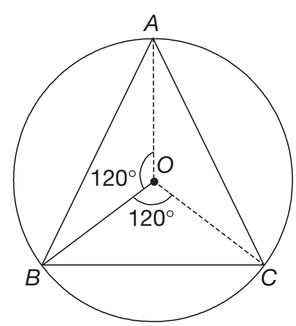


Figure 13.85

Construction 3: Construct an equilateral triangle circumscribed over a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm with centre O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ such that $\angle POR = \angle ROQ = 120°$.

$(\therefore\ \angle QOP = 120°)$

Step 3: At P, Q and R draw perpendiculars to $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ respectively to form ΔABC. ΔABC is the required circumscribing equilateral triangle.

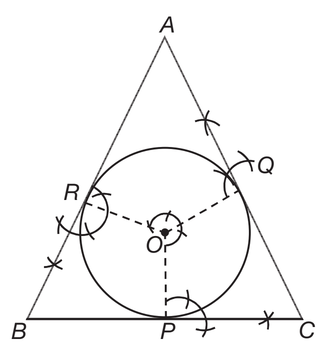


Figure 13.86

Construction 4: Draw the circum-circle of a given triangle.

Step 1: Draw ΔABC with the given measurements.

Step 2: Draw perpendicular bisectors of the two of the sides, say $\overline{AB}$ and AC to intersect at S.

Step 3: Taking S as centre, and the radius equal to AS or BS or CS, draw a circle. The circle passes through all the vertices A, B and C of the triangle.

$\therefore$ The circle drawn is the required circum-circle.

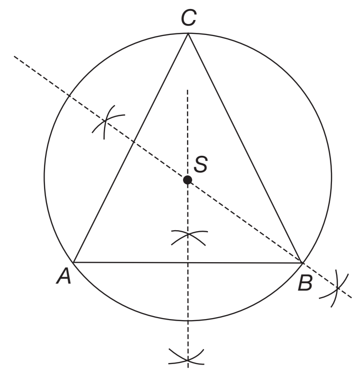


Figure 13.87

> **Note** The circum-centre is equidistant from the vertices of the triangle.

Construction 5: Construct the in circle of a given triangle ABC.

Step 1: Draw a ΔABC with the given measurements.

Step 2: Draw bisectors of two of the angles, say $\angle B$ and $\angle C$ to intersect at I.

Step 3: Draw perpendicular $\overline{IM}$ from I onto $\overline{BC}$.

Step 4: Taking I as centre and IM as the radius, draw a circle.

This circle touches all the sides of the triangle. This is the in circle of the triangle.

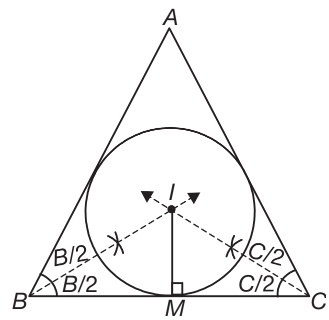


Figure 13.88

> **Note** The in-centre is equidistant from all the sides of the triangle.

Construction 6: Construct a square inscribed in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm and mark centre as 'O'.

Step 2: Draw diameters $\overline{AC}$ and $\overline{BD}$ such that $\overline{AC} \perp \overline{BD}$.

Step 3: Join A, B, C and D. Quadrilateral $ABCD$ is the required square inscribed in the given circle.

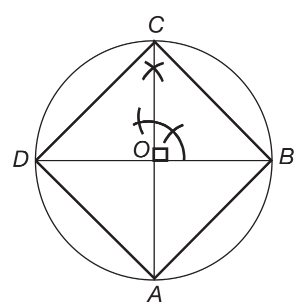


Figure 13.89

Construction 7: Construct a square circumscribed over the given circle of radius 2.5 cm.

Step 1: Draw a circle of radius 2.5 cm.

Step 2: Draw two mutually perpendicular diameters $\overline{PR}$ and $\overline{SQ}$.

Step 3: At P, Q, R and S, draw lines perpendicular to OP, OQ, OR and OS respectively to form square $ABCD$ as shown in the figure.

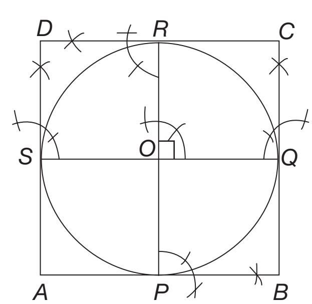


Figure 13.90

Construction 8: Construct a regular pentagon of side 4 cm. Circumscribe a circle to it.

Step 1: Draw a line segment, $AB = 4$ cm.

Step 2: Draw $\overrightarrow{BX}$ such that $\angle ABX = 108°$.

Step 2: Mark the point C on $\overrightarrow{BX}$ such that $BC = 4$ cm.

Step 4: Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ and mark their intersecting point as O.

Step 5: With O as centre and OA as radius draw a circle. And it passes through the points B and C.

Step 6: With C as centre and 4 cm as radius draw an arc which cuts the circle at the point D.

Step 7: With D as centre and 4 cm as radius draw an arc which cuts the circle at the point E.

Step 8: Join CD, DE and AE.

Step 9: $ABCDE$ is the required pentagon.

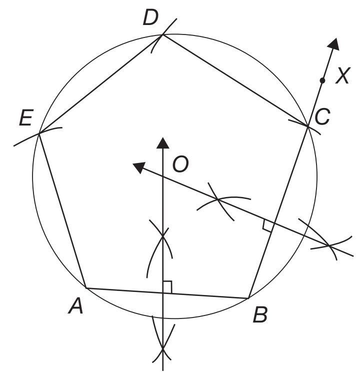


Figure 13.91

Construction 9: Construct a regular pentagon in a circle of radius 4 cm.

Step 1: Construct a circle with radius 4 cm.

Step 2: Draw two radii $\overline{OA}$ and $\overline{OB}$ such that $\angle AOB = 72°$.

Step 3: Join AB.

Step 4: With A as centre and AB as radius draw an arc which cuts the circle at the point E.

Step 5: With E as centre and AB as radius draw an arc which cuts the circle at the point D.

Step 6: With D as centre and AB as radius draw an arc which cuts the circle at the point C.

Step 7: Join AE, ED, DC and CB.

Step 8: $ABCDE$ is the required pentagon.

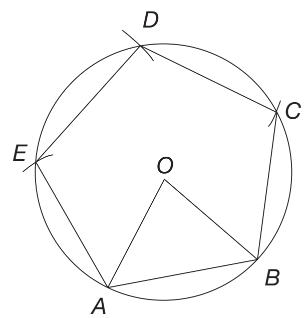


Figure 13.92

Construction 10: Construct a regular pentagon about a circle of radius 3.5 cm.

Step 1: Construct a circle with radius 3.5 cm.

Step 2: Draw radii OP, OQ, OR, OS and OT such that the angle between any two consecutive radii is 72°.

Step 3: Draw tangents to the circle at the points P, Q, R, S and T.

Step 4: Mark the intersecting points of the tangents as A, B, C, D and E.

Step 5: $ABCDE$ is the required pentagon.

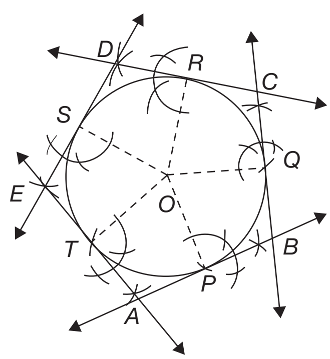


Figure 13.93

Construction 11: Construct a regular hexagon circumscribed over a circle of radius 3.5 cm.

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ such that the angle between any two adjacent radii is 60°.

Step 3: Draw lines at P, Q, R, S, T and U perpendicular to $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ respectively, to form the required circumscribed hexagon $ABCDEF$.

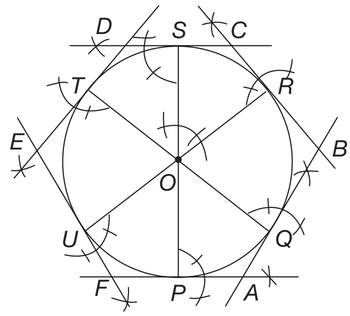


Figure 13.94

Construction 12: Inscribe a regular hexagon in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm taking the centre as *O*.

Step 2: Draw radius $\overline{OA}$. With radius, equal to *OA* and starting with *A* as the centre, mark points *B*, *C*, *D*, *E* and *F* one after the other.

Step 3: Join *A*, *B*, *C*, *D*, *E* and *F*. Polygon *ABCDEF* is the required hexagon.

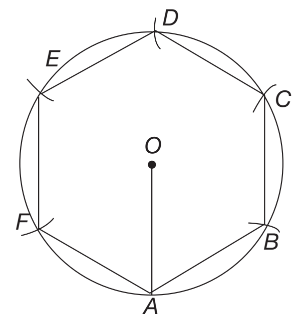


Figure 13.95

Construction 13:

1. To construct a tangent to a circle at a point on it.

   When the location of the centre is known.

Step 1: Draw a circle with centre *O* with any radius. Let *P* be a point on the circle.

Step 2: Join *OP*.

Step 3: Construct $\angle OPY = 90°$.

Step 4: Produce *YP* to *X* and *XPY* is the required tangent to the given circle at point *P*.

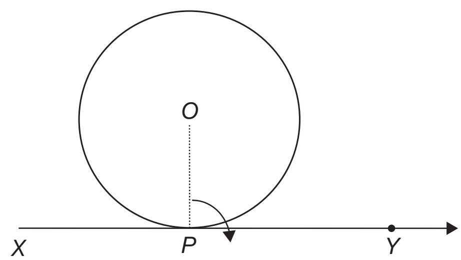


Figure 13.96

2. We have a circle, but we do not know where the centre is located.

Step 1: Draw a chord *AB*.

Step 2: Mark point *C* on major arc *ACB*. Join *BC* and *AC*.

Step 3: Draw $\angle BAY = \angle ACB$.

Step 4: Produce *YA* to *X* as shown in the figure. *XAY* is the required tangent at *A*.

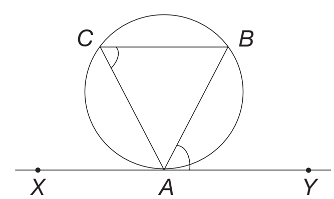


Figure 13.97

Construction 14: To construct the two tangents to a circle with centre at *O*, from an external point *P*.

Draw a circle with *OP* as diameter.

1. When the location of the centre is known.

Step 1: Bisect *OP* at *A*. With centre *A* and radius equal to *OA* or *AP*, draw a circle that intersects the given circle at two points *T* and *Q*.

Step 2: Join *PT* and *PQ*, which are the required tangents from *P*.

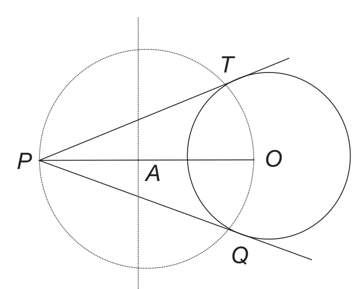


Figure 13.98

2. When the location of the centre is not known.

Step 1: Draw a circle of radius 2.5 cm. Mark a point P outside the circle.

Step 2: Draw a secant PQR through 'P' to intersect the circle at Q and R.

Step 3: Produce QP to S, such that $PS = QP$.

Step 4: With SR as diameter, construct a semi-circle.

Step 5: Construct a line perpendicular to SR at P to intersect the semi-circle at U.

Step 6: Draw arcs with PU as radius and P as the centre to intersect the given circle at T and T_1.

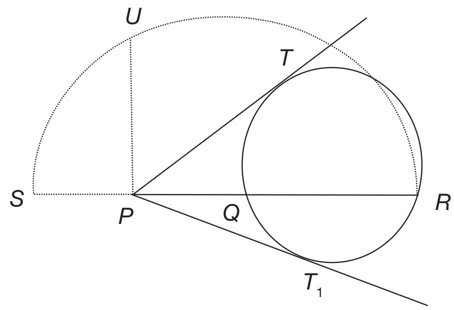


Figure 13.99

Step 7: Join PT and PT_1. These are the required tangents from P to the given circles.

Construction 15: To construct the direct common tangents to the circles of radii 3.5 cm and 1.5 cm whose centres are 6.5 cm apart.

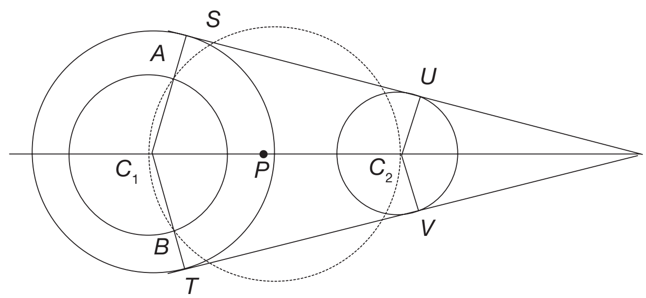


Figure 13.100

Step 1: Draw line segment, $C_1C_2 = 6.5$ cm.

Step 2: Draw circles with radii 3.5 cm and 1.5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2 at P and draw a circle taking P as centre and radius equal to PC_1 or PC_2.

Step 4: Draw another circle with radius equal to $(3.5 - 1.5) = 2$ cm, taking the centre of larger circle C_1 intersecting the circle drawn in step 3 at A and B.

Step 5: Join C_1A and C_1B and produce C_1A and C_1B to meet the outer circle at S and T respectively.

Step 6: Draw line segments C_2U and C_2V, such that $C_2U \| C_1S$ and $C_2V \| C_1T$, where U, V are points on the circle with centre C_2.

Step 7: Join SU and TV. These are the required direct common tangents.

Note The length of a direct common tangent to two circles with radii r_1 and r_2 and centres d units apart, is given by $\sqrt{d^2 - (r_1 - r_2)^2}$.

Construction 16: To construct the transverse common tangents to two circles with radii 2.5 cm and 1.5 cm, with centres at a distance of 6 cm from each other.

Step 1: Draw a line segment, $C_1C_2 = 6$ cm.

Step 2: Draw circles of radii 2.5 cm and 1.5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2. Let M be the mid-point of C_1C_2.

Step 4: Draw a circle with M as centre and C_1M or C_2M as radius.

Step 5: With C_2 as centre and radius equal to (2.5 + 1.5) or 4 cm, mark off 2 points A and B on the circle in step 4.

Step 6: Join C_2A and C_2B to meet the other circle with centre C_2 at S and T respectively.

Step 7: Draw lines, C_1U and C_1V such that $C_1U \,||\, C_2B$ and $C_1V \,||\, C_2A$, where U, V are points on the circle with centre at C_1.

Step 8: Join TU and SV. These are the required transverse common tangents.

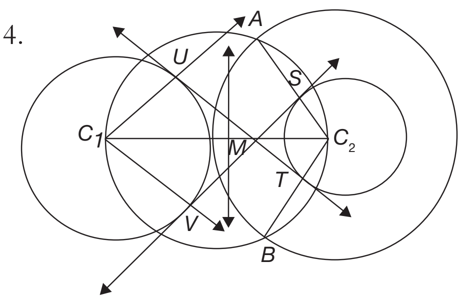


Figure 13.101

Notes

1. The length of the transverse common tangent of two circles with radii r_1 and r_2 and centres at a distance d ($d > r_1 + r_2$) is $\sqrt{d^2 - (r_1 + r_2)^2}$.
2. Transverse common tangents can be drawn to non intersecting and non touching circles only or to the circles that satisfy the condition, $d > r_1 + r_2$.

Construction 17: To construct a circle of given radius passing through two given points.

Example: Construct a circle of radius 2.5 cm passing through two points A and B 4 cm apart.

Step 1: Draw a line segment, $AB = 4$ cm.

Step 2: Bisect $\overline{AB}$ by drawing a perpendicular bisector of $\overline{AB}$.

Step 3: Draw an arc with A or B as centre and with radius 2.5 cm which intersect the perpendicular bisector of $\overline{AB}$ at O.

Step 4: Draw a circle that passes through A and B with O as centre and radius 2.5 cm.

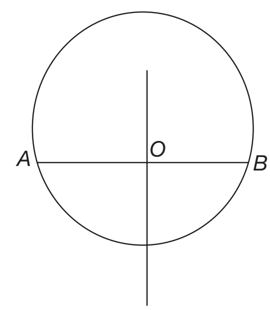


Figure 13.102

∴ This is the required circle with given radius and passing through the given points A and B.

Construction 18: To draw a circle touching a given line AB at a given point P and passing through a given point Q.

Step 1: Draw a line AB and mark point P on it.

Step 2: Draw PR at P such that $PR \perp AB$, i.e., $\angle APR = 90°$.

Step 3: Join PQ and draw the perpendicular bisector of PQ to intersect PR at O.

Step 4: Draw a circle, touches line AB at P and passes through the given point Q with O as centre and OP as radius.

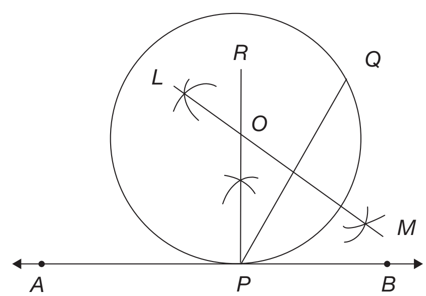


Figure 13.103

∴ The circle drawn is the required circle.

Construction 19: To construct a circle with a given radius and touching the two given intersecting lines.

Example: Construct an angle of 70° and draw a circle of radius 2.5 cm touching the arms of the angle.

Step 1: Draw an angle $\angle ABC = 70°$.

Step 2: Draw BD, the bisector of the angle ABC.

Step 3: Draw a line $\overline{PD} \parallel \overline{BC}$ at a distance of 2.5 cm (equal to the radius of the circle) to intersect the angle bisector of step 2 at O.

Step 4: Draw a circle with O as centre and radius 2.5 cm

$\therefore$ The circle drawn touches the lines at X and Y.

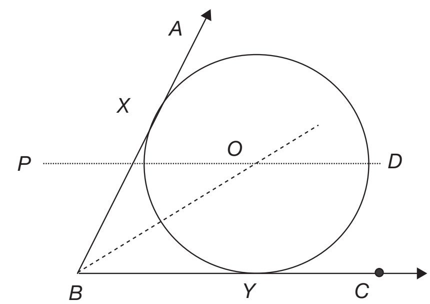


Figure 13.104

Construction 20: To construct a circle of given radius which touches a given circle and a given line.

Example: Draw a circle of radius 3 cm which touches a given line AB and a given circle of radius 2 cm with centre C.

Step 1: Let AB be the given line.

Step 2: Draw a line PQ parallel to $\overline{AB}$ at a distance 3 cm from $\overline{AB}$.

Step 3: Draw an arc with C as centre and with radius equal to $(2 + 3) = 5$ cm to cut PQ at point O.

Step 4: Draw a circle which touches the given circle at E and the given line AB at F with O as centre and radius 3 cm.

$\therefore$ The circle drawn in step 4 is the required circle.

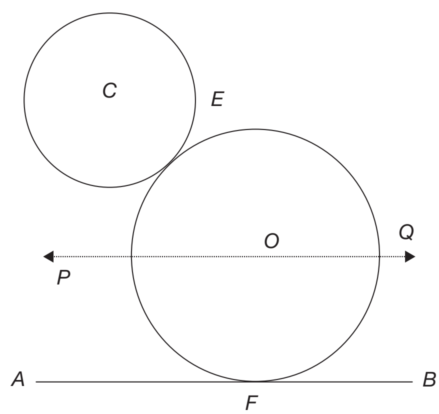


Figure 13.105

Construction 21: Construct a triangle ABC in which $BC = 6$ cm, $\angle A = 60°$ and altitude through A is 4 cm.

Step 1: Draw a line segment, $BC = 6$ cm.

Step 2: Draw $\overrightarrow{BL}$ such that $\angle CBL = 60°$.

Step 3: Draw $\overrightarrow{BM}$ such that $\angle MBL = 90°$.

Step 4: Draw a perpendicular bisector $\left(\overleftrightarrow{XY}\right)$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point P.

Step 5: Mark the intersecting point of $\overleftrightarrow{XY}$ and $\overrightarrow{BM}$ as O.

Step 6: With O as centre and OB as radius draw a circle.

Step 7: Mark the point N on $\overleftrightarrow{XY}$ such that $PN = 4$ cm.

Step 8: Through N, draw a line parallel to $\overline{BC}$ which intersects the circle at points A and A'.

Step 9: Join AB, AC and $A'B$, $A'C$.

Step 10: Now, ABC or $A'BC$ is required triangle.

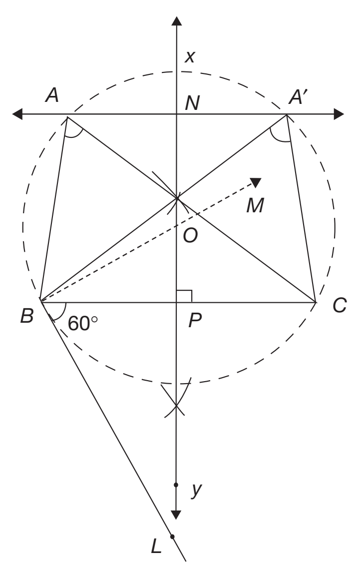


Figure 13.106

Proof: By alternate segment theorem, $\angle BAC = \angle CBL = \angle BA'C = 60°$.

Altitude through A = Altitude through A' = $PN = 4$ cm.

Construction 22: Construct a triangle ABC in which $BC = 7$ cm, $\angle A = 65°$ and median AT is 5 cm.

Step 1: Draw a line segment, $BC = 7$ cm.

Step 2: Draw $\overrightarrow{BP}$ such that $\angle CBP = 65°$.

Step 3: Draw $\overrightarrow{BQ}$ such that $\angle PBQ = 90°$.

Step 4: Draw a perpendicular bisector $(\overleftrightarrow{RS})$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point T.

Step 5: Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{BQ}$ as O.

Step 6: With O as centre and OB as radius draw circle.

Step 7: With T as centre 5 cm as radius draw two arcs which intersect the circle at the points A and A'.

Step 8: Join AB and AC and A' and $A'C$.

Step 9: Now, ABC or $A'BC$ is the required triangle.

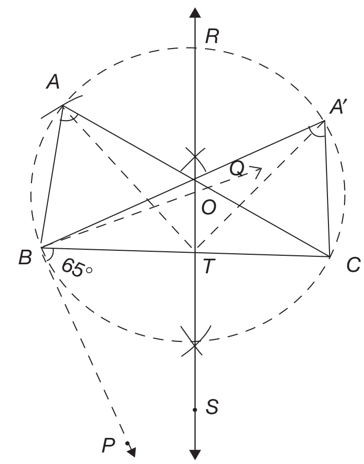


Figure 13.107

Proof: By alternate segment theorem, $\angle BAC = \angle CBP = \angle BA'C = 65°$.

Construction 23: Construct a cyclic quadrilateral $ABCD$ in which $AB = 3$ cm, $AD = 4$ cm, $AC = 6$ cm and $\angle D = 70°$.

Rough figure:

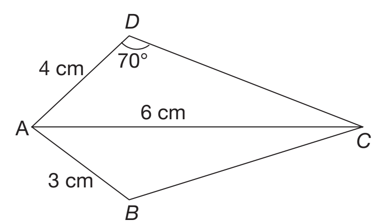


Figure 13.108

Step 1: Draw a line segment, $AC = 6$ cm.

Step 2: Draw $\overrightarrow{AP}$ such that $\angle CAP = 70°$.

Step 3: Draw $\overrightarrow{AQ}$ such that $\angle PAQ = 90°$.

Step 4: Draw a perpendicular bisectors $(\overleftrightarrow{RS})$ of $\overline{AC}$ which intersects $\overline{AC}$ at the point T.

Step 5: Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{AQ}$ as O.

Step 6: With O as centre and OA as radius draw a circle.

Step 7: With A as centre, 4 cm as radius draw an arc which intersects the circle atthe point D.

Step 8: With A as centre, 3 cm as radius draw an arc which intersects the circle at the point B.

Step 9: Join AB, BC, CD and AD.

Step 10: $ABCD$ is the required cyclic quadrilateral.

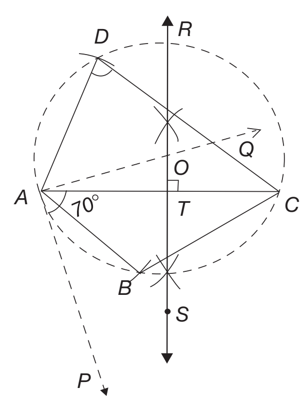


Figure 13.109

Construction 24: Find the mean proportional of the given segments p cm and q cm.

Step 1: Draw $\overleftrightarrow{XZ}$.

Step 2: Mark the point Y on $\overleftrightarrow{XZ}$ such that $XY = p$ cm and $YZ = q$ cm.

Step 3: Draw the perpendicular bisector $\left(\overleftrightarrow{AB}\right)$ of $\overline{XZ}$ which meets $\overleftrightarrow{XZ}$ at the point O.

Step 4: With O as centre and OX as radius draw a semi circle. $\overleftrightarrow{XZ}$ is the diameter of the semi circle.

Step 5: Draw $\overleftrightarrow{YM}$ perpendicular to $\overleftrightarrow{XZ}$ which meets semicircle at the point M.

Step 6: Now MY, is the mean proportional of XY and YZ, i.e., p and q.

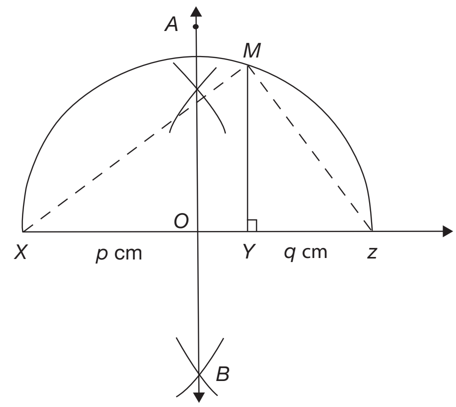


Figure 13.110

Proof:

$$\Delta XYM \sim \Delta MYZ$$

$$\Rightarrow \frac{XY}{YM} = \frac{MY}{YZ}$$

$$\Rightarrow MY = \sqrt{(XY)(YZ)} = \sqrt{pq}.$$

Uses: By using this construction, square root ($\sqrt{12}, \sqrt{15}, \ldots$) can be found.

Construction 25: Construct a square whose area is equal to the area of the rectangle.

Step 1: Produce $\overline{AB}$ of rectangle $ABCD$ to the point E such that $BE = BC$.

Step 2: Construct mean proportional (BF) of AB and BE.

Step 3: Construct a square with side BF.

Step 4: $BFGH$ is the required square.

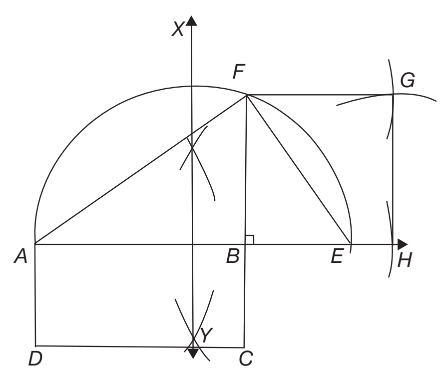


Figure 13.111

Proof: BF is the mean proportional of AB and BE.

$$\Rightarrow (BF)^2 = AB \times BE$$

$$\Rightarrow (BF)^2 = AB \times BC \; (\because BC = BE)$$

$\Rightarrow$ Area of square $BFGH$ = Area of rectangle $ABCD$.

Construction 26: Construct a triangle similar to a given triangle ABC with its sides equal to $\left(\frac{5}{9}\right)$th of the corresponding sides of ΔABC.

Step 1: ABC is the given triangle.

Step 2: Draw $\overrightarrow{BD}$ which makes non-zero angle with $\overline{BC}$ and D is 9 cm away from the point B.

Step 3: Mark the point E on $\overrightarrow{BD}$ such that $BE = 5$ cm.

Step 4: Join $\overline{CD}$.

Step 5: Draw $\overline{EC'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.

Step 6: Draw $\overline{C'A'}$ parallel to $\overline{CA}$ which meets $\overline{BA}$ at the point A'.

Step 7: $A'BC'$ is the required triangle.

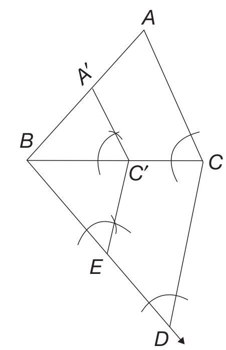


Figure 13.112

Proof: In ΔBCD, $\dfrac{BE}{BD} = \dfrac{5}{9}$ and $\overline{DC} \parallel \overline{EC'}$

$\therefore\ \Delta BEC' \sim \Delta BDC$

$$\therefore\ \frac{BC'}{BC} = \frac{5}{9}$$

In ΔBCA, $\dfrac{BC'}{BC} = \dfrac{5}{9}$ and $\overline{C'A'} \parallel \overline{CA}$

$\therefore\ \Delta BCA \sim \Delta BC'A'$

$$\therefore\ \frac{BC'}{BC} = \frac{BA'}{BA} = \frac{A'C'}{AC} = \frac{5}{9}$$

Construction 27: Construct a quadrilateral similar to a given quadrilateral $ABCD$ with its sides equal to $\left(\dfrac{6}{10}\right)$th of the corresponding sides of $ABCD$.

Step 1: $ABCD$ is the given quadrilateral.

Step 2: Join BD.

Step 3: Draw $\overrightarrow{BX}$ which makes non zero angle with $\overline{BD}$ and X is 10 cm away from the point B.

Step 4: Mark the point Y on $\overrightarrow{BX}$ such that $BY = 6$ cm.

Step 5: Join XD.

Step 6: Draw $\overline{YD'}$ parallel to $\overline{XD}$ which meets BD at the point D'.

Step 7: Draw $\overline{D'A'}$ parallel to $\overline{DA}$ which meets $\overline{AB}$ at the point A'.

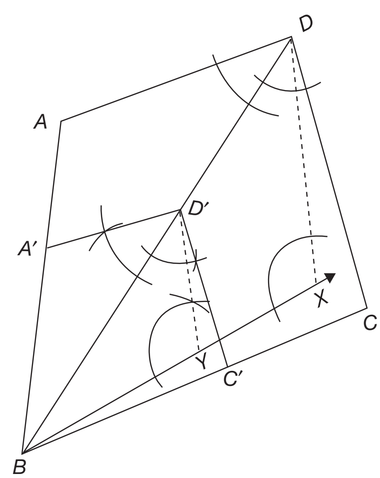


Figure 13.113

Step 8: Draw $\overline{D'C'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.

Step 9: $A'BC'D'$ is the required quadrilateral.

Construction 28: Construction of a pentagon similar to the given pentagon on a side of given length.

Step 1: *ABCDE* is a given pentagon and $\overline{XY}$ is the given line segment.

Step 2: Mark the point B' on $\overline{AB}$ such that $AB' = XY$.

Step 3: Join *AC* and *AD*.

Step 4: Draw $\overline{B'C'}$ parallel to $\overline{BC}$ which meets $\overline{AC}$ at the point C'.

Step 5: Draw $\overline{C'D'}$ parallel to $\overline{CD}$ which meets $\overline{AD}$ at the point D'.

Step 6: Draw $\overline{D'E'}$ parallel to $\overline{DE}$ which meets *AE* at the point E'.

Step 7: $AB'C'D'E'$ is the required pentagon.

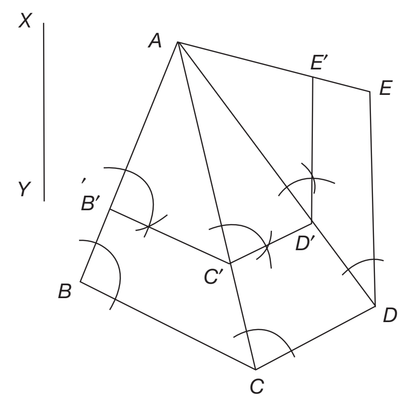


Figure 13.114

LOCUS

Mark a fixed point *O* on a sheet of paper. Now, start marking points P_1, P_2, P_3, P_4, … on the sheet of paper such that $OP_1 = OP_2 = OP_3 = \ldots = 4$ cm. What do we observe on joining these points by a smooth curve? We observe a pattern, which is circular in shape such that any point on the circle obtained is at a distance of 4 cm from the point *O*.

It can be said that whenever a set of points satisfying a certain condition are plotted, a pattern is formed. This pattern formed by all possible points satisfying the given condition is called the locus of points. In the above given example, we have a locus of points which are equidistant (4 cm) from the given point *O*.

The collection (set) of all points and only those points which satisfy certain given geometrical conditions is called **locus of a point**.

Alternatively, locus can be defined as the path or curve traced by a point in a plane when subjected to some geometrical conditions.

Consider the following examples:

1. The locus of the point in a plane which is at a constant distance '*r*' from a fixed point '*O*' is a circle with centre *O* and radius *r* units.

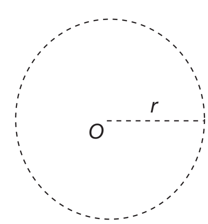


Figure 13.115

2. The locus of the point in a plane which is at a constant distance from a fixed straight line is a pair of lines, parallel to the fixed line. Let the fixed line be *l*. The lines *m* and *n* form the set of all points which are at a constant distance from *l*.

3. The locus of a point in a plane, which is equidistant from a given pair of parallel lines is a straight line, parallel to the two given lines and lying midway between them.

   In the given above, *m* and *n* are the given lines and line *l* is the locus.

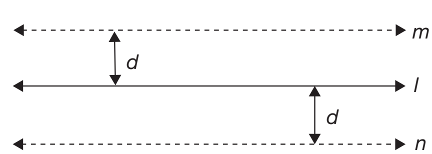


Figure 13.116

To prove that a given path or curve is the desired locus, it is necessary to prove that

(i) every point lying on the path satisfies the given geometrical conditions.

(ii) every point that satisfies the given conditions lies on the path.

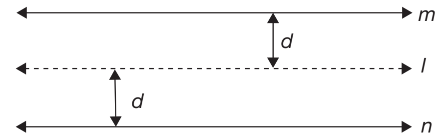


Figure 13.117

EXAMPLE 13.15

Show that the locus of a point equidistant from the end points of a line segment is the perpendicular bisector of the segment.

SOLUTION

The proof will be taken up in two steps:

Step 1: We, initially prove that any point equidistant from the end points of a line segment lies on the perpendicular bisector of the line segment.

Given: M and N are two points on a plane. A is a point in the same plane such that $AM = AN$.

RTP: A lies on the perpendicular bisector of MN.

Proof: Let M and N be the two fixed points in a plane.

Let A be a point such that $AM = AN$ and L be the mid-point of $\overline{MN}$.

If A coincides with L, then A lies on the bisector of MN.

Suppose A is different from L.

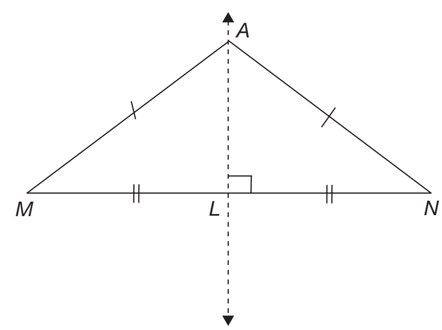


Figure 13.118

Then, in ΔMLA and ΔNLA,

$ML = NL$, $AM = AN$ and AL is a common side.

$\therefore$ By SSS congruence property, $\Delta MLA \cong \Delta NLA$.

$\Rightarrow \angle MLA = \angle NLA$ ($\because$ Corresponding elements of congruent triangles are equal) (1)

But $\angle MLA + \angle NLA = 180°$ ($\because$ They form a straight angle)

$\Rightarrow 2\ \angle MLA = 180°$ (using (1))

$\therefore\ \angle MLA = \angle NLA = 90°$

So, $\overline{AL} \perp \overline{MN}$ and hence $\overline{AL}$ is the perpendicular bisector of $\overline{MN}$.

$\therefore$ A lies on the perpendicular bisector of $\overline{MN}$.

Step 2: Now, we prove that any point on the perpendicular bisector of the line segment is equidistant from the end points of the line segment.

Given: MN is a line segment and P is a point on the perpendicular bisector. L is the mid-point of MN.

RTP: $MP = NP$.

Proof: If P coincides with L, then $MP = NP$.

Suppose P is different from L. Then, in ΔMLP and NLP,

$ML = LN$

LP is the common side and

$\angle MLP = \angle NLP = 90°$

$\therefore$ By the SAS congruence property, $\Delta MLP \cong \Delta NLP$.

So, $MP = PN$ ($\because$ The corresponding elements of congruent triangles are equal)

$\therefore$ Any point on the perpendicular bisector of $\overline{MN}$ is equidistant from the points M and N.

Hence, from the steps 1 and 2 of the proof it can be said that the locus of the point equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

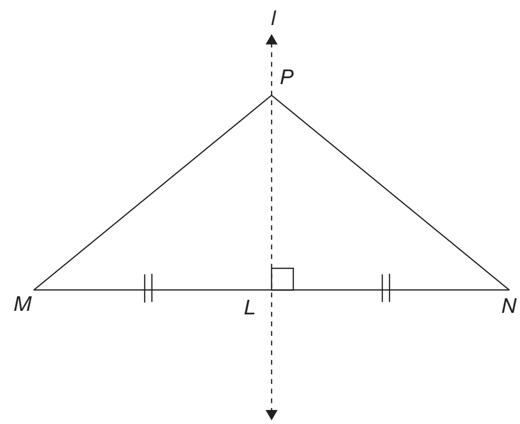


Figure 13.119

EXAMPLE 13.16

Show that the locus of a point equidistant from two intersecting lines in the plane determined by the lines is the union of the pair of lines bisecting the angles formed by given lines.

SOLUTION

Step 1: We initially prove that any point equidistant from two given intersecting lines lies on one of the lines bisecting the angles formed by given lines.

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines intersecting at O. P is the point on the plane such that $PM = PN$. Line l is the bisector of $\angle BOD$ and $\angle AOC$.

Line m is the bisector of $\angle BOC$ and $\angle AOD$.

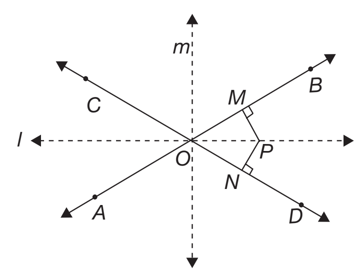


Figure 13.120

RTP: P lies on either on line l or line m.

Proof: In ΔPOM and ΔPON,

$PM = PN$,

OP is a common side and $\angle PMO = \angle PNO = 90°$

$\therefore$ By RHS congruence property, $\Delta POM \cong \Delta PON$.

So, $\angle POM = \angle PON$, i.e., P lies on the angle bisector of $\angle BOD$.

As l is the bisector of $\angle BOD$ and $\angle AOC$, P lies on the line l.

Similarly if P lies in any of the regions of $\angle BOC$, $\angle AOC$ or $\angle AOD$, such that it is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, then we can conclude that P lies on the angle bisector l or on the angle bisector m.

Step 2: We prove that any point on the bisector of one of angles formed by two intersecting lines is equidistant from the lines.

Given: Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O. Lines l and m are the angle bisectors.

Proof: Let l be the angle bisector of $\angle BOD$ and $\angle AOC$, and m be the angle bisector of $\angle BOC$ and $\angle AOD$.

Let P be a point on the angle bisector l, as shown in the figure.

If P coincides with O, then P is equidistant from the line $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Suppose P is different from O.

Draw the perpendiculars $\overline{PM}$ and $\overline{PN}$ from the point P onto the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ respectively.

Then in ΔPOM and ΔPON, $\angle POM = \angle PON$, $\angle PNO = \angle PMO = 90°$ and OP is a common side.

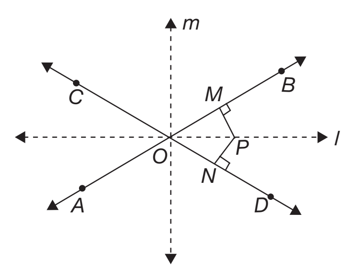


Figure 13.121

$\therefore$ By the AAS congruence property $\Delta POM \cong \Delta PON$

So, $PN = PM$ ($\because$ Corresponding sides), i.e.,

P is equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Hence, from the steps 1 and 2 of the proof it can be said that the locus of the point which is equidistant from the two intersecting lines is the pair of the angle bisectors of the two pairs of vertically opposite angles formed by the lines.

Equation of a Locus

We know a locus is the set of points that satisfy a given geometrical condition. When we express the geometrical condition in the form of an algebraic equation that equation is called the equation of the locus.

Steps to Find the Equation of a Locus

Step 1: Consider any point (x_1, y_1) on the locus.

Step 2: Express the given geometrical condition in the form of an equation using x_1 and y_1.

Step 3: Simplify the equation obtained in step 2.

Step 4: Replace (x_1, y_1) by (x, y) in the simplified equation obtained in step 3, which gives the required equation of the locus.

The following formulae will be helpful in finding the equation of a locus.

1. Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. Area of the triangle formed by joining the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$, where the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
3. Equation of the circle with centre (a, b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$.
4. The perpendicular distance from a point $P(x_1, y_1)$ to a given line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

EXAMPLE 13.17

Find the equation of the locus of a point that forms a triangle of area 5 units with the points $A(2, 3)$and $B(-1, 4)$.

SOLUTION

Let $P(x_1, y_1)$ be point on the locus, $(x_2, y_2) = (2, 3)$ and $(x_3, y_3) = (-1, 4)$

Given area of $\Delta PAB = 5$ sq. units.

$$\therefore \ \frac{1}{2}\begin{vmatrix} x_1 - 2 & 2-(-1) \\ y_1 - 3 & 3-4 \end{vmatrix} = 5$$

$$\begin{vmatrix} x_1 - 2 & 3 \\ y_1 - 3 & -1 \end{vmatrix} = 10$$

$-(x_1 - 2) - 3(y_1 - 3) = \pm 10$

$x_1 + 3y_1 + 11 = \pm 10$

Required equation is $x + 3y = 10 - 11$ or $x + 3y = -10 - 11$

$\therefore \ x + 3y + 1 = 0$ or $x + 3y + 21 = 0.$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. There are no congruent figures which are similar. (True/False)

2. Two triangles with measures 3 cm, 4 cm, 5 cm and 0.60 cm, 0.8 cm, 1 cm are similar. (Agree/disagree)

3. A triangle is formed by joining the mid-points of the sides of a given triangle. This process is continued indefinitely. All such triangles formed are similar to one another. (True/False)

4. All the similar figures are congruent if their areas are equal. (Yes/No)

5. The ratio of corresponding sides of two similar triangles is 2 : 3, then the ratio of the perimeters of two triangles is 4 : 9. (True/False)

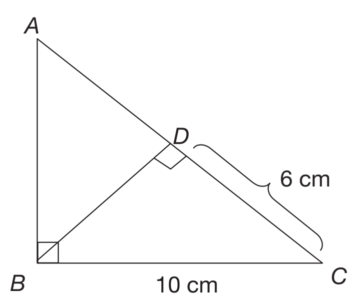
6. Which of the following is/are true?
   (a) All triangles are similar.
   (b) All circles are similar.
   (c) All squares are similar.

7. Two equal chords of a circle are always parallel. (True/False)

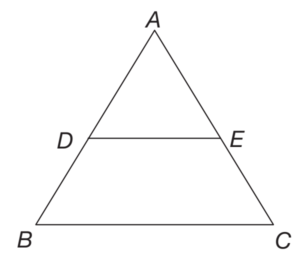
8. In a circle, chord PQ subtends an angle of 80° at the centre and chord RS subtends 100°, then which chord is longer?

9. Number of circles that pass through three collinear points is ______.

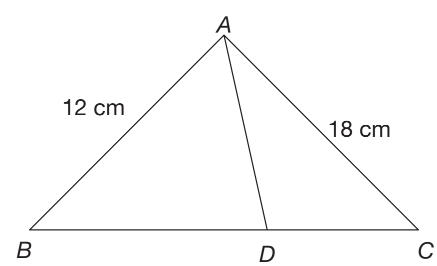
10. In the following figure, $\angle ABC = 90°$, $BC = 10$ cm, $CD = 6$ cm, then $AD =$ ______.



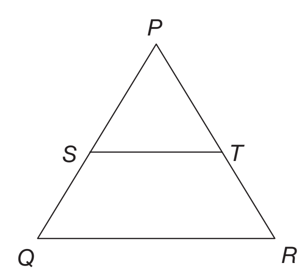
11. In the following figure, $AD = DB$ and $DE \parallel BC$, then $AE = EC$. (True/False)



12. In the following figure (not to scale), if $BC = 20$ cm and $\angle BAD = \angle CAD$, then $BD =$ ______.



13. In the following figure, $ST \parallel QR$, then $\Delta PST \sim \Delta PQR$. (Yes/No)

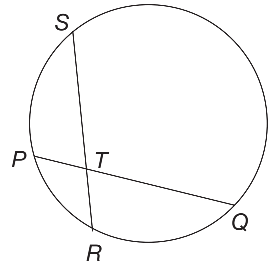


14. In the above figure, if $ST \parallel QR$ and $PS : PQ = 2 : 5$ and $TR = 15$ cm, then $PT =$ ______.

15. The number of tangents drawn from an external point to a circle is ______.

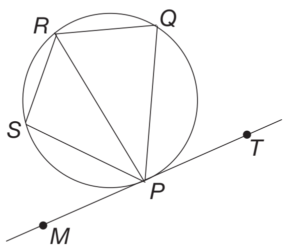
16. If two circles intersect at two distinct points, then the number of common tangents is ______.

17. In the following figure, $PT = 4$ cm, $TQ = 6$ cm and $RT = 3$ cm, then $TS =$ ______.



18. If two circles touch each other externally, then the number of transverse common tangents is ______.

19. In the following figure, to find $\angle PQR$, ______ must be given. ($\angle PRQ/\angle QPT/\angle RPT$)



20. From a point P which is at a distance of 13 cm from the centre of the circle of radius 5 cm, a tangent is drawn to the circle. The length of the tangent is ______.

21. A line drawn from the centre of a circle to a chord always bisects it. (True/False)

22. Distance between two circles with radii R and r is d units. If $d^2 = R^2 + r^2$ then the two circles ______ (intersect at one point/do not intersect/intersect at two distinct points).

23. Two circles with radii r_1 and r_2 touch externally. The length of their direct common tangent is ______.

24. In a circle, angle made by an arc in the major segment is 60°. Then the angle made by it in the minor segment is ______.

25. If a trapezium is cyclic, then its ______ are equal. (parallel sides/oblique sides)

26. In a circle, two chords PQ and RS bisect each other. Then $PRQS$ is ______.

27. Line joining the centers of two intersecting circles always bisect their common chord. (True/False)

28. The locus of the tip of a seconds hand of a watch is a ______.

29. A parallelogram has no line of symmetry. (True/False)

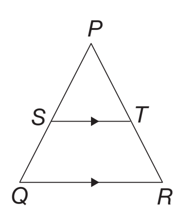
30. If the length of an enlarged rectangle is 12 cm and the scale factor is $\frac{3}{2}$, then the length of the original rectangle is ______.

Short Answer Type Questions

31. If the altitude of an equilateral triangle is 6 units, then the radius of its incircle is ________.

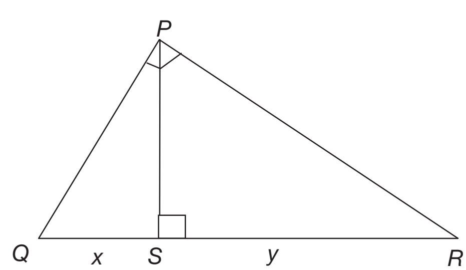
32. In a rhombus $PQRS$, PR and QS are the diagonals of the rhombus. If $PQ = 10$ cm, then find the value of $PR^2 + QS^2$.

33. In a triangle PQR, ST is parallel to QR. Show that $RT(PQ + PS) = SQ(PR + PT)$.



34. In a right angled triangle PQR, angle $Q = 90°$ and QD is the altitude. Find DR, if $PD = 17$ cm and $QD = 21$ cm.

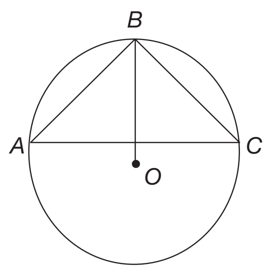
35. In the following figure, $QS = x$, $SR = y$, $\angle QPR = 90°$ and $\angle PSR = 90°$, then find $(PQ)^2 - (PR)^2$ interms of x and y.



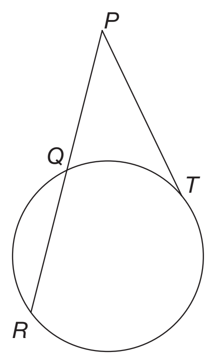
36. In the following figure, AB and BC are equidistant from the centre 'O' of the circle. Show that

(a) ABC is an isosceles triangle.

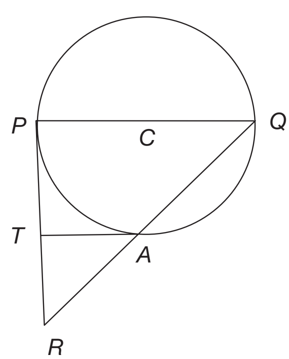
(b) OB bisects angle ABC.



37. In the following figure, PR is a secant and PT is a tangent to the circle. If $PT = 6$ cm and $QR = 5$ cm, then $PQ =$ ______ cm.



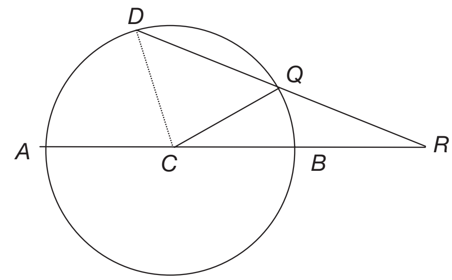
38. In the following figure, PQ is the diameter of the circle with radius 5 cm. If AT is the tangent and equal to the radius of the circle, then find the length of AR.



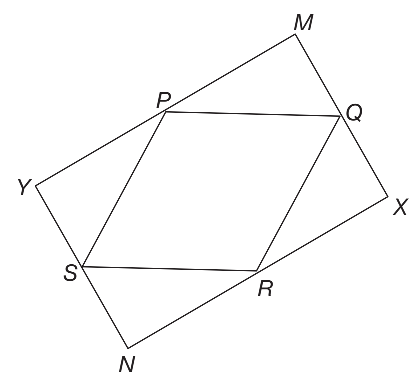
39. The distance between two buildings is 24 m. The height of the buildings are 12 m and 22 m. Find the distance between the tops.

40. Find the distance between the centres of the two circles, if their radii are 11 cm and 7 cm, and the length of the transverse common tangent is $\sqrt{301}$ cm.

41. In the figure given along side, AB is the diameter of the circle, C is the centre of the circle and CQR is an isosceles triangle, such that $CQ = QR$. Prove that $\angle DCA = 3 \cdot \angle QCR$.



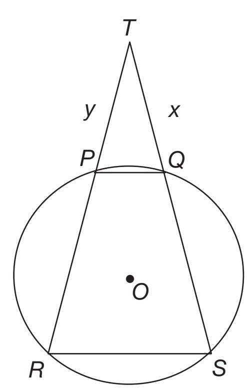
42. In the following figure, $PQRS$ is a rhombus formed by joining the mid-points of a quadrilateral $YMXN$, show that $3PQ^2 = SN^2 + NR^2 + QX^2 + XR^2 + PY^2 + YS^2$.



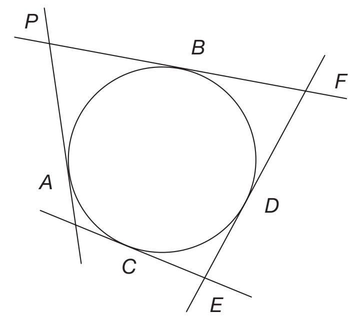
43. Find the locus of a point which is at a distance of 5 units from $(-1, -2)$.

44. In the following figure, PR and SQ are chords, of the circle with centre O, intersecting at T and $TQ = x$; $TP = y$.

Show that $(TS + TR) : (TS - TR) = (x + y) : (y - x)$.



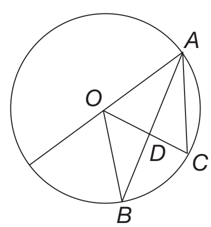
45. In the given figure, PA, PB, EC and ED are tangents to the circle. If $PA = 13$ cm, $CE = 4.5$ cm and $FE = 9$ cm, then find PF.



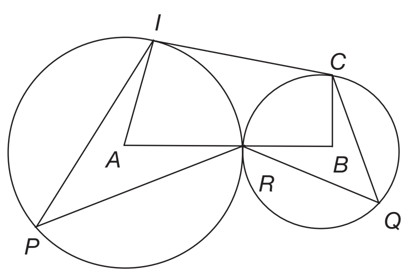
Essay Type Questions

46. EF and EH are the two chords of a circle with centre O intersecting at E. The diameter ED bisects the angle HEF. Show that the triangle FEH is an isosceles triangle.

47. In the following figure, O is the centre of the circle, AC are parallel lines. If $\angle ACO = 80°$, then find $\angle ADO$.

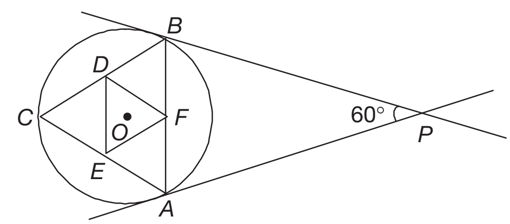


48.



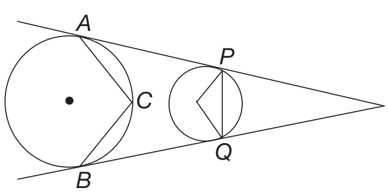
In the above figure, A and B are the centres of two circles, DC being the common tangent. If $\angle DPR = 35°$, then $\angle RQC =$ ______.

49.



In the diagram given above, D, E and F are midpoints of BC, CA and AB. If the angle between the tangents drawn at A and B is 60°, find $\angle EFD$.

50.



In the diagram above, A, B, P and Q are points of contacts of direct common tangents of the two circles. If $\angle ACB$ is 120°, then find the angle between the two tangents and angle made by PQ at the centre of same circle.

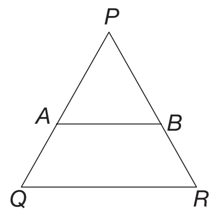
CONCEPT APPLICATION

Level 1

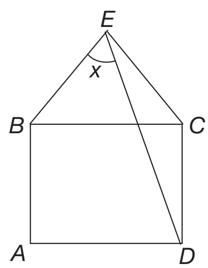
1. In the triangle PQR, AB is parallel to QR. The ratio of the areas oftwo similar triangles PAB and PQR is 1 : 2. Then $PQ : AQ =$ ______.

(a) $\sqrt{2} : 1$ (b) $1 : \sqrt{2} - 1$
(c) $1 : (\sqrt{2} + 1)$ (d) $\sqrt{2} : \sqrt{2} - 1$

PRACTICE QUESTIONS



2. In the figure given below, equilateral triangle ECB surmounts square $ABCD$. Find the angle BED represented by x.

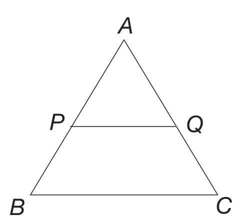


(a) 15° (b) 30°
(c) 45° (d) 60°

3. In two triangles ABC and DEF, $\angle A = \angle D$. The sum of the angles A and B is equal to the sum of the angles D and E. If $BC = 6$ cm and $EF = 8$ cm, find the ratio of the areas of the triangles, ABC and DEF.

(a) 3 : 4 (b) 4 : 3
(c) 9 : 16 (d) 16 : 9

4. In the following figure, PQ is parallel to BC and $PQ : BC = 1 : 3$. If the area of the triangle ABC is 144 cm^2, then what is the area of the triangle APQ?



(a) 48 cm^2 (b) 36 cm^2
(c) 16 cm^2 (d) 9 cm^2

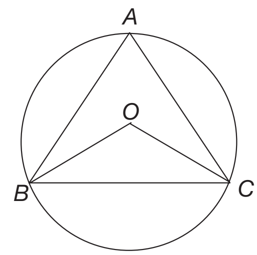
5. In triangle ABC, sides AB and AC are extended to D and E, respectively, such that $AB = BD$ and $AC = CE$. Find DE, if $BC = 6$ cm.

(a) 3 cm (b) 6 cm
(c) 9 cm (d) 12 cm

6. A man travels on a bicycle, 10 km east from the starting point A to reach point B, then he cycles 15 km south to reach point C. Find the shortest distance between A and C.

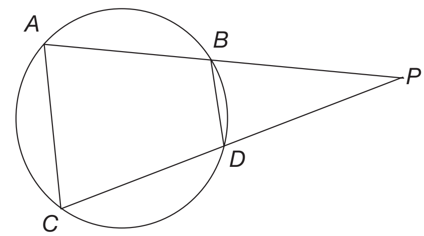
(a) 25 km (b) 5 km
(c) $25\sqrt{13}$ km (d) $5\sqrt{13}$ km

7. In the following figure, O is the centre of the circle. If $\angle BAC = 60°$, then $\angle OBC =$ _______.



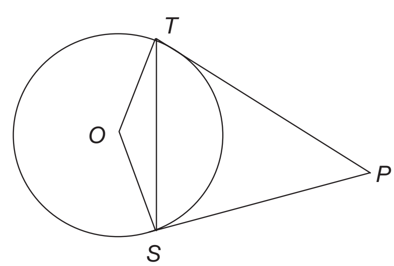
(a) 120° (b) 30°
(c) 40° (d) 60°

8. In the following figure (not to scale), $AB = CD$ and $\overline{AB}$ and $\overline{CD}$ are produced to meet at the point P. If $\angle BAC = 70°$, then find $\angle P$.



(a) 30° (b) 40°
(c) 45° (d) 50°

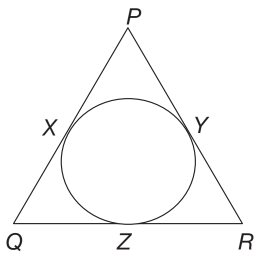
9.



PT and PS are the tangents to the circle with centre O. If $\angle TPS = 65°$, then $\angle OTS =$ _______.

(a) 32° (b) 45°
(c) $57\frac{1}{2}°$ (d) $32\frac{1}{2}°$

10. In the following figure X, Y and Z are the points at which the incircle touches the sides of the triangle. If $PX = 4$ cm, $QZ = 7$cm and $YR = 9$ cm, then the perimeter of triangle PQR is



(a) 20 cm (b) 46 cm

(c) 40 cm (d) 80 cm

11. The locus of the point P which is at a constant distance of 2 units from the origin and which lies in the first or the second quadrants is _______.

(a) $y = -\sqrt{4 - x^2}$ (b) $y = \sqrt{4 - x^2}$

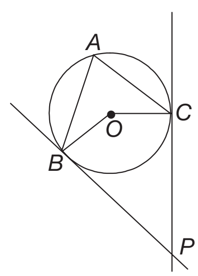
(c) $x = \sqrt{4 - y^2}$ (d) $x = -\sqrt{4 - y^2}$

12. If PAB is a triangle in which $\angle B = 90°$ and $A(1, 1)$ and $B(0, 1)$, then the locus of P is _______.

(a) $y = 0$ (b) $xy = 0$

(c) $x = y$ (d) $x = 0$

13. In the following figure, if the angle between two chords AB and AC is 65°, then the angle between two tangents which are drawn at B and C is _______.



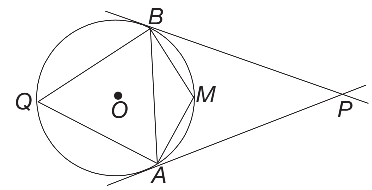
(a) 50° (b) 30°

(c) 60° (d) 40°

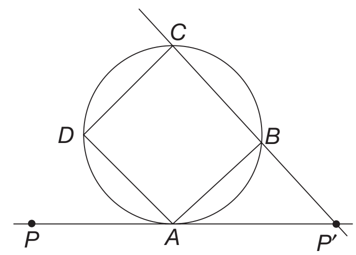
14. In the following figure, O is the centre of the circle and $\angle AMB = 120°$, Find the angle between the two tangents AP and BP.

(a) 30° (b) 45°

(c) 70° (d) 60°



15.

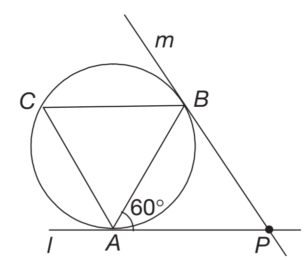


If $ABCD$ is a square inscribed in a circle and PA is a tangent, then the angle between the lines $P'A$ and $P'B$ is _______.

(a) 30° (b) 20°

(c) 40° (d) 45°

16. In the following figure, if l and m are two tangents and AB is a chord making an angle of 60° with the tangent l, then the angle between l and m is _______.



(a) 45° (b) 30°

(c) 60° (d) 90°

17. Find the length of a transverse common tangent of the two circles whose radii are 3.5 cm, 4.5 cm and the distance between their centres is 10 cm.

(a) 36 cm (b) 6 cm

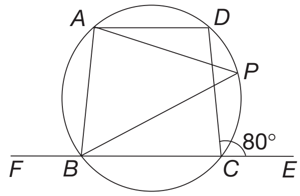
(c) 64 cm (d) 8 cm

18. If $ABCD$ is a trapezium, AC and BD are the diagonals intersecting each other at point O. Then $AC : BD =$ _______.

(a) $AB : CD$ (b) $AB + AD : DC + BC$

(c) $AO^2 : OB^2$ (d) $AO - OC : OB - OD$

19. In the following figure (not to scale), $\overline{PA}$ and $\overline{PB}$ are equal chords and $ABCD$ is a cyclic quadrilateral. If $\angle DCE = 80°$, $\angle DAP = 30°$, then find $\angle APB$.



(a) 40° (b) 80°

(c) 90° (d) 160°

20. In trapezium $KLMN$, KL and MN are parallel sides. A line is drawn, from the point A on KN, parallel to MN meeting LM at B. $KN : LM$ is equal to ______.

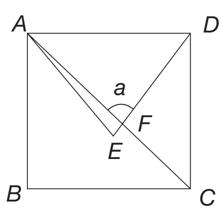
(a) $KL : NM$

(b) $(KL + KA) : (NM + BM)$

(c) $(KA - AN) : (LB - BM)$

(d) $KL^2 : MN^2$

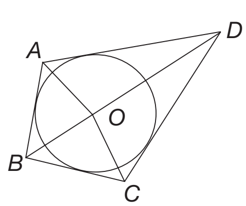
21. In the following figure, $ABCD$ is a square and AED is an equilateral triangle. Find the value of a.



(a) 30° (b) 45°

(c) 60° (d) 75°

22. A circle with centre O is inscribed in a quadrilateral $ABCD$ as shown in the figure. Which of the following statements is/are true?



(A) $\angle AOD + \angle BOC = 180°$

(B) $\angle AOB$ and $\angle COD$ are complementary.

(C) OA, OB, OC and OD are the angle bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively.

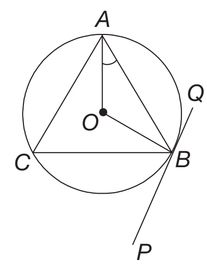
(a) Both (A) and (B)

(b) Both (B) and (C)

(c) Both (A) and (C)

(d) All the three

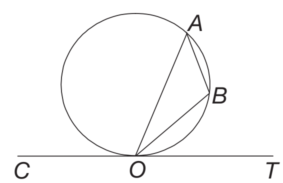
23. In the following figure, O is the centre of the circle and if $\angle OAB = 30°$, then the acute angle between AB and the tangent PQ at B is ______.



(a) 30° (b) 60°

(c) 45° (d) 90°

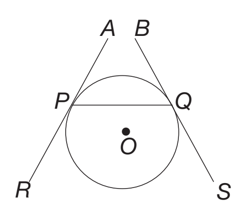
24. In the following figure, $AB = OB$ and CT is the tangent to the circle at O. If $\angle COA = 125°$, then $\angle OAB$ is ______.



(a) 55° (b) $27\frac{1}{0}°$

(c) $82\frac{1}{2}°$ (d) 45°

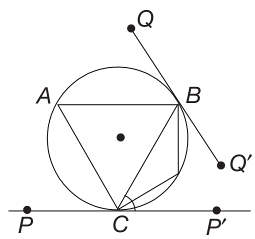
25. AR and BS are the tangents to the circle, with centre O, touching at P and Q respectively and PQ is the chord. If $\angle OQP = 25°$, then $\angle RPQ =$ ______.



(a) 100° (b) 115°

(c) 150° (d) 90°

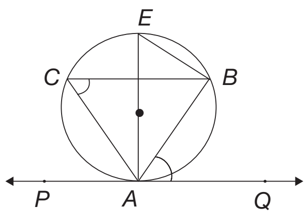
26. In the diagram, if $\angle BCP' = \angle ABQ = 60°$, then the triangle ABC is ______.



(a) scalene (b) equilateral

(c) right angled (d) acute angled

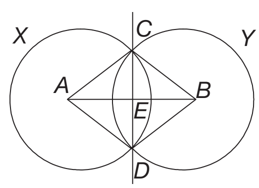
27. In the following figure, AQ is a tangent to the circle at A. If $\angle ACB = 60°$, then $\angle BAQ =$ _______.



(a) 30° (b) 60°

(c) 120° (d) 45°

28. In the following figure, two circles X and Y with centres A and B respectively intersect at C and D. The radii AC and AD of circle X are tangents to the circle Y. Radii BC and BD of circle Y are tangents to the circle X. Find $\angle AEC$.



(a) 45° (b) 60°

(c) 90° (d) Cannot be determined

29. The tangent AB touches a circle, with centre O, at the point P. If the radius of the circle is 5 cm, OB = 10 cm and $OB = AB$, then find AP.

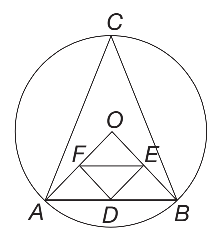
(a) $5\sqrt{5}$ cm

(b) $10\sqrt{5}$ cm

(c) $(10 - 5\sqrt{3})$ cm

(d) $\left(10 - \dfrac{5}{\sqrt{3}}\right)$ cm

30. In the following figure, O is the centre of the circle and D, E and F are mid-points of AB, BO and OA respectively. If $\angle DEF = 30°$, then find $\angle ACB$.

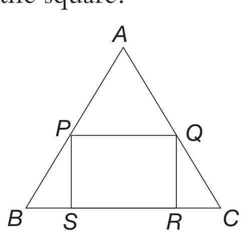


(a) 30° (b) 60°

(c) 90° (d) 120°

Level 2

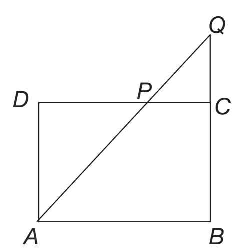
31. In the figure given below, ABC is an equilateral triangle and $PQRS$ is a square of side 6 cm. By how many cm^2 is the area of the triangle more than that of the square?



(a) $\dfrac{21}{\sqrt{3}}$ (b) 21

(c) $21\sqrt{3}$ (d) 63

32. In the following figure (not to scale) $ABCD$ is a rectangle, $BC = 24$ cm, $DP = 10$ cm and $CD = 15$ cm. Then, AQ and CQ respectively are _______.



(a) 39 cm, 13 cm (b) 13 cm, 12 cm

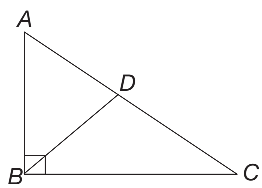
(c) 25 cm, 13 cm (d) 39 cm, 12 cm

33. At a particular time, the shadow cast by a tower is 6 m long. If the distance from top of the tower to the end of the shadow is 10 m long, determine the height of the tower.

(a) 4 m (b) 8 m

(c) 16 m (d) 12 m

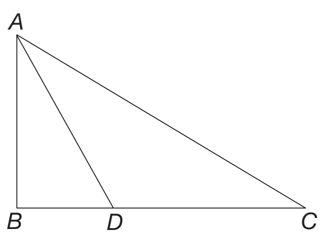
34. In the following figure, $\angle ABC = 90°$, $AD = 15$ and $DC = 20$. If BD is the bisector of $\angle ABC$, what is the perimeter of the triangle ABC?



(a) 74 (b) 84

(c) 91 (d) 105

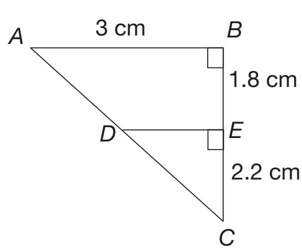
35. In the triangle ABC, $\angle ABC$ or $\angle B = 90°$. $AB : BD : DC = 3 : 1 : 3$. If $AC = 20$ cm, then what is the length of AD (in cm)?



(a) $5\sqrt{2}$ (b) $6\sqrt{3}$

(c) $4\sqrt{5}$ (d) $4\sqrt{10}$

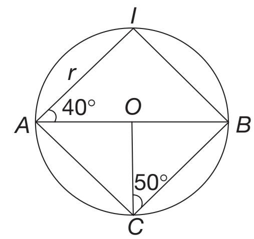
36. In the following figure, find AD.



(a) 1.8 cm (b) 2.25 cm

(c) 2.2 cm (d) 1.85 cm

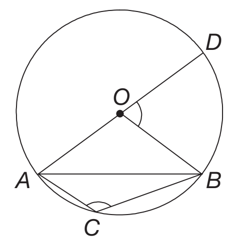
37. In the given figure, AB is a diameter, O is the centre of the circle and $\angle OCB = 50°$, then find $\angle DBC$.



(a) 80° (b) 100°

(c) 120° (d) 140°

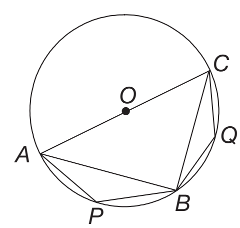
38. In the following figure, O is the centre of the circle and AD is the diameter. If $\angle ACB = 135°$, then find $\angle DOB$.



(a) 135° (b) 60°

(c) 90° (d) 45°

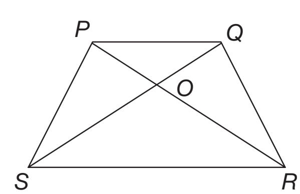
39. In the following figure, O is the centre of the circle, AC is the diameter and if $\angle APB = 120°$, then find $\angle BQC$.



(a) 30° (b) 150°

(c) 90° (d) 120°

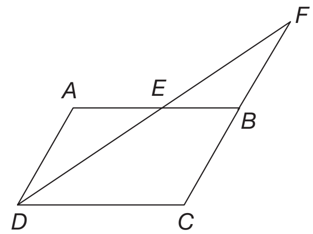
40. In the trapezium $PQRS$, PQ is parallel to RS and the ratio of the areas of the triangle POQ to triangle ROS is 225 : 900. Then SR = ______.



(a) 30 PQ (b) 25 PQ

(c) 2 PQ (d) PQ

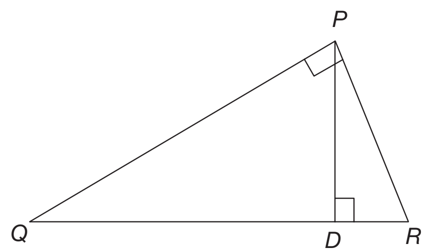
41. In the following figure, $ABCD$ is a parallelogram, CB is extended to F and the line joining D and F intersect AB at E. Then ______.



(a) $\dfrac{AD}{AE} = \dfrac{BF}{BE}$ (b) $\dfrac{AD}{AE} = \dfrac{CF}{CD}$

(c) $\dfrac{BF}{BE} = \dfrac{CF}{CD}$ (d) All of these

42.

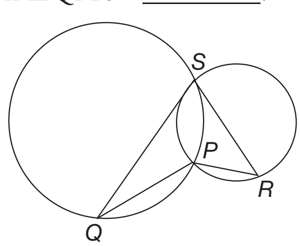


PQR is a right angled triangle, where $\angle P = 90°$. $\overline{PD}$ is perpendicular to $\overline{QR}$. $PQ : \sqrt{PR} =$ ______.

(a) $QD : \sqrt{DR}$ (b) $\sqrt{QD} : \sqrt{DR}$

(c) $QD^2 : \sqrt{RD^2}$ (d) None of these

43. Two circles intersect at two points P and S. QR is a tangent to the two circles at Q and R. If $\angle QSR = 72°$, then $\angle QPR =$ ________.



(a) 84° (b) 96°

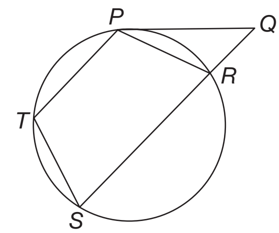
(c) 102° (d) 108°

44. An equilateral triangle CDE is constructed on a side CD of square $ABCD$. The measure of $\angle AEB$ can be ________.

(a) 150° (b) 45°

(c) 30° (d) 20°

45. In the figure above (not to scale), PQ is a tangent segment to the circle at P. If P, R, S and T are concyclic points, $\angle QPR = 40°$ and $PR = RQ$, then find $\angle PTS$.



(a) 80° (b) 100°

(c) 60° (d) 120°

46. Construct the incircle of a given triangle ABC. The following sentences are the steps involved in the above construction. Arrange them in sequencial order from first to left.

(A) Draw perpendicular $\overline{IM}$ from I onto $\overline{BC}$.

(B) Taking I as centre and IM as the radius, draw a circle.

(C) Draw a $DABC$ with the given measurements.

(D) Draw bisectors of two of the angles, say $\angle B$ and $\angle C$ to intersect at I.

(a) DCAB (b) CDAB

(c) CADB (d) DACB

47. Construct a regular pentagon in a circle of radius 4 cm. The following sentences are the steps involved in the following constructions. Arrange them in sequential order from first to last.

(A) With D as centre and AB as radius draw an arc which cuts the circle at the point C.

(B) With A as centre and AB as radius draw an arc which cuts the circle at the point E.

(C) Construct a circle with radius 4 cm.

(D) Join AB.

(E) Draw two radii $\overline{OA}$ and $\overline{OB}$ such that $\angle AOB = 72°$.

(F) Join AE, ED, DC and CB.

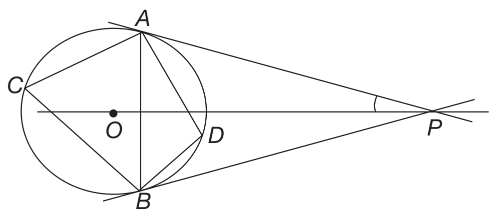
(G) With E as centre and AB as radius draw an arc which cuts the circle at the point D.

(a) CEDGBAF (b) CEBDGAF

(c) CEDBGAF (d) CEBGADF

Level 3

48. In the diagram, O is the centre of the circle and $\angle OPA = 30°$. Find $\angle ACB$ and $\angle ADB$ respectively.



(a) 120°, 60° (b) 60°, 120°
(c) 75°, 105° (d) 35°, 145°

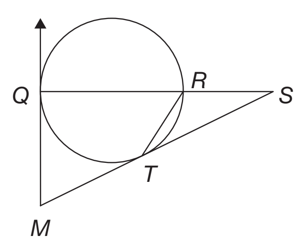
49. Side of a square $PQRS$ is 4 cm long. $\overline{PR}$ is produced to the point M such that $PR = 2RM$. Find SM.

(a) $\sqrt{10}$ cm (b) $\sqrt{5}$ cm
(c) $2\sqrt{5}$ (d) $2\sqrt{10}$ cm

50. ABC is an equilateral triangle of side 6 cm. If a circle of radius 1 cm is moving inside and along the sides of the triangle, then locus of the centre of the circle is an equilateral triangle of side __________.

(a) 5 cm (b) 4 cm
(c) $(6-2\sqrt{3})$ cm (d) $(3+\sqrt{3})$ cm

51. In the following figure (not to scale), STM and MQ are tangents to the circle at T and Q respectively. SRQ is a straight line. $SR = TR$ and $\angle TSR = 25°$. Find $\angle QMT$.



(a) 55° (b) 60°
(c) 75° (d) 80°

52. PQ is the direct common tangent of two circles (S, 9 cm) and (R, 4 cm) which touch each other externally. Find the area of the quadrilateral $PQRS$. (in cm^2)

(a) 72 (b) 65
(c) 78 (d) 69

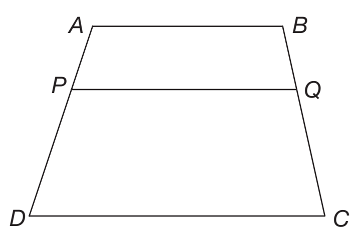
53. Diagonal AC of a rectangle $ABCD$ is produced to the point E such that $AC : CE = 2 : 1$. $AB = 8$ cm and $BC = 6$ cm. Find the length of DE.

(a) $2\sqrt{19}$ cm
(b) 15 cm
(c) $3\sqrt{17}$ cm
(d) 13 cm

54. In ΔPQR, $PQ = 6$ cm, $PR = 9$ cm and M is a point on QR such that it divides QR in the ratio 1 : 2. $PM \perp QR$. Find QR.

(a) $\sqrt{18}$ cm
(b) $3\sqrt{12}$ cm
(c) $3\sqrt{15}$ cm
(d) $\sqrt{20}$ cm

55. In the following figure (not to scale), $ABCD$ is an isosceles trapezium. $\overline{AB} \parallel \overline{CD}$, $AB = 9$ cm and $CD = 12$ cm. $AP : PD = BQ : QC = 1 : 2$. Find PQ.

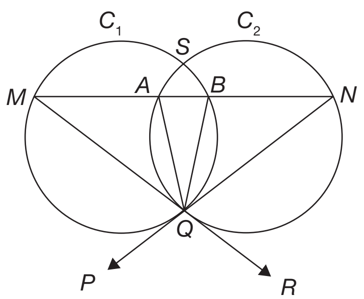


(a) 11 cm (b) 10.5 cm
(c) 10 cm (d) 9.5 cm

56. P, Q and R are on AB, BC and AC of the equilateral triangle ABC respectively. $AP : PB = CQ : QB = 1 : 2$. G is the centroid of the triangle PQB and R is the mid-point of AC. Find $BG : GR$.

(a) 1 : 2 (b) 2 : 3
(c) 3 : 4 (d) 4 : 5

57. In the given figure (not to scale), two circles C_1 and C_2 intersect at S and Q. PQN and RQM are tangents drawn to C_1 and C_2 respectively at Q. MAB and ABN are the chords of the circles C_1 and C_2. If $\angle NQR = 85°$, then find $\angle AQB$.



(a) 5° (b) 10°

(c) 15° (d) Cannot be determined

58. Two sides of a triangle are 5 cm and 12 cm long. The measure of third side is an integer in cm. If the triangle is an obtuse triangle, then how many such triangles are possible?

(a) 9 (b) 8

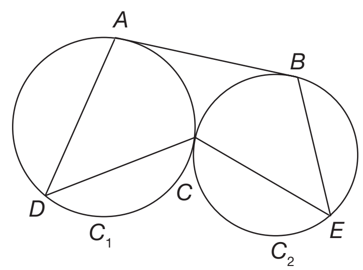
(c) 7 (d) 6

59. In a ΔPQR, M lies on PR and between P and R such that $QR = QM = PM$. If $\angle MQR = 40°$, then find $\angle P$.

(a) 35° (b) 25°

(c) 45° (d) 55°

60. In the following figure, (not to scale), AB is the common tangent to the circles C_1 and C_2. C_1 and C_2 are touching externally at C. AD and DC are two chords of the circle C_1 and BE and CE are two chords of the circle C_2. Find the measure of $\angle ADC + \angle BEC$.



(a) 60° (b) 90°

(c) 120° (d) 100°

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. False
2. Agree
3. True
4. Yes
5. False
6. 2 and 3.
7. False
8. RS
9. zero
10. $\frac{32}{3}$ cm
11. True
12. 8 cm
13. Yes
14. 10 cm
15. 2
16. 2
17. 8 cm
18. 1
19. $\angle RPT$
20. 12 cm
21. False
22. intersect at two distinct points
23. $2\sqrt{r_1 r_2}$
24. 120°
25. oblique sides
26. rectangle
27. True
28. circle
29. True
30. 8 cm

Short Answer Type Questions

31. 2 units
32. 400 cm^2
34. 26 cm (approximately)
35. $x^2 - y^2$
37. 4 cm
38. $5\sqrt{2}$
39. 26 m
40. 25 cm
43. $x^2 + y^2 + 2x + 4y = 0.$
45. 17.5 cm

Essay Type Questions

47. 120°
48. 55°
49. 60°
50. 60°, 120°

CONCEPT APPLICATION

Level 1

1. (d)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (b)	**9.** (d)	**10.** (c)
11. (b)	**12.** (d)	**13.** (a)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (c)
21. (d)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (c)	**30.** (b)

Level 2

31. (c)	**32.** (d)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (c)
41. (d)	**42.** (b)	**43.** (d)	**44.** (c)	**45.** (b)	**46.** (b)	**49.** (c)			

Level 3

48. (b)	**49.** (d)	**50.** (c)	**51.** (d)	**52.** (c)	**53.** (c)	**54.** (c)	**55.** (c)	**56.** (d)	**57.** (b)
58. (d)	**59.** (a)	**60.** (b)							

CONCEPT APPLICATION

Level 1

1. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
2. $BC = CD = EC$ and proceed using $\angle ECD = 90° + 60° = 150°$.
3. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
4. The ratio of areas of two similar trianglesis equal to the ratio of the squares of their corresponding sides.
5. Apply basic proportionality theorem.
6. Apply Pythagoras theorem.
7. $\angle BOC = 2\angle BAC$ and proceed.
8. Exterior angle of a cyclic quadrilateral is equal to its interior opposite angle, i.e., $\angle BAC = \angle DCA$ and proceed.
9. Radius of a circle is perpendicular to the tangent at the point of contact and tangents drawn to a circle from an external point are equal.
10. Tangents drawn to a circle from an external point are equal.
11. Recall the definition of locus. In the first or the second quadrant, y is positive.
12. Apply Pythagoras theorem.
13. Radius is perpendicular to the tangent at the point of contact.

    In the quadrilateral $BOCP$, $\angle BOC + \angle BPC = 180°$.
14. Recall the properties of cyclic quadrilateral and also of tangents.

    $\angle BOA + \angle BPA = 180°$ and $\angle BOA = 2\angle BQA$.
15. Apply 'alternate segment theorem'.
16. Tangents drawn to a circle from an external point are equal.
17. Length of the transverse common tangent $= \sqrt{d^2 - (R + r)^2}$.
18. Diagonals of a trapezium divide each other proportionally.
19. Recall the properties of cyclic quadrilateral.

    $\angle PAB = \angle PBA$ and $\angle DAB = \angle DCE$.
20. Apply BPT and use componondo–dividendo after drawing the complete figure.
21. Diagonal of a square bisects the angle at the vertices.

    $\angle FDC = 30°$ and $\angle FCD = 45°$.
22. Evaluate the solution from the options.
23. Apply 'alternate segment theorem'.
24. Recall 'alternate segment theorem'.
25. Radius is perpendicular to the tangent at the point of contact.
26. Apply 'alternate segment theorem'.
27. Apply 'alternate segment theorem'.
28. Recall the properties of a kite.
29. Apply Pythagoras theorem.
30. (i) $ADEF$ is a parallelogram.

    (ii) $\angle FAD = 30°$ and

    $\angle OAD = \angle OBA$

    (angles opposite to equal sides).

Level 2

31. (i) In triangle PBS, $\angle B = 60°$.

    $\therefore \angle P = 30°$ and $\angle S = 90°$.

    (ii) The sides of the triangle PBS, i.e., BS, SP and PB are in the ratio $1 : \sqrt{3} : 2$.

    (iii) Given $PS = 6$ cm.
32. (i) Use Pythagoras theorem to find AP.

    (ii) Triangle QAB and triangle QPC are similar.

    $\therefore \dfrac{QP}{PA} = \dfrac{QC}{CB}$.

33. (i) Apply 'Pythagoras theorem'.

(ii) The tower is perpendicular to the surface of the ground.

34. (i) Apply 'angle bisector theorem'.

(ii) $AB : BC = AD : DC$.

(iii) Apply Pythagoras theorem, and find AB and BC.

35. (i) Apply 'Pythagoras theorem' for triangle ABC.

(ii) Let $AB = 3x$, $BD = x$ and $CD = 3x$.

(iii) First find AB and BC using $AB^2 + BC^2 = AC^2$.

(iv) Then find AD by using $AD^2 = AB^2 + BD^2$.

36. (i) Apply 'basic proportionality theorem'.

(ii) Apply Pythagoras theorem to find AC.

(iii) Then apply basic proportionality theorem, i.e., $CE/EB = CD/DA$.

37. (i) In triangle OBC, angles opposite to equal sides are equal.

(ii) $\angle ADB = \angle ACB = 90°$ (Since angle in a semi-circle is 90°).

(iii) $\angle OCB = \angle OBC$ and $\angle OAC = \angle OCA$ (Angles opposite to equal sides).

38. (i) $ACBD$ is a cyclic quadrilateral.

(ii) $\angle ACB$ and $\angle ADB$ are supplementary angles.

(iii) $\angle AOB = 2\angle ADB$.

39. (i) $APBC$ is a cyclic quadrilateral.

(ii) $\angle ABC$ is an angle in a semi circle.

(iii) $ABQC$ is a cyclic quadrilateral.

40. (i) POQ and ROS are similar triangles.

(ii) SR and PQ are proportional to the square roots of the areas of similar triangles SOR and POQ.

41. (i) Triangles FEB and FDC are similar.

(ii) Triangles AED and EFB are similar.

42. (i) Triangle PDR, QDP and QPR are similar.

(ii) Corresponding sides of similar triangles are proportional.

(iii) $\Delta PDQ \sim \Delta RDP$.

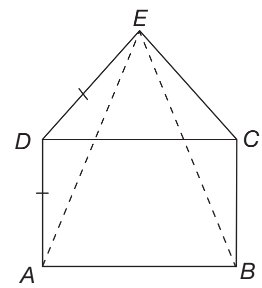
43. (i) Apply 'alternate segment theorem'.

(ii) Join QR and join PS.

$\angle PQR = \angle PSQ$ and

$\angle PRQ = \angle PSR$ (By alternate segment theorem)

44.



$\angle ADE = \angle ADC + \angle CDE = 90° + 60°$ ($\because$ Angles in a square and equilateral triangle) $= 150°$

In ΔADE, $AD = DE$

$$\therefore \angle DAE = \angle DEA = \frac{1}{2}(180° - 150°)$$

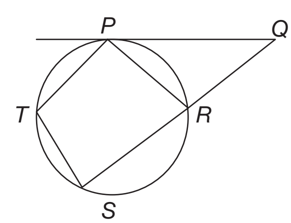
$$= 15°.$$

Similarly, $\angle CEB = 15°$

$$\therefore \angle AEB = 60° - (\angle DEA + \angle CEB)$$

$$= 60° - (15° + 15°) = 30°.$$

45.



Given $PR = RQ$ and $\angle Q = 40°$

$\Rightarrow \angle RPQ = \angle Q$

$\therefore \angle RPQ = 40°$

In ΔPQR,

$\angle PRQ = 180° - (40° + 40°) = 100°$.

We have, exterior angle of cyclic quadrilateralis equal to its interior opposite angle.

$\therefore \angle PTS = \angle PRQ$

($\because$ $PRST$ is a cyclic quadrilateral)

$\therefore \angle PTS = 100°$.

46. CDAB is the required sequential order.

47. CEDBGAF is the required sequential order.

Level 3

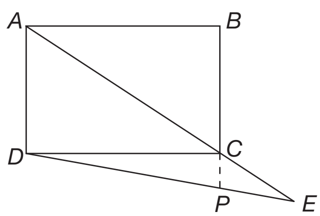
48. (i) Tangents drawn from an external point to the same circle are equal, i.e., $PA = PB$.

(ii) $\angle APO = \angle OPB$ and $\angle AOB + \angle APB = 180°$.

(iii) $\angle ACB = \frac{1}{2}\angle AOB$.

(iv) $ACBD$ is a cyclic quadrilateral.

53.



Given, $AB = 8$ cm and $BC = 6$ cm

$\therefore\ AC = \sqrt{8^2 + 6^2} = 10$ cm

And also given $AC : CE = 2 : 1$

Produce BC to meet DE at the point P

As CP is parallel to AD,

$\Delta ECP \sim \Delta EAD$ (1)

$$\therefore\ \frac{CP}{AD} = \frac{CE}{AE} = \frac{CP}{6} = \frac{1}{3}$$

$\Rightarrow$ CP = 2 cm.

ΔCPD is a right triangle.

$\therefore\ DP = \sqrt{CD^2 + CP^2} = \sqrt{68} = 2\sqrt{17}$ cm

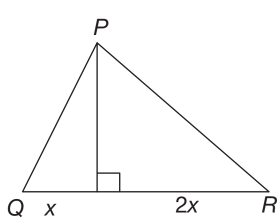
But $DP : PE = 2 : 1$ (from (1))

$PE = \sqrt{17}$ cm

$\therefore\ DE = DP + PE$

$= 2\sqrt{17} + \sqrt{17} = 3\sqrt{17}$ cm.

54.



Given $PQ = 6$ cm, PR 9 cm and $QM : MR = x : 2x$

Let $QM = x$ and $MR = 2x$.

As PM is perpendicular QR, ΔPMQ and ΔPMR are right triangles.

$\therefore\ (PM)^2 = (OQ)^2 - (QM)^2$

$(PM)^2 = (PR)^2 - (MR)^2$ (2)

From Eqs. (1) and (2), we $(PQ)^2 - (QM)^2 = (PR)^2 - (MR)^2$

$(6)^2 - (x)^2 = (9)^2 - (2x)^2$

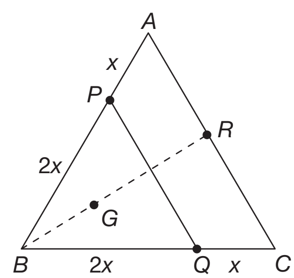
$\Rightarrow 3x^2 = 45 \Rightarrow x^2 = 15 \Rightarrow x = 15$ cm

$\therefore\ QR = 3\sqrt{15}$ cm.

55. $PQ = \frac{1}{3}(CD) + \frac{2}{3}(AB)$

$= \frac{1}{3}(12) + \frac{2}{3}(9) = 10$ cm.

56.



Let $AB = BC = AC = 3x$

$\therefore\ BP = BQ = PQ = 2x$

($\because\ AP : BP = CQ : BQ = 1 : 2$)

As ΔBPQ and ΔBAC are equilateral triangle, the centroid of ΔBPQ lies on BR (where BR is median drawn on to AC)

We know that centroid divides the median in the ratio 2 : 1.

$$\therefore BG = \frac{2}{3}\frac{\left[\sqrt{3}(2x)\right]}{2} = \frac{2\sqrt{3}x}{3}$$

But $BR = \frac{\sqrt{3}(3x)}{2} = \frac{3\sqrt{3}x}{2}$

Now $GR = BR - BG$

$$= \frac{3\sqrt{3}x}{2} - \frac{2\sqrt{3}x}{3} = \frac{5\sqrt{3}x}{6}$$

Now $BG : GR = \frac{2\sqrt{3}x}{3} : \frac{5\sqrt{3}x}{6} = 4 : 5$.

57. Given $\angle NQR = 85°$

But $\angle NQR = \angle MQP$ ($\because$ Vertically opposite angles)

$\therefore \angle MQP = 85°$

By alternate segment theorem $\angle NQR = \angle QAB$

$\therefore \angle QAB = 85°$ and $\angle MQP = \angle QBA$

$\therefore \angle QBA = 85°$

In ΔAQB, $\angle AQB = 180° - (85° + 85°) = 10°$.

58. Two sides of *a* triangle 5 cm and 12 cm.

Let $a = 5$ cm and $b = 12$ cm

Let the third side be x cm

$\therefore 12 - 5 < x < 12 + 5$

$\Rightarrow 7 < x < 17$

$\therefore$ Possible integer values for x are 8, 9, 10, 11, 12, 13, 14, 15 and 16.

Case 1: If b is the longest side then $b^2 > a^2 + x^2$

$\Rightarrow 12^2 > 5^2 + x^2$

$\Rightarrow 144 - 25 > x^2$

$\Rightarrow x^2 < 119$

$\therefore x$ can be 8, 9 or 10.

Case 2: If x is the longest side, then $x^2 > a^2 + b^2$

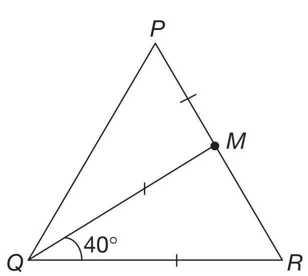
$\Rightarrow x^2 > 5^2 + 12^2 \Rightarrow x^2 \Rightarrow 169$

$\therefore x$ can be 14, 15 or 16.

$\therefore$ Number of possible triangles

$= 6$ ($\because$ The measurement third side is an integer in cm).

59.



Given $QR = QM = PM$ and $\angle MQR = 40°$

In ΔQMR, $QM = QR$

$\therefore \angle QRM = \angle QMR$

Now $\angle QRM = \angle QMR$

$= \frac{1}{2}(180° - 40°) = 70°$

In ΔMPQ, $PM = MQ$

$\angle PQM = \angle MPQ$ (1)

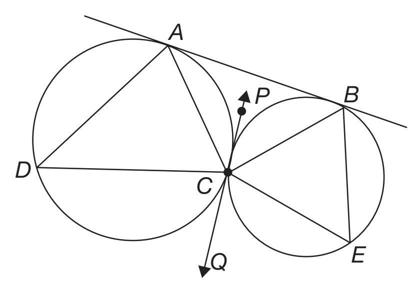
But $\angle PQM + \angle MPQ = \angle QMR$

$= 70°$ (2)

From Eqs. (1) and (2)

$\angle MPQ = \frac{1}{2}(70°) = 35°$.

60.



Join AC and BC

Draw a common tangent to the circles through the point C.

Let

$\angle ADC = x$ and $\angle BEC = y$ (1)

By alternate segment theorem, $\angle ADC = \angle BAC$,

$\angle ADC = \angle ACP$

$\therefore \angle BAC = \angle ACP = X$

and also $\angle BEC = \angle CBA$, $\angle BEC = \angle BCP$

$\therefore \angle CBA = \angle BCP = y$

In ΔABC, $\angle BAC + \angle CBA + \angle ACB = 180°$

$x + y + (x + y) = 180° \Rightarrow x + y = 90°$ (2)

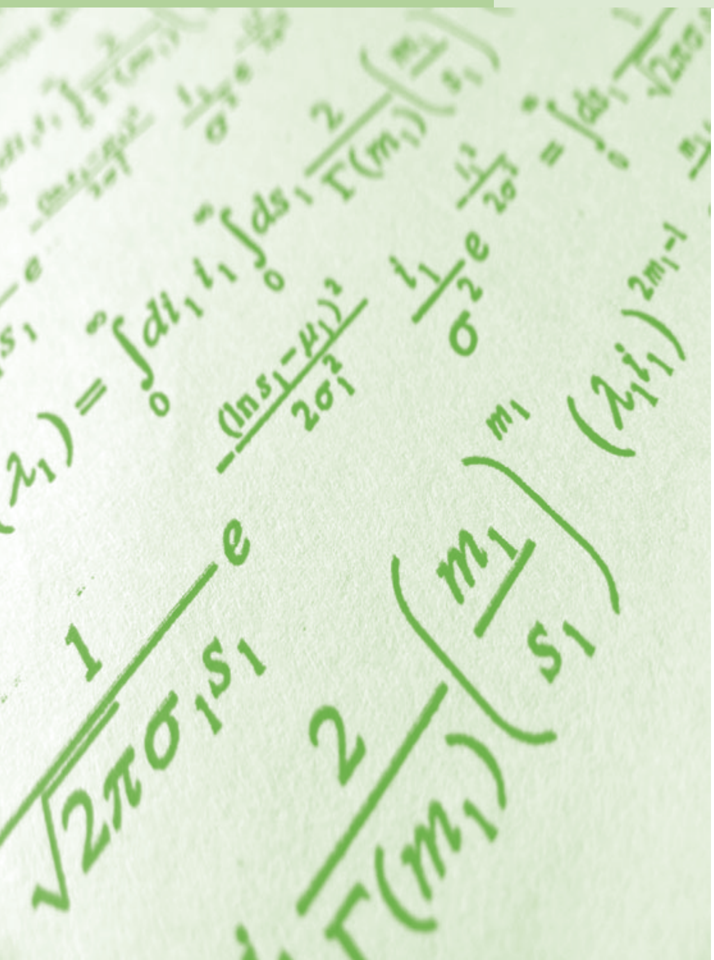
From (1) and (2), we have

$\angle ADC + \angle BEC = 90°$.

Chapter

14

Mensuration



REMEMBER

Before beginning this chapter, you should be able to:

- Explain polygons and their properties
- Use circles, circumference of a circle, sector of a circle in solving mathematical problems

KEY IDEAS

After completing this chapter, you would be able to:

- Calculate perimeters and areas of plane figures
- Study types of prisms, and pyramids and calculate lateral, total surface areas and volumes of prisms and pyramids
- Compute curved, total surface area and volume of cylinder and cone
- Obtain surface area and volume of sphere, hemisphere and hollow sphere

INTRODUCTION

Mensuration is a branch of mathematics that deals with the computation of geometric magnitudes, such as the length of a line, the area of a surface and the volume of a solid. In this chapter we will deal with the areas and volumes of three dimensional figures like prisms, pyramids, cones, spheres, hemispheres, etc. However, some problems on plane figures like circles, sectors, segments, etc., as an exercise of revision.

Circle and Semi-Circle

1. Area of circle $= \pi r^2$ sq. units.
2. Area of the semicircle $= \dfrac{\pi r^2}{2}$ sq. units.
3. Circumference of the circle $= 2\pi r$ units $= \pi d$ units
4. Circumference of the semicircle $= (\pi + 2)r$ units $= \dfrac{36r}{7}$ units

(Where r is radius and d is diameter)

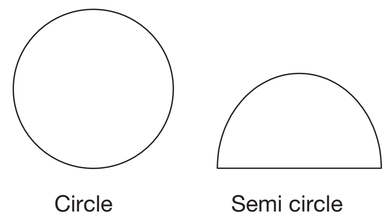


Figure 14.1

Circular Ring

Area of the ring $= \pi(R^2 - r^2) = \pi(R + r)(R - r)$
(Where R and r are outer radius and inner radius of a ring and $(R - r)$ is the width of the ring)

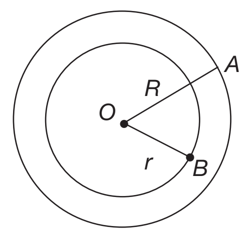


Figure 14.2

Sectors and Segements

1. Length of the arc ACB
$$l = \left(\frac{\theta}{360°}\right) 2\pi r \text{ units.}$$
2. Area of the sector $AOBC$
$$A = \left(\frac{\theta}{360°}\right) \pi r^2 \text{ sq. units.}$$
3. Perimeter of the sector $= (l + 2r)$ units.
4. Area of the segment $ACB = (A -$ Area of the $\Delta AOB)$ sq. units.
5. Perimeter of the segment $ACB =$ (Length of arc $ACB + AB$) units.

(Where r is the radius of the circle and θ is sector angle)

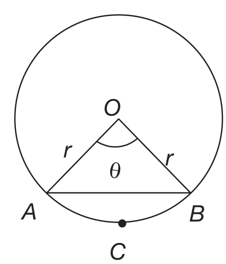
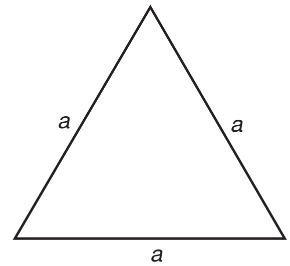


Figure 14.3

Rotations Made by a Wheel

1. Distance covered by a wheel in one revolution = Circumference of the wheel.
2. Number of rotations made by a wheel in unit time $= \dfrac{\text{Distance covered by it in unit time}}{\text{Circumference of the wheel}}$.
3. Angle made by minute hand in one minute $= \dfrac{360°}{60} = 6°$.
4. Angle made by hour hand in one minute $= \dfrac{30°}{60} = \left(\dfrac{1}{2}\right)^{\circ}$.

Equilateral Triangle



1. Circumference of an equilateral triangle = $3a$ units.
2. Area of the equilateral triangle $= \dfrac{\sqrt{3}a^2}{4}$ sq. units.
3. Height of the equilateral triangle $= \dfrac{\sqrt{3}a}{2}$ units.
4. Radius of in-circle of equilateral triangle $= \dfrac{1}{3}\left(\dfrac{\sqrt{3}a}{2}\right) = \dfrac{a}{2\sqrt{3}}$ units.
5. Circum-radius of equilateral triangle $= \dfrac{2}{3}\left(\dfrac{\sqrt{3}a}{2}\right) = \dfrac{a}{\sqrt{3}}$ units.

(Where a is side of the triangle)

EXAMPLE 14.1

The hour hand of a clock is 6 cm long. Find the area swept by it between 11:20 am and 11:55 am. (in cm^2)

(a) 2.75 **(b)** 5.5 **(c)** 11 **(d)** None of these

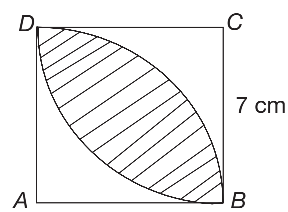
SOLUTION

Angle made by the hours hand of the clock in 35 minutes is 17.5° (angle of sector).

EXAMPLE 14.2

In the figure given below, $ABCD$ is a square of side 7 cm. BD is an arc of a circle of radius AB. What is the area of the shaded region?

(a) 14 cm^2 **(b)** 21 cm^2 **(c)** 28 cm^2 **(d)** 35 cm^2



SOLUTION

Area of shaded region = 2(Area of sector $\overline{BAD}$ − Area of ΔABD).

PRISMS

Prism is a solid in which two congruent and parallel polygons form the top and the bottom faces. The lateral faces are parallelograms.

The line joining the centres of the two parallel polygons is called the axis of the prism and the length of the axis is referred to as the height of prism.

If two parallel and congruent polygons are regular and if the axis is perpendicular to the base, then the prism is called a right prism. The lateral surfaces of a right prism are rectangles.

Consider two congruent and parallel triangular planes ABC and PQR. If we join the corresponding vertices of both the planes, i.e., A to P, B to Q and C to R, then the resultant solid formed is a triangular prism. A right prism, the base of which is a rectangle is called a cuboid and the one, the base of which is a pentagon is called a pentagonal prism. If all the faces of the solid are congruent, it is a cube. In case of a cube or a cuboid, any face may be the base of the prism. A prism whose base and top faces are squares but the lateral faces are rectangular is called a square prism.

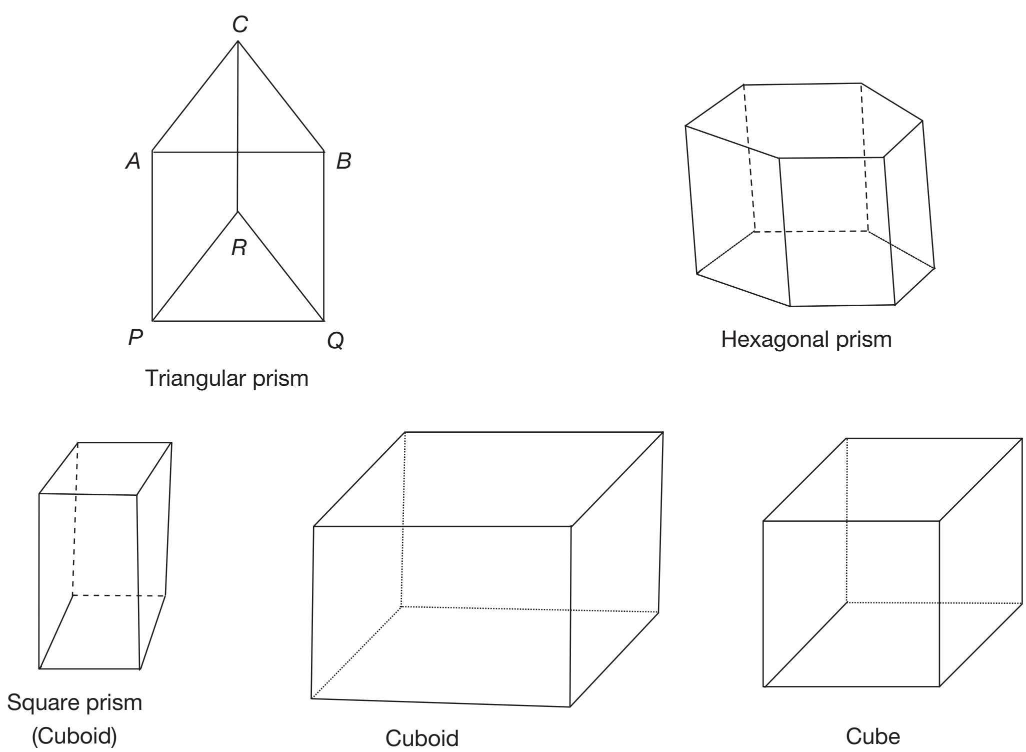


Figure 14.4

Notes The following points hold good for all prisms.

1. The number of lateral faces = The number of sides of the base.
2. The number of edges of a prism = Number of sides of the base × 3.
3. The sum of the lengths of the edges = 2(Perimeter of base) + Number of sides × Height.

Lateral Surface Area (LSA) of a Prism

LSA = Perimeter of base × Height = ph

Total Surface Area (TSA) of a Prism

TSA = LSA + 2(Area of base)

Volume of a Prism

Volume = Area of base × Height = Ah

Note The volume of water flowing in a canal = The cross section area of the canal × The speed of water.

EXAMPLE 14.3

The base of a right prism is a right angled triangle. The measure of the base of the right angled triangle is 3 m and its height 4 m. If the height of the prism is 7 m, then find

(a) the number of edges of the prism.

(b) the volume of the prism.

(c) the total surface area of the prism.

SOLUTION

(a) The number of the edges = The number of sides of the base × 3 = 3 × 3 = 9.

(b) The volume of the prism = Area of the base × Height of the prism $=\frac{1}{2}(3\times 4)\times 7 = 42\text{ m}^3$.

(c) TSA = LSA + 2(Area of base)

$= ph + 2(\text{Area of base})$

where, p = Perimeter of the base = sum of lengths of the sides of the given triangle.

As, hypotenuse of the triangle $\sqrt{3^2+4^2} = \sqrt{25} = 5$ m

∴ Perimeter of the base = 3 + 4 + 5 = 12 m

$\Rightarrow \text{LSA} = ph = 12 \times 7 = 84\text{ m}^2$.

$\text{TSA} = \text{LSA} + 2(\text{Area of base}) = 84 + 2\left(\frac{1}{2}\times 3\times 4\right) = 84 + 12 = 96\text{ m}^2$.

CUBES AND CUBOIDS

Cuboid

In a right prism, if the base is a rectangle, then it is called a cuboid. A match box, a brick, a room, etc., are in the shape of a cuboid.

The three dimensions of the cuboid, its length (l), breadth (b) and height (h) are generally denoted by $l \times b \times h$.

1. The lateral surface area of a cuboid $= ph = 2(l + b)h$ sq. units, where p is the perimeter of the base.
2. The total surface area of a cuboid = LSA + 2(Base area) $= 2(l + b)h + 2lb = 2(lb + bh + lh)$ sq. units.
3. The volume of a cuboid $= Ah = (lb)h = lbh$ cubic units, where A is the area of the base.
4. Diagonal of cuboid $=\sqrt{l^2+b^2+h^2}$ units

Note If a box made of wood of thickness t has inner dimensions of l, b and h, then

The outer length $= l + 2t$,

The outer breadth $= b + 2t$ and

The outer height $= h + 2t$.

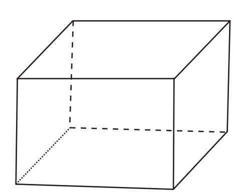


Figure 14.5 Cuboid

Cube

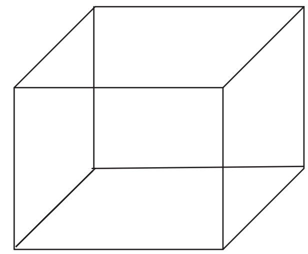


Figure 14.6 cube

In a cuboid, if all the dimensions, i.e., its length, breadth and height are equal, then the solid is called a cube.

All the edges of a cube are equal in length and each edge is called the side of the cube.

Thus, the size of a cube is completely determined by its side.

If the side of cube is 'a' units, then

1. The lateral surface area of a cube = $4a^2$ sq. units.
2. The total surface area of a cube = LSA + 2(Area of base) = $4a^2 + 2a^2 = 6a^2$ sq. units.
3. The volume of a cube = a^3 cubic units.
4. The diagonal of a cube $= \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a$ units.

Note If the inner edge of a cube made of wood of thickness 't' is 'a' units, then the outer edge of the cube is given by $(a + 2t)$ units.

EXAMPLE 14.4

The dimensions of a room are 12 m × 7 m × 5 m. Find

(a) the diagonal of the room.

(b) the cost of flooring at the rate of ₹2 per m^2.

(c) the cost of whitewashing the room excluding the floor at the rate of ₹3 per m^2.

SOLUTION

(a) The diagonal of the room $= \sqrt{l^2 + b^2 + h^2} = \sqrt{12^2 + 7^2 + 5^2} = \sqrt{144 + 49 + 25} = \sqrt{218}$ m.

(b) To find the cost of the flooring, we should know the area of the base.

Base area = $lb = 12 \times 7 = 84$ m^2

$\therefore$ The cost of flooring = 84×2 = ₹168.

(c) The total area that is to be whitewashed

= LSA + Area of roof = $2(l + b)h + lb$

$= 2(12 + 7)5 + 12 \times 7 = 2(19)(5) + 84$

$= 190 + 84$

$= 274$ m^2

$\therefore$ The cost of whitewashing = 274×3 = ₹822.

EXAMPLE 14.5

A box is in the form of a cube. Its edge is 5 m long. Find

(a) the total length of the edges.

(b) the cost of painting the outside of the box, on all the surfaces, at the rate of ₹5 per m^2.

(c) the volume of liquid which the box can hold.

SOLUTION

(a) Length of edges = Number of edges of base × 3 × Length of each edge = $4 \times 3 \times 5 = 60$ m.

(b) To find the cost of painting the box, we need to find the total surface area.

TSA = $6a^2 = 6 \times 5^2 = 6 \times 25 = 150$ m^2

∴ Cost of painting = 150×5 = ₹750.

(c) Volume = $a^3 = 5^3 = 125$ m^3.

EXAMPLE 14.6

The sum of the length, breadth and the height of a cuboid is $5\sqrt{3}$ cm and length of its diagonal is $3\sqrt{5}$ cm. Find the total surface area of the cuboid.

(a) 30 cm^2 **(b)** 20 cm^2 **(c)** 15 cm^2 **(d)** 18 cm^2

HINTS

(i) Use suitable algebraic identity to find the LSA of the cuboid.

(ii) $l + b + h = 5\sqrt{3}$ and $l^2 + b^2 + h^2 = 3\sqrt{5}$.

(iii) Square the first equation and evaluate $2(lb + bh + hl)$.

RIGHT CIRCULAR CYLINDER

A cylinder has two congruent and parallel circular planes which are connected by a curved surface. Each of the circular planes is called the base of the cylinder. A road roller, water pipe, power cables, round pillars are some of the objects which are in the shape of a cylinder.

In the Fig. 14.7, a right circular cylinder is shown. Let A be the centre of the top face and A' be the centre of the base. The line joining the centres (i.e., AA') is called the axis of cylinder. The length AA' is called the height of the cylinder. If the axis is perpendicular to the base, then it is a right circular cylinder. The radius r of the base of the cylinder and the height h, completely describe the cylinder.

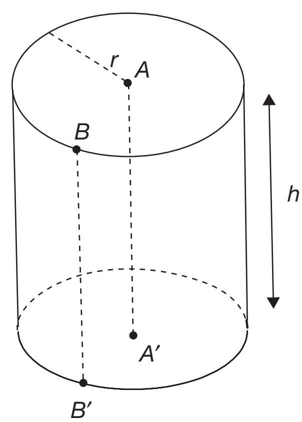


Figure 14.7

Lateral (curved) surface area = Perimeter of base × Height = $2\pi rh$ sq. units.

The total surface area = LSA + 2(Base area) = $2\pi rh + 2(\pi r^2)$ = $2\pi r(h + r)$ sq. units.

Volume = Area of base × Height = $\pi r^2 h$ cubic units.

Hollow Cylinder

The part of a cylinder from which a smaller cylinder of the same axis is cut out is a hollow cylinder. Let R and r be the external and internal radii of the hollow cylinder and h be the height.

Volume of the material used = $\pi R^2h - \pi r^2h = \pi h(R + r)(R - r)$ cubic units.

Curved surface area = $2\pi Rh + 2\pi rh = 2\pi h(R + r)$ sq. units.

Total surface area = Curved surface area + Area of the two ends.

$$= 2\pi h(R + r) + 2\pi(R^2 - r^2) = 2\pi(R + r)(R - r + h) \text{ sq. units.}$$

Note If a plastic pipe of length l is such that its outer radius is R and the inner radius is r, then the volume of the plastic content of the pipe = $l\pi(R^2 - r^2)$ cubic units.

EXAMPLE 14.7

A closed cylindrical container, the radius of which is 7 cm and height 10 cm is to be made out of a metal sheet. Find

(a) the area of metal sheet required.

(b) the volume of the cylinder made.

(c) the cost of painting the lateral surface of the cylinder at the rate of ₹4 per cm^2.

SOLUTION

(a) The area of the metal sheet required = The total surface area of the cylinder = $2\pi r(r + h)$

$$= 2 \times \frac{22}{7} \times 7(7 + 10) = 44(17) = 748 \text{ cm}^2.$$

(b) Volume = $\pi r^2h = \frac{22}{7} \times 7 \times 7 \times 10 = 22 \times 70 = 1540 \text{ cm}^3$.

(c) To find the cost of painting the lateral surface, we need to find the curved (lateral) surface area.

$$\therefore \text{LSA} = 2\pi rh = 2 \times \frac{22}{7} \times 7 \times 10 = 440 \text{ cm}^2.$$

Cost of painting = 440 × 4 = ₹1760.

EXAMPLE 14.8

A cylindrical tank with radius 60 cm is being filled by a circular pipe with internal diameter of 4 cm at the rate of 11 m/s. Find the height of the water column in 18 minutes.

(a) 66 m **(b)** 12.2 m **(c)** 13.2 m **(d)** 6.1 m

HINT

Volume of water in the tank = Area of the cross sections of the pipe × Rate × Time.

PYRAMID

A pyramid is a solid obtained by joining the vertices of a polygon to a point in the space by straight lines. The base of the solid obtained is the polygon and lateral faces are triangles. The fixed point in space where all the triangles (i.e., lateral faces) meet is called its vertex.

In the Fig. 14.8, the base $ABCD$ is a quadrilateral. All the vertices of the base are joined to a fixed point O in space, by straight lines. The resultant solid obtained is called a pyramid.

The straight line joining the vertex and the centre of the base is called the axis of the pyramid. If the axis is not perpendicular to the base, it is an oblique pyramid.

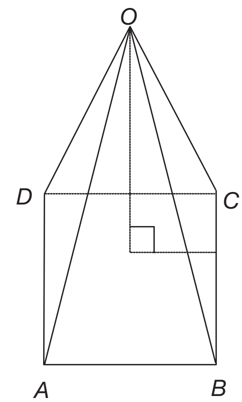


Figure 14.8

Right Pyramid

If the base of a pyramid is a regular polygon and if the line joining the vertex to the centre of the base is perpendicular to the base, then the pyramid is called a right pyramid.

The length of the line segment joining the vertex to the centre of the base of a right pyramid is called the height of the pyramid and is represented by 'h'.

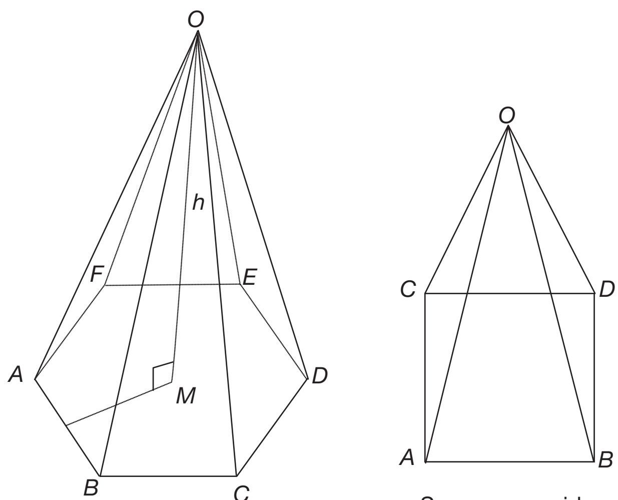


Figure 14.9

The perpendicular distance between the vertex and the mid-point of any of the sides of the base (i.e., regular polygon) of a right pyramid is called its slant height and is represented by 'l'.

For a right pyramid with perimeter of base = p, Height = h and Slant height = l,

1. Lateral surface area = $\frac{1}{2}$ (Perimeter of base) × (Slant height) = $\frac{1}{2}pl$.
2. Total surface area = Lateral surface area + Area of base.
3. Volume of a pyramid = $\frac{1}{3}$ × Area of base × Height.

EXAMPLE 14.9

An hexagonal pyramid is 20 m high. Side of the base is 5 m. Find the volume and the slant height of the pyramid.

SOLUTION

Given $h = 20$ m,

Side of base = $a = 5$ m

$\therefore$ Area of base $= \dfrac{\sqrt{3}}{4} \times a^2 \times 6 = \dfrac{6\sqrt{3}}{4} \times 5^2 = \dfrac{3\sqrt{3}}{2} \times 25\ \text{m}^2$.

Volume $= \dfrac{1}{3} Ah$, where A = Area of the base and h = Height

$$= \frac{1}{3} \times \frac{3\sqrt{3}}{2}(25) \times 20 = \sqrt{3} \times 250 = 250\sqrt{3}\ \text{m}^3.$$

To find slant height, refer to the figure shown. In the figure,

O is the vertex of the pyramid and G is the centre of the hexagonal base. H is the mid-point of AB.

OG is the axis of the pyramid.

OH is the slant height of the pyramid.

ΔOGH is a right angled triangle.

$\therefore OH^2 = GH^2 + OG^2$

GH = Altitude of $\Delta AGB = \dfrac{\sqrt{3}}{2} a = \dfrac{\sqrt{3}}{2} \times 5 = \dfrac{5\sqrt{3}}{2}$ m

$$\therefore OH^2 = \left(\frac{5\sqrt{3}}{2}\right)^2 + (20)^2 = \frac{25 \times 3}{4} + 400$$

$$= \frac{75 + 1600}{4} = \frac{1675}{4}$$

$$\Rightarrow OH = \frac{\sqrt{1675}}{2}\ \text{m}$$

$\therefore$ Slant height $= \dfrac{\sqrt{1675}}{2}$ m.

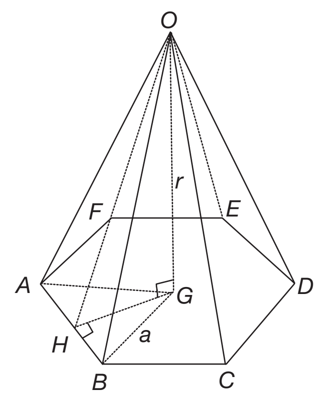


Figure 14.10

CONE

A cone is a solid pointed figure with a circular base. A cone is a kind of pyramid whose base is a circle.

A cone has one vertex, one plane surface (i.e., the base) and a curved surface (i.e., the lateral surface).

The line joining the vertex to the centre of base (i.e., AO) is called the axis of the cone. The length of the line segment AO is called the height or perpendicular height of the cone. An icecream cone and a conical tent are some of the examples of conical objects.

Right Circular Cone

In a cone, if the line joining the vertex and the centre of the base of the cone is perpendicular to the base, then it is a right circular cone. In other words, if the axis of the cone is perpendicular to the base of the cone, then it is a right circular cone. We generally deal with problems on right circular cones.

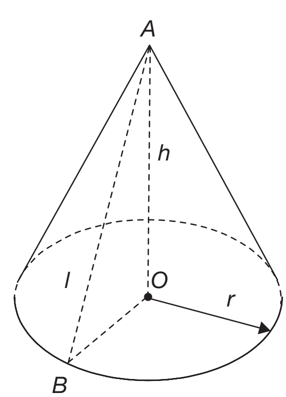


Figure 14.11 Cone

A cone is generally defined as a solid obtained by the revolution of a right angled triangle about one of its two perpendicular sides.

If we consider any point B on the periphery of the base of the cone, then the line joining B and the vertex A is called the slant height of the cone and is denoted by 'l'.

From the figure it is clear that ΔAOB is right angled.

$\therefore l = \sqrt{r^2 + h^2}$.

Hollow Cone

In earlier classes we have studied about sector. We may recall that sector is an area bounded by an arc of a circle and its two radii. (as shown in Fig. 14.12)

Now consider the sector AOB. If we roll the sector up and bring (join) together the radii OA and OB such that they coincide, then the figure formed is called a hollow cone. The radius of the circle becomes the slant height of the cone and the length of the arc of the sector becomes the perimeter of the base of the cone.

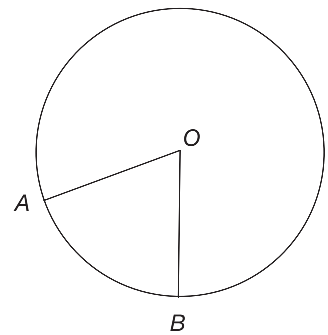


Figure 14.12

For a cone of radius r, height h and slant height l,

1. Curved surface area of a cone = πrl sq. units.
2. Total surface area of a cone = Curved surface area + area of base = $\pi rl + \pi r^2 = \pi r(r + l)$ sq. units.
3. Volume of a cone = $\frac{1}{3}\pi r^2 h$ cubic units.

EXAMPLE 14.10

Find the volume of the greatest right circular cone, which can be cut from a cube of a side 4 cm. (in cm^3)

(a) $\frac{12\pi}{5}$ (b) $\frac{20\pi}{3}$ (c) $\frac{18\pi}{5}$ (d) $\frac{16\pi}{3}$

SOLUTION

Let the diameter of cone be the edge of the square

$\therefore l = 4$ cm

$h = 4$ cm

$r = 2$ cm

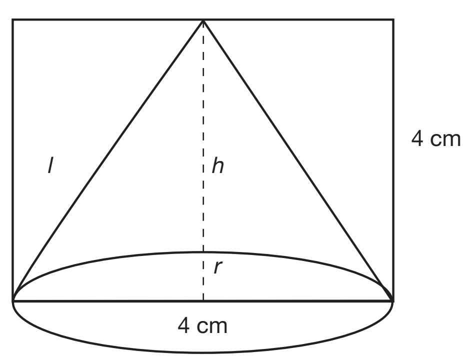
Volume of a cone $= \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi (2)^2 \cdot 4$

$\Rightarrow V = \frac{1}{3}\pi \cdot 16$

$\therefore V = \frac{16}{3}\pi$ cm^3

$\therefore$ Correct option is (d).



Cone Frustum (or a Conical Bucket)

If a right circular cone is cut by a plane perpendicular to its axis (i.e., a plane parallel to the base), then the solid portion containing the base of the cone is called the frustum of the cone.

From the Fig. 14.13, we observe that a frustum is in the shape of a bucket.

Let,

Radius of upper base be R,

Radius of lower base = r,

Height of frustum = h,

Slant height of frustum = l

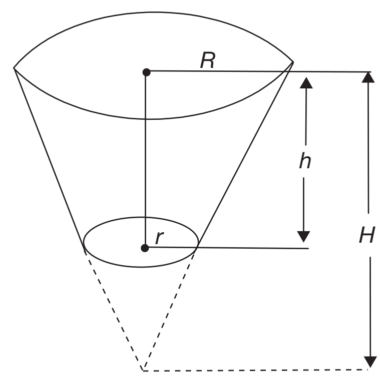


Figure 14.13

1. Curved surface area of a frustum = $\pi l(R + r)$ sq. units.
2. Total surface area of a frustum = Curved surface area + Area of upper base + Area of lower base = $\pi l(R + r) + \pi r^2 + \pi R^2$ sq. units.
3. Volume of a frustum = $\frac{1}{3}\pi h(R^2 + Rr + r^2)$ cubic units.
4. Slant height (l) of a frustum $= \sqrt{(R-r)^2 + h^2}$ units.

EXAMPLE 14.11

A joker's cap is in the form of a cone of radius 7 cm and height 24 cm. Find the area of the cardboard required to make the cap.

SOLUTION

Area of the cardboard required = Curved surface area of the cap (or cone) = πrl

Now, $l = \sqrt{r^2 + h^2} = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$ cm

$\Rightarrow$ Curved surface area = $\frac{22}{7} \times 7 \times 25 = 550$ cm^2

$\therefore$ Area of the cardboard required = 550 cm^2.

EXAMPLE 14.12

The diameter of an icecream cone is 7 cm and its height is 12 cm. Find the volume of icecream that the cone can contain.

SOLUTION

Volume of icecream $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 12 = 22 \times 7 = 154$ cm^3.

EXAMPLE 14.13

The diameters of top and bottom portions of a milk can are 56 cm and 14 cm respectively. The height of the can is 72 cm. Find the

(a) area of metal sheet required to make the can (without lid).

(b) volume of milk which the container can hold.

SOLUTION

The milk can is in the shape of a frustum with $R = 28$ cm, $r = 7$ cm and $h = 72$ cm.

(a) Area of metal sheet required = Curved surface area + Area of bottom base

$$= \pi l(R + r) + \pi r^2$$

Now, $l = \sqrt{(R - r^2) + h^2} = \sqrt{(28-7)^2 + 72^2} = \sqrt{21^2 + 72^2} = \sqrt{9(7^2 + 24^2)}$

$= 3\sqrt{49 + 576} = 3 \times \sqrt{625} = 3 \times 25 = 75$ cm

$\therefore$ Area of metal sheet $= \dfrac{22}{7} \times 75(28 + 7) + \dfrac{22}{7} \times 7^2 = 22 \times 75 \times 5 + 22 \times 7$

$= 22(375 + 7)$

$= 22(382) = 8404 \text{ cm}^2.$

(b) Amount of milk which the container can hold $= \dfrac{1}{3}\pi h(R^2 + Rr + r^2)$

$= \dfrac{1}{3} \times \dfrac{22}{7} \times 72(28^2 + 7 \times 28 + 7^2)$

$= \dfrac{22}{7} \times 24(7 \times 4 \times 28 + 7 \times 28 + 7 \times 7)$

$= \dfrac{22}{7} \times 24 \times 7(112 + 28 + 7)$

$= 22 \times 24 \times (147) = 77616 \text{ cm}^3.$

EXAMPLE 14.14

From a circular canvas of diameter 56 m, a sector of 270° was cut out and a conical tent was formed by joining the straight ends of this piece. Find the radius and the height of the tent.

SOLUTION

As shown in the figure, when the free ends of the torn canvas are joined to form a cone, the radius of sector becomes slant height.

$\therefore l = \dfrac{56}{2} = 28$ m.

The length of the arc of the sector becomes the circumference of the base of the cone.

Let the radius of the base of the cone $= r$.

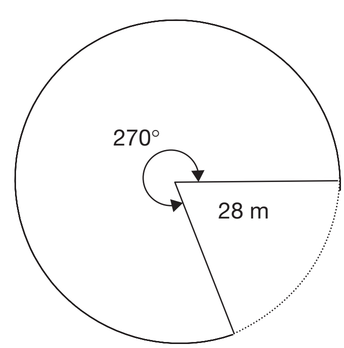


Figure 14.14

$$\Rightarrow 2\pi r = 2 \times \frac{22}{7} \times \frac{56}{2} \times \left(\frac{270}{360}\right)$$

$$\Rightarrow 2\pi r = 2 \times \frac{22}{7} \times 28 \times \frac{3}{4} \Rightarrow r = 21 \text{ m}$$

$$\therefore \text{Height, } h = \sqrt{l^2 - r^2} = \sqrt{28^2 - 21^2} = \sqrt{7^2(4^2 - 3^2)} = 7\sqrt{16-9} = 7\sqrt{7} \text{ m}$$

$$\therefore h = 7\sqrt{7} \text{ m and } r = 21 \text{ m.}$$

SPHERE

Sphere is a set of points in the space which are equidistant from a fixed point. The fixed point is called the centre of the sphere, and the distance is called the radius of the sphere. A lemon, a foot ball, the moon, globe, the Earth, small lead balls used in cycle bearings are some objects which are spherical in shape.

A line joining any two points on the surface of sphere and passing through the centre of the sphere is called its diameter.

The size of sphere can be completely determined by knowing its radius or diameter.

Solid Sphere

A solid sphere is the region in space bounded by a sphere. The centre of a sphere is also a part of solid sphere whereas the centre is not a part of hollow sphere. Marbles, lead shots, etc., are the examples of solid spheres while a tennis ball is a hollow sphere.

Hollow Sphere

From a solid sphere a smaller sphere having the same centre of the solid sphere, is cut off, then we obtain a hollow sphere. This can also be called a spherical shell.

Hemisphere

If a sphere is cut into two halves by a plane passing through the centre of sphere, then each of the halves is called a hemisphere.

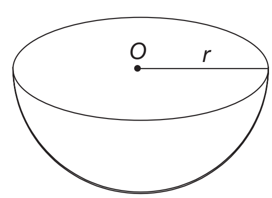


Figure 14.15 Hemisphere

Hemispherical Shell

A hemispherical shell is shown in the figure given.

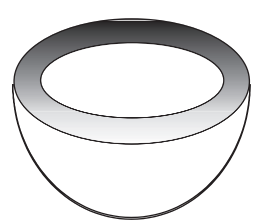


Figure 14.16 Hemispherical shell

Formulae to Memorize

Sphere

1. Surface area of a sphere = $4\pi r^2$ sq. units
2. Volume of a sphere = $\frac{4}{3}\pi r^3$ sq. units

Spherical Shell/Hollow Sphere

1. Thickness = $R - r$, where R = Outer radius, r = Inner radius
2. Volume = $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$ cubic units
3. Total surface area of a hemispherical shell = $\frac{1}{2}$(Surface area of outer hemisphere + Surface area of inner hemisphere + Area of ring)

Hemisphere

1. Curved surface area of a hemisphere = $2\pi r^2$ sq. units
2. Total surface area of a hemisphere = $3\pi r^2$ sq. units
3. Volume of a hemisphere = $\frac{2}{3}\pi r^3$ cubic units

EXAMPLE 14.15

The cost of painting a solid sphere at the rate of 50 paise per square metre is ₹1232. Find the volume of steel required to make the sphere.

SOLUTION

Cost of painting = Surface area × Rate of painting

$\therefore$ Surface area = $\dfrac{\text{Cost of painting}}{\text{Rate of painting}} = \dfrac{1232}{0.5} = 2464\ \text{m}^2$.

$$\Rightarrow 4\pi r^2 = 2464 \Rightarrow r^2 = \frac{2464}{4\pi} = \frac{616}{\left(\frac{22}{7}\right)} = \frac{616\times 7}{22} = 28\times 7$$

$\Rightarrow r = 7\times 2 = 14$ m

$\therefore$ Volume of steel required = $\frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times 14\times 14\times 14 = \frac{34496}{3}\ \text{m}^3$.

EXAMPLE 14.16

A hollow hemispherical bowl of thickness 1 cm has an inner radius of 6 cm. Find the volume of metal required to make the bowl.

SOLUTION

Inner radius, $r = 6$ cm

thickness, $t = 1$ cm

$\therefore$ Outer radius, $R = r + t = 6 + 1 = 7$ cm

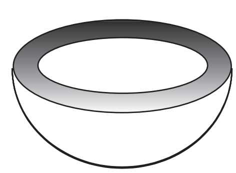


Figure 14.17

$\therefore$ Volume of steel required $= \frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times 7^3 - \frac{2}{3} \times \frac{22}{7} \times 6^3 = \frac{44}{21}(7^3 - 6^3)$$

$$= \frac{44}{21}(343 - 216) = \frac{44}{21} \times 127 = \frac{5588}{21} \text{ cm}^2.$$

EXAMPLE 14.17

A thin hollow hemispherical sailing vessel is made of metal covered by a conical canvas tent. The radius of the hemisphere is 14 m and total height of vessel (including the height of tent) is 28 m. Find area of metal sheet and the canvas required.

SOLUTION

The vessel (with the conical tent) is shown in figure.

Total height, $H = 28$ m

Radius of hemisphere $= r = 14$ m

$\therefore$ Height of conical tent $= h = H - r$

$= 28 - 14 = 14$ m

We can observe that radius of base of cone = Radius of the hemisphere = 14 m

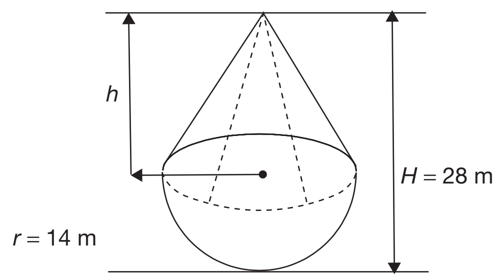


Figure 14.18

$\therefore$ Area of canvas required $= \pi rl = \frac{22}{7} \times 14 \times \sqrt{14^2 + 14} = 44 \times 14\sqrt{2} = 616\sqrt{2}$ m^2

Area of metal sheet required = Surface area of hemisphere $= 2\pi r^2 = 2 \times \frac{22}{7} \times 14 \times 14 = 1232$ m^2.

EXAMPLE 14.18

A wafer cone is completely filled with icecream forms a hemispherical scoop, just covering the cone. The radius of the top of the cone, as well as the height of the cone are 7 cm each. Find the volume of the icecream in it (in cm^3). (Take $\pi = 22/7$ and ignore the thickness of the cone)

(a) 1176 **(b)** 1980 **(c)** 1078 **(d)** 1274

SOLUTION

Required volume = Volume of the icecream forming the hemisphere + Volume of the icecream within the cone.

Radius of the hemisphere shape = Radius of the cone = 7 cm

$\therefore$ Required volume $= \frac{2}{3}\pi(7)^3 + \frac{1}{3}\pi(7)^3$

$$= \pi(7)^3 = \frac{22}{7}(7)^3 = (22)(49)$$

$$= 1078 \text{ cm}^3.$$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If the length of the side of an equilateral triangle is 12 cm, then what is its in-radius?

2. The radius of a circle is 8 cm and O is its centre. If $\angle AOB = 60°$ and AB is a chord, then what is the length of the chord AB?

3. The circum-radius of an equilateral triangle is x cm. What is the perimeter of the triangle in terms of x?

4. If the difference between the outer radius and the inner radius of a ring is 14 cm, then what is the difference between its outer circumference and inner circumference?

5. The area of a ring is 22 cm^2. What is the difference of the square of the outer radius and the square of the inner radius?

6. A cone is formed by joining together the two straight edges of a sector, so that they coincide with each other. The length of the arc of the sector becomes the _______ of the circular base and radius of sector becomes the _______ of the cone.

7. The volume of a cube with diagonal d is _______.

8. If the total surface area of a cube is $\frac{50}{3}$ m^3, then find its side.

9. Find the maximum number of soaps of size 2 cm × 3 cm × 5 cm that can be kept in a cuboidal box of dimensions 6 cm × 3 cm × 15 cm.

10. Total number of faces in a prism which has 12 edges is _______.

11. W, P, H and A are whole surface area, perimeter of base, height and area of the base of a prism respectively. The relation between W, P, H and A is _______.

12. If s is the perimeter of the base of a prism, n is the number of sides of the base, S is the total length of the edges and h is the height, then $S =$ _______.

13. If the number of lateral surfaces of a right prism is equal to n, then the number of edges of the base of the prism is _______.

14. If f, e, and v represent the number of rectangular faces, number of edges and number of vertices respectively of a cuboid, then the values of f, e, and v respectively are _______.

15. Find the number of vertices of a pyramid, whose base is a pentagon.

16. A and B are the volumes of a pyramid and a right prism respectively. If the pyramid and the prism have the same base area and the same height, then what is the relation between A and B?

17. If the ratio of the base radii of two cones having the same curved surface areas is 6 : 7, then the ratio of their slant heights is _______.

18. The heights of two cones are equal and the radii of their bases are R and r. The ratio of their volumes is _______.

19. If the heights of two cylinders are equal and their radii are in the ratio of 7 : 5, then the ratio of their volumes is _______.

20. Volumes of two cylinders of radii R, r and heights H, h respectively are equal. Then $R^2H =$ _______.

21. The volumes of two cylinders of radii R, r and heights H, h respectively are equal. If $R : r = 2 : 3$, then $H : h =$ _______.

22. A sector of a circle of radius 6 cm and central angle 30° is folded into a cone such that the radius of the sector becomes the slant height of the cone. What is the radius of the base of the cone thus formed?

23. If R and r are the external and the internal radii of a hemispherical bowl, then what is the area of the ring, which forms the edge of the bowl (in sq. units)?

24. What is the volume of a hollow cylinder with R, r and h as outer radius, inner radius and height respectively?

25. The side of a cube is equal to the radius of the sphere. Find the ratio of their volumes.

26. A sphere and the base of a cylinder have equal radii. The diameter of the sphere is equal to the height of the cylinder. The ratio of the curved surface area of the cylinder and surface area of the sphere is _______.

27. A road roller of length $3l$ m and radius $\frac{l}{3}$ m can cover a field in 100 revolutions, moving once over. The area of the field in terms of l is _____ m^3.

28. What is the volume of sand to be spread uniformly over a ground of dimensions. $10x$ m × $8x$ m up to a height of $0.1x$ m?

29. The outer radius and the inner radius of a hollow cylinder are $(2 + x)$ cm and $(2 - x)$ cm. What is its thickness?

30. The slant height, outer radius and inner radius of a cone frustum are $2a$ cm, $(a + b)$ cm and $(a - b)$ cm. What is its curved surface area?

Short Answer Type Questions

31. A circle is inscribed in an equilateral triangle. If the in-radius is 21 cm, what is the area of the triangle?

32. Three cubes each of side 3.2 cm are joined end to end. Find the total surface area of the resulting cuboid.

33. A square is drawn with the length of side equal to the diagonal of a cube. If the area of the square is 72075 cm^2, then find the side of the cube.

34. What is the area of a ground that can be levelled by a cylindrical roller of radius 3.5 m and 4 m long by making 10 rounds?

35. A square of side 28 cm is folded into a cylinder by joining its two sides. Find the base area of the cylinder thus formed.

36. Find the number of cubes of side 2 m to be dropped in a cylindrical vessel of radius 14 m in order to increase the water level by 5 m.

37. Find the capacity of a closed cuboidal cistern whose length is 3 m, breadth is 2 m and height is 6 m. Also find the area of iron sheet required to make the cistern.

38. An open metallic conical tank is 6 m deep and its circular top has diameter of 16 m. Find the cost of tin plating its inner surface at the rate of ₹0.8 per 100 cm^2. (Take $\pi = 3.14$)

39. The total surface area of a hemisphere is 3768 cm^2. Find the radius of the hemisphere. (Take π = 3.14)

40. The base radius of a conical tent is 120 cm and its slant height is 750 cm. Find the area of the canvas required to make 10 such tents (in m^2). (Take π = 3.14)

41. From a cylindrical wooden log of length 30 cm and base radius $7\sqrt{2}$ cm, biggest cuboid of square base is made. Find the volume of wood wasted.

42. A right circular cone is such that the angle at its vertex is 90° and its base radius is 49 cm, then find the curved surface area of the cone.

43. The base of a right pyramid is an equilateral triangle, each side of which is $6\sqrt{3}$ cm long and its height is 4 cm. Find the total surface area of the pyramid in cm^2.

44. If the thickness of a hemispherical bowl is 12 cm and its outer diameter is 10.24 m, then find the inner surface area of the hemisphere. (Take π = 3.14)

45. A spherical piece of metal of diameter 6 cm is drawn into a wire of 4 mm in diameter. Find the length of the wire.

Essay Type Questions

46. The cost of the canvas required to make a conical tent of base radius 8 m at the rate of ₹40 per m^2 is ₹10048. Find the height of the tent. (Take $\pi = 3.14$)

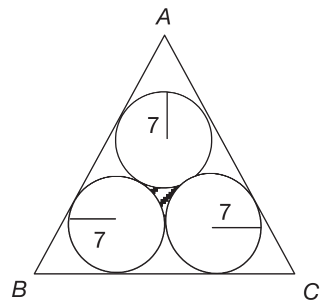
47. A hollow sphere which has internal and external diameter as 16 cm and 14 cm respectively is melted into a cone with a height of 16 cm. Find the diameter of the base of the cone.

48. A drum in the shape of a frustum of a cone with radii 24 ft and 15 ft and height 5 ft is full of water. The drum is emptied into a rectangular tank of

base 99 ft × 43 ft. Find the rise in the height of the water level in the tank.

49. A cylindrical tank of radius 7 m, has water to some level. If 110 cubes of side 7 dm are completely immersed in it, then find the rise in the water level in the tank. (in metres)

50. Find the area of the shaded portion in the figure given below, where *ABC* is an equilateral triangle and the radius of each circle is 7 cm.



CONCEPT APPLICATION

Level 1

1. The area of a sector whose perimeter is four times its radius (r units) is

(a) $\sqrt{r}$ sq. units. (b) r^4 sq. units.

(c) r^2 sq. units. (d) $\frac{r^2}{2}$ sq. units.

2. A chord of a circle of radius 28 cm makes an angle of 90° at the centre. Find the area of the major segment.

(a) 1456 cm^2 (b) 1848 cm^2

(c) 392 cm^2 (d) 2240 cm^2

3. The area of a circle inscribed in an equilateral triangle is 48π square units. What is the perimeter of the triangle?

(a) $17\sqrt{3}$ units (b) 36 units

(c) 72 units (d) $48\sqrt{3}$ units

4. Two circles touch each other externally. The distance between the centres of the circles is 14 cm and the sum of their areas is 308 cm^2. Find the difference between radii of the circles. (in cm)

(a) 1 (b) 2

(c) 0 (d) 0.5

5. If the outer and the inner radii of a circular track are 7 m and 3.5 m respectively, then the area of the track is ______.

(a) 100 m^2 (b) 178 m^2

(c) 115.5 m^2 (d) 135.5 m^2

6. The base of a right pyramid is an equilateral triangle of perimeter 8 dm and the height of the pyramid is $30\sqrt{3}$ cm. Find the volume of the pyramid.

(a) 16000 cm^3 (b) 1600 cm^3

(c) $\frac{16000}{3}$ cm^3 (d) $\frac{5}{4}$ cm^3

7. The volume of a cuboid is $20\sqrt{42}$ m^3. Its length is $5\sqrt{2}$ m, breadth and height are in the ratio $\sqrt{3} : \sqrt{7}$. Find its height.

(a) $\sqrt{7}$ m (b) $3\sqrt{7}$ m

(c) $4\sqrt{7}$ m (d) $2\sqrt{7}$ m

8. A metal cube of edge $\frac{3\sqrt{2}}{\sqrt{5}}$ m is melted and formed into three smaller cubes. If the edges of the two smaller cubes are $\frac{3}{\sqrt{10}}$ m and $\frac{\sqrt{5}}{\sqrt{2}}$ m, find the edge of the third smaller cube.

(a) $\frac{3}{\sqrt{7}}$ m (b) $\frac{6}{\sqrt{15}}$ m

(c) $\frac{5}{\sqrt{11}}$ m (d) $\frac{4}{\sqrt{10}}$ m

9. Find the volume of the space covered by rotating a rectangular sheet of dimensions 16.1 cm × 7.5 cm along its length.

(a) 2846.25 cm^3 (b) 2664 cm^3

(c) 2864.25 cm^3 (d) 2684 cm^3

PRACTICE QUESTIONS

10. The base of a right prism is an equilateral triangle of edge 12 m. If the volume of the prism is $288\sqrt{3}$ m^3, then its height is ______.

(a) 6 m (b) 8 m

(c) 10 m (d) 12 m

11. A roller levelled an area of 165000 m^2 in 125 revolutions, whose length is 28 m. Find the radius of the roller.

(a) 7.5 m (b) 8.5 m

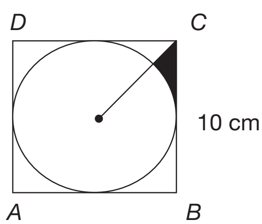
(c) 6.5 m (d) 7 m

12. A large sphere of radius 3.5 cm is carved from a cubical solid. Find the difference between their surface areas.

(a) 122 cm^2 (b) 80.5 cm^2

(c) 144.5 cm^2 (d) 140 cm^2

13. In the figure given below, *ABCD* is a square of side 10 cm and a circle is inscribed in it. Find the area of the shaded part as shown in the figure.



(a) $\left(\dfrac{100-36\pi}{41}\right)$ cm^2

(b) $\left(\dfrac{100-25\pi}{8}\right)$ cm^2

(c) $\left(\dfrac{100+25\pi}{8}\right)$ cm^2

(d) None of these

14. The outer curved surface area of a cylindrical metal pipe is 1100 m^2 and the length of the pipe is 25 m. The outer radius of the pipe is ______.

(a) 8 m (b) 9 m

(c) 7 m (d) 6 m

15. The volume of a hemisphere is 2.25π cm^3. What is the total surface area of the hemisphere?

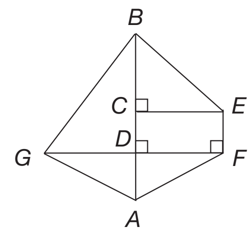
(a) 2.25π cm^2 (b) 5π cm^2

(c) 6.75π cm^2 (d) 4.5π cm^2

16. Find the area of the figure given below, in which $AB = 100$ m, $CE = 30$ m, C is mid-point of $\overline{AB}$ and D is mid-point of $\overline{AC}$ and $\overline{GF}$.

(a) 5250 m^2 (b) 3750 m^2

(c) 3375 m^2 (d) 3175 m^2



17. The area of the base of a right equilateral triangular prism is $16\sqrt{3}$ cm^2. If the height of the prism is 12 cm, then the lateral surface area and the total surface area of the prism respectively are

(a) 288 cm^2, $(288 + 32\sqrt{3})$ cm^2

(b) 388 cm^2, $(388 + 32\sqrt{3})$ cm^2

(c) 288 cm^2, $(288 + 24\sqrt{3})$ cm^2

(d) 388 cm^2, $(388 + 24\sqrt{3})$ cm^2

18. A metallic cone of diameter 32 cm and height 9 cm is melted and made into identical spheres, each of radius 2 cm. How many such spheres can be made?

(a) 72 (b) 64

(c) 52 (d) 48

19. A cylindrical vessel open at the top has a base radius of 28 cm. If the total cost of painting the outer part of the vessel is ₹357 at the rate of ₹0.2 per 100 cm^2, then find the height of the vessel. (approximately)

(a) 10 m (b) 9 m

(c) 8 m (d) 4 m

20. The radii of the ends of a bucket 16 cm high are 20 cm and 8 cm. Find the curved surface area of the bucket.

(a) 1760 cm^2 (b) 2240 cm^2

(c) 880 cm^2 (d) 3120 cm^2

21. A cylindrical vessel of radius 8 cm contains water. A solid sphere of radius 6 cm is lowered into the water until it is completely immersed. What is the rise in the water level in the vessel?

(a) 3 cm (b) 3.5 cm

(c) 4 cm (d) 4.5 cm

22. What is the difference in the areas of the regular hexagon circumscribing a circle of radius 10 cm and the regular hexagon inscribed in the circle?

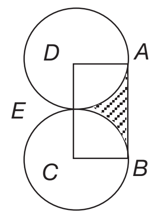
(a) 50 cm^2 (b) $50\sqrt{3}$ cm^2

(c) $100\sqrt{3}$ cm^2 (d) $100\sqrt{3}$ cm^2

23. In the shown figure, two circles of radii of 7 cm each, are shown. *ABCD* is rectangle and *AD* and *BC* are the radii. Find the area of the shaded region (in cm^2).

(a) 20 (b) 21

(c) 19 (d) 18



24. There is a closed rectangular shed of dimensions 10 m × 4 m inside a field. A cow is tied at one corner of outside of the shed with a 6 m long rope. What is the area that the cow can graze in the field?

(a) 66 m^2 (b) 88 m^2

(c) 0.8π m^2 (d) 27π m^2

25. The base of a right prism is a square of perimeter 20 cm and its height is 30 cm. What is the volume of the prism?

(a) 700 cm^3 (b) 750 cm^3

(c) 800 cm^3 (d) 850 cm^3

26. A conical cup when filled with icecream forms a hemispherical shape on its open end. Find the volume of icecream (approximately), if radius of the base of the cone is 3.5 cm, the vertical height of cone is 7 cm and width of the cone is negligible.

(a) 120 cm^3 (b) 150 cm^3

(c) 180 cm^3 (d) 210 cm^3

27. A hemispherical bowl of internal diameter 24 cm contains water. This water is to be filled in cylindrical bottles, each of radius 6 cm and height 8 cm. How many such bottles are required to empty the bowl?

(a) 3 (b) 4

(c) 5 (d) 6

28. A dome of a building is in the form of a hemisphere. The total cost of white washing it from inside, was ₹1330.56. The rate at which it was white washed is ₹3 per square metre. Find the volume of the dome approximately.

(a) 1150.53 m^3 (b) 1050 m^3

(c) 1241.9 m^3 (d) 1500 m^3

Level 2

29. A circular garden of radius 15 m is surrounded by a circular path of width 7 m. If the path is to be covered with tiles at a rate of ₹10 per m^2, then find the total cost of the work. (in ₹)

(a) 8410 (b) 7140

(c) 8140 (d) 7410

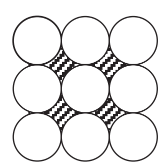
30. Find the area of the shaded region, given that the radius of each circle is equal to 5 cm.

(a) $(400 - 100\pi)$ cm^2

(b) $(360 - 100\pi)$ cm^2

(c) 231 cm^2

(d) $(400 - 50\pi)$ cm^2



31. The volume of a right prism, whose base is an equilateral triangle, is $1500\sqrt{3}$ cm^3 and the height of the prism is 125 cm. Find the side of the base of the prism.

(a) $8\sqrt{3}$ cm (b) $4\sqrt{3}$ cm

(c) $16\sqrt{3}$ cm (d) $24\sqrt{3}$ cm

32. A right circular cylinder of volume 1386 cm^3 is cut from a right circular cylinder of radius 4 cm and height 49 cm, such that a hollow cylinder of

PRACTICE QUESTIONS

uniform thickness, with a height of 49 cm and an outer raidus of 4 cm is left behind. Find the thickness of the hollow cylinder left behind.

(a) 0.5 cm (b) 2 cm

(c) 1.5 cm (d) 1 cm

33. The volume of a hemisphere is 18π cm^3. What is the total surface area of the hemisphere?

(a) 18π cm^2

(b) 27π cm^2

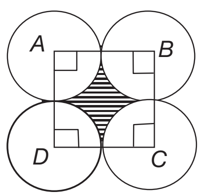
(c) 21π cm^2

(c) 24π cm^2

34. The diagram shown above has four circles of 7 cm radius with centres at A, B, C and D. If the quadrilateral $ABCD$ represents a square, then find the area of the shaded region.

(a) 42 cm^2 (b) 21 cm^2

(c) 63 cm^2 (d) 84 cm^2



35. Find the total surface area of a hollow metallic hemisphere whose internal radius is 14 cm and the thickness of the metal is 7 cm.

(a) 4774 cm^2

(b) 4477 cm^2

(c) 4747 cm^2

(d) 7744 cm^2

36. A metal cube of edge $\frac{3}{10}$ m is melted and formed into three smaller cubes. If the edges of the two smaller cubes are $\frac{1}{5}$ m and $\frac{1}{4}$ m, find the edge of the third smaller cube.

(a) $\frac{7}{20}$ m (b) $\frac{1}{20}$ m

(c) $\frac{3}{20}$ m (d) None of these

37. Two hemispherical vessels can hold 10.8 litres and 50 litres of liquid respectively. The ratio of their inner curved surface areas is ______.

(a) 16 : 25 (b) 25 : 9

(c) 9 : 25 (d) 4 : 3

38. A cylindrical drum 1.5 m in diameter and 3 m in height is full of water. The water is emptied into another cylindrical tank in which water rises by 2 m. Find the diameter of the second cylinder up to 2 decimal places.

(a) 1.74 m (b) 1.94 m

(c) 1.64 m (d) 1.84 m

39. Curved surface area of a conical cup is $154\sqrt{2}$ cm^2 and base radius is 7 cm. Find the angle at the vertex of the conical cup.

(a) 90° (b) 60°

(c) 45° (d) 30°

40. An equilateral triangle has a circle inscribed in it and is circumscribed by a circle. There is another equilateral triangle inscribed in the inner circle. Find the ratio of the areas of the outer circle and the inner equilateral triangle.

(a) $\frac{16\pi}{3\sqrt{3}}$ (b) $\frac{8\pi}{2\sqrt{3}}$

(c) $\frac{24\pi}{3\sqrt{3}}$ (d) None of these

41. A triangle has sides of 48 cm, 14 cm and 50 cm. Find its circum-radius (in cm).

(a) 25 (b) 12.5

(c) 20 (d) 17.5

42. The base of a pyramid is an n-sided regular polygon of area 360 cm^2. The total surface area of the pyramid is 900 cm^2. Each lateral face of the pyramid has an area of 30 cm^2. Find n.

(a) 20 (b) 18

(c) 16 (d) 24

43. In a right prism, the base is an equilateral triangle. Its volume is $80\sqrt{3}$ cm^3 and its lateral surface area is 240 cm^2. Find its height (in cm).

(a) 10 (b) 5

(c) 15 (d) 20

PRACTICE QUESTIONS

44. A goat is tied to one corner of a field of dimensions 16 m × 10 m with a rope 7 m long. Find the area of the field that the goat can graze.

The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) Required area = 38.5 m^2

(B) Area of the field that the goat can graze

= Area of the sector of radius 7 m and a sector angle of 90°

(C) $\frac{90°}{360} \times \frac{22}{7} \times (7)^2$

(a) ABC (b) BCA

(c) BAC (d) CBA

45. A right prism has a triangular base. If its perimeter is 24 cm and lateral surface area is 192 cm^2, find its height.

The following are the steps involved in solving the above problem. Arrange them in sequential order.

(A) $192 = 24 \times h$

(B) Given, LSA = 192 cm^2, Perimeter of the base = 24 cm

(C) $h = 8$ cm

(D) Lateral surface area of a prism = Perimeter of the base × Height

(a) BADC (b) BCAD

(c) DABC (d) BDAC

Level 3

46. An ink pen, with a cylindrical barrel of diameter 2 cm and height 10.5 cm, and completely filled with ink, can be used to write 4950 words. How many words can be written using 400 ml of ink?
(Take 1 litre = 1000 cm^3)

(a) 40000 (b) 60000

(c) 45000 (d) 80000

47. Each of height and side of the base of a regular hexagonal pyramid is equal to x cm. Find its lateral surface area in terms of x (in cm^2).

(a) $\frac{9\sqrt{7}}{2}x^2$ (b) $\frac{7\sqrt{7}}{2}x^2$

(c) $\frac{5\sqrt{7}}{2}x^2$ (d) $\frac{3\sqrt{7}}{2}x^2$

48. The diameters of the top and the bottom portions of a bucket are 42 cm and 28 cm. If the height of the bucket is 24 cm, then find the cost of painting its outer surface at the rate of 5 paise/cm^2.

(a) ₹158.25

(b) ₹172.45

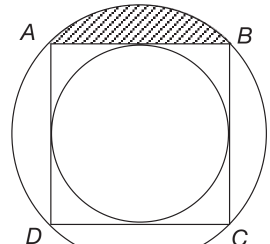
(c) ₹168.30

(d) ₹164.20

49. In the following figure, a circle is inscribed in square $ABCD$ and the square is circumscribed by a circle. If the radius of the smaller circle is r cm, then find the area of the shaded region (in cm^2).

(a) $\left(\frac{\pi - 2}{4}\right)r^2$ (b) $\left(\frac{3\pi - 4}{2}\right)r^2$

(c) $\left(\frac{\pi + 2}{4}\right)r^2$ (d) $\left(\frac{\pi - 2}{2}\right)r^2$

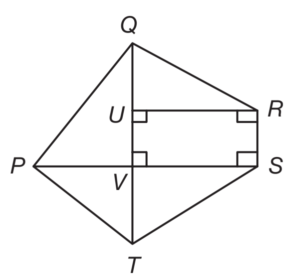


50. $ABCD$ is a square of side 4 cm. If E is a point in the interior of the square such that ΔCED is equilateral, then find the area of ΔACE (in cm^2).

(a) $2(\sqrt{3} - 1)$ (b) $4(\sqrt{3} - 1)$

(c) $6(\sqrt{3} - 1)$ (d) $8(\sqrt{3} - 1)$

51.



In the given figure (not to scale), $QT = 90$ m and $UR = 50$ m. $QU : UT = UV : VT = 1 : 2$. $PV : VS = 4 : 5$. Find the area of the figure. (in m^2)

(a) 4550 (b) 4200

(c) 4250 (d) 4100

52. H_1 is a regular hexagon circumscribing a circle. H_2 is a regular hexagon inscribed in the circle. Find the ratio of areas of H_1 and H_2.

(a) 4 : 3 (b) 2 : 1

(c) 3 : 1 (d) 3 : 2

53. A dish, in the shape of a frustum of a cone, has a height of 6 cm. Its top and its bottom have radii of 24 cm and 16 cm respectively. Find its curved surface area (in cm^2).

(a) 240π (b) 400π

(c) 180π (d) 160π

54. Two circles touch each other externally. The sum of their area is 490π cm^2. Their centres are separated by 28 cm. Find the difference of their radii (in cm).

(a) 14 (b) 7

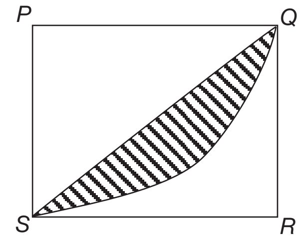
(c) 10.5 (d) 3.5

55. A closed rectangular shed has dimensions 21 m × 14 m. It is inside a field. A cow is tied outside the shed at one of its corners with a 21 m rope. Find the area over which the cow can graze (in m^2).

(a) 342π (b) 294π

(c) 343π (d) 441π

56. In the given figure, $PQRS$ is a square of diagonal $7\sqrt{2}$ cm. With P as the centre, the arc QS is drawn. Find the area of the shaded region (in cm^2).



(a) $\frac{49}{4}(\pi - 2)$ (b) $\frac{49}{4}(\pi - 1)$

(c) $\frac{49}{4}(\pi - 3)$ (d) $\frac{49}{2}(\pi - 2)$

57. Three solid cubes have a face diagonal of $4\sqrt{2}$ cm each. Three other solid cubes have a face diagonal of $8\sqrt{2}$ cm each. All the cubes are melted together to form a cube. Find the side of the cube formed (in cm).

(a) $\sqrt[3]{324}$ (b) $\sqrt[3]{576}$

(c) 12 (d) 24

58. The outer radius and inner radius of a 30 cm long cylindrical gold pipe are 14 cm and 7 cm respectively. It is filled with bronze. The densities of gold and bronze are 20 gm/cm^3 and 30 gm/cm^3 respectively. Find the weight of the cylinder formed. (in gm)

(a) 66150π (b) 99225π

(c) 132300π (d) 198450π

59. A rectangular sump has an inner length and breadth of 24 m and 20 m respectively. Water flows through an inlet pipe at 180 m per minute. The cross-sectional area of the pipe is 0.5 m^2. The tank takes half an hour to get filled. Find the depth of the sump (in m).

(a) 4.625 (b) 6.125

(c) 5.625 (d) 5.125

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $2\sqrt{3}$ cm
2. 8 cm
3. $3\sqrt{3}x$ cm
4. 88 cm
5. 7 cm
6. 0.5 cm
7. $\dfrac{d^3}{3\sqrt{3}}$ cu units.
8. $\dfrac{5}{3}$ m
9. 9
10. 6
11. $W = P \times H + 2A$
12. $nh + 2s$
13. n
14. 6, 12 and 8
15. 6
16. $A = \dfrac{B}{3}$
17. 7 : 6
18. $R^2 : r^2$
19. 49 : 25
20. r^2h
21. 9 : 4
22. circumference, slant height
23. $\pi(R^2 - r^2)$
24. $\pi(R^2 - r^2)h$
25. 21 : 88
26. 1 : 1
27. $200\pi l^2$ cm^2
28. $8x^3$ m^3
29. $2x$ cm
30. $4\pi a^2$ cm^2

Short Answer Type Questions

31. $1323\sqrt{3}$ sq. units
32. 143.36 cm^2
33. 155 cm
34. 880 m^2
35. $\dfrac{686}{11}$ cm^2
36. 385
37. 72 m^2
38. ₹20096
39. 157 m^2
40. 282.6 m^2
41. 3360 cm^3
42. $7546\sqrt{2}$ cm^2
43. $72\sqrt{3}$ cm^2
44. 20 cm
45. 900 cm

Essay Type Questions

46. 251.2 m^2, 6 m
47. 13 cm
48. $1\dfrac{3}{7}$ ft
49. 0.245
50. 7.87 cm^2

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (c)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (c)	**7.** (c)	**8.** (c)	**9.** (d)	**10.** (b)
11. (a)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (a)	**18.** (a)	**19.** (a)	**20.** (a)
21. (d)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (b)	**26.** (c)	**27.** (b)	**28.** (c)		

Level 2

29. (c)	**30.** (a)	**31.** (b)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (a)	**36.** (c)	**37.** (c)	**38.** (d)
39. (a)	**40.** (a)	**41.** (a)	**42.** (b)	**43.** (d)	**44.** (b)	**45.** (d)			

Level 3

46. (b)	**47.** (d)	**48.** (c)	**49.** (d)	**50.** (b)	**50.** (a)	**51.** (a)	**53.** (b)	**54.** (a)	**55.** (c)
56. (a)	**57.** (c)	**58.** (c)	**59.** (c)						

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. Perimeter of a sector = $l + 2r = 4r$.
2. Find the area of triangle and the area of sector formed by the chord.
3. Radius of the circle $= \frac{1}{3}$ of the median.
4. Use $(a + b)^2 = a^2 + b^2 + 2ab$ and $a - b = \sqrt{(a+b)^2 - 4ab}$.
5. Area of the track = Area of outer circle − Area of inner circle.
6. Volume of the pyramid $= \frac{1}{3} \times$ Area of the base $\times$ Height.
7. Breadth $= \sqrt{3}x$ m and Height $= \sqrt{7}x$ m.
8. Volume of large cube is equal to the sum of volumes of three small cubes.
9. Find the volume of cylinder. Whose radius is breadth of rectangle and height is equal to length of the rectangle.
10. Volume of the prism = Area of the base × Height.
11. Area levelled by the roller in one revolution = CSA of the cylinder.
12. Diameter of the sphere = Length of the edge of a cube.
13. Draw $\overline{OP}$ from the centre to the midpoint of $\overline{BC}$.
14. CSA of a cylinder $= 2\pi Rh$.
15. Find 'r' by using volume formula.
16. Find individual areas of different parts of the figure.
17. Find the edge of the prism.
18. Volume of the cone = Volume of all the spheres formed.
19. TSA of outer part $= 357 \times \frac{100}{0.2}\text{cm}^2$.
20. CSA of bucket $= \pi l(R + r)$.
21. Equate the two volumes.
22. Find the area of two hexagon.
23. Required area is the difference of areas of rectangle and sum of areas of two sectors.
24. Draw the diagram and proceed.
25. Volume of the prism = Area of the base × Height.
26. Required volume = Volume of cone + Volume of sphere.
27. Volume of bowl = Volume of a bottle × Number of bottles.
28. Volume of a hemisphere $= \frac{3}{2}\pi r^3$ cubic units.

Level 2

29. (i) Area of circular path = Area of ring.
    (ii) Area of path $= \pi(R^2 - r^2)$.
    (iii) Total cost = Area × cost/m^2.
30. (i) Find the area of the square formed by joining the centers of all outer circles.
    (ii) The required area = Area of the square − 16 $\left(\frac{1}{4} \times \text{Area of circle}\right)$.
31. (i) Volume of prism $= \frac{\sqrt{3}}{4}a^2 \times$ Height.
    (ii) Use the above formula and get the value of a.
32. Find the radius of the cylinder which is cut. (i.e., $\pi r^2 h = 1386$).
33. Radius of the hemisphere = Radius of the sphere.
34. Area of shaded region = Area of square ABCD − 4 $\left(\frac{1}{4} \times \text{Area of circle}\right)$.
35. TSA $= 3\pi R^2 + \pi r^2$.
36. Volume of big cube = Sum of the volumes of smaller cubes.
37. Ratio of their CSA's is $r_1^2 : r_2^2$.
38. Volume of water in the cylindrical drum = Volume of the second cylinder up to the water risen.
39. Use, $\tan\left(\frac{\theta}{2}\right) = \frac{r}{h}$ and find θ.

HINTS AND EXPLANATION

40. (i) Circum radius $= \frac{a}{\sqrt{3}}$ and inradius $= \frac{a}{2\sqrt{3}}$, where a is the side of the outer equilateral triangle.

(ii) For an equilateral triangle of side a, if an in-circle and circum circle are drawn whose radii are r and R then $r = \frac{a}{2\sqrt{3}}$ and $R = \frac{a}{\sqrt{3}}$.

41. $48^2 = 2304$

$14^2 = 196$

$50^2 = 2500$

$48^2 + 14^2 = 50^2$

$\therefore$ The triangle is right angled and its hypotenuse is 50 cm. Its circum radius $= \frac{50}{2}$ cm = 25 cm.

42. Total area of the lateral faces = 900 − 360 = 540 cm^2.

Number of lateral faces it has $\frac{540}{30} = 18$.

43. Let the side of the base be a cm. Let the height of the prism be h cm.

$$\left(\frac{\sqrt{3}}{4}a^2\right)(h) = 80\sqrt{3} \qquad (1)$$

$$(3a)(h) = 240 \qquad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sqrt{3}}{12}a = \frac{\sqrt{3}}{3}$$

$a = 4$

From Eq. (2), $\Rightarrow h = 20$.

44. BCA

45. BDAC

Level 3

46. (i) 1 ml = 1 cm^3.

(ii) Find the number of words per 1 ml of ink.

47. CSA of a pyramid $= \frac{1}{2}$ × Perimeter of the base × Slant height.

48. CSA of a bucket $= \pi l(R + r)$.

49. Diagonals of the square meet at the centre of the circles.

50. (i) Draw the figure according to the data and draw $\overline{EF} \parallel \overline{CD}$.

(ii) Area of ΔACE = Area of the triangle ABC − {Area of ΔECD + Area of $ABED$}.

51. $QT = 90$ m

$QU : UT = 1 : 2$

$\therefore UT = 60$ m

$UV : VT = 1 : 2$ and $QU = 30$ m

$\therefore UV = 20$ m and $VT = 40$ m

$URSV$ is a rectangle.

$\therefore VS = UR = 50$ m.

$PV : VS = 4 : 5$

$\therefore PV = 40$ m.

Area of the figure = Area of ΔPQT + Area of $QRST$

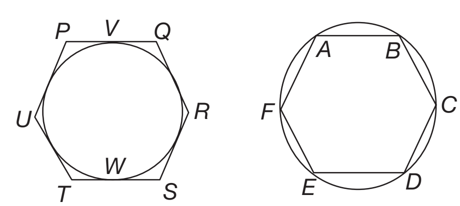
$$= \frac{1}{2}(PV)(QT) + \frac{1}{2}(QT + RS)(UR)$$

$$= \frac{1}{2}(40)(90) + \frac{1}{2}(90 + 20)(50)$$

$(\because RS = UV = 20$ m$)$

$= 1800 + 2750 = 4550$ m^2.

52.



Let H_1 be $PQRSTU$ and H_2 be $ABCDEF$.

Let R be the radius of the circle.

$$\therefore R = \frac{\sqrt{3}(PU)}{2} \Rightarrow PU = \frac{2R}{\sqrt{3}}$$

$AB = R$ ($\because$ A hexagon inscribed in a circle must have its side equal to the radius of the circle)

Area of $H_1 = \dfrac{3\sqrt{3}}{2}\left(\dfrac{2}{\sqrt{3}}R\right)^2$

And area of $H_2 = \dfrac{3\sqrt{3}}{2}(R)^2$

($\because$ Area of hexagon $= \dfrac{3\sqrt{3}}{2} \times$ (Its side)^2)

Required ratio $= \dfrac{\left(\dfrac{2}{\sqrt{3}}R\right)^2}{R^2} = \dfrac{4}{3}$.

53. The bucket is in shape of a frustum. The slant height of a frustum

$= \sqrt{(\text{Top radius} - \text{Bottom Radius})^2 + (\text{Height})^2}$.

$\therefore$ Slant height of the bucket

$l = \sqrt{(24-16)^2 + 6^2} = 10$ m.

$\therefore$ Its curved surface area $= \pi l(R + r)$

$= \pi\,(10)(24 + 16) = 400\,\pi$ m^2.

54. Let the radii of the circles be denoted by r_1 cm and r_2 cm where $r_1 \geq r_2$. As the circles touch each other externally, distance between their centres = Sum of their radii.

$\therefore\ r_1 + r_2 = 28$ (1)

Also $\pi(r_1)^2 + \pi(r_2)^2 = 490\,\pi$

$\therefore\ (r_1)^2 + (r_2)^2 = 490$ (2)

Squaring both sides of Eq. (1), we get

$(r_1)^2 + (r_2)^2 + 2(r_1)(r_2) = 784$

From Eq. (2), $490 + 2(r_1)(r_2) = 784$

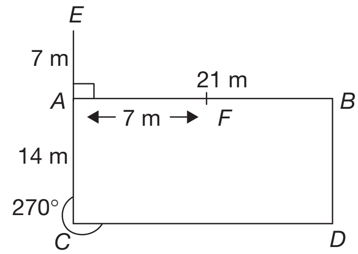
$r_1 \cdot r_2 = 147$ (3)

Solving Eqs. (1) and (3), we get

$r_1 = 21$ and $r_2 = 7$. ($\therefore\ r_1 \geq r_2$)

$\therefore$ Required difference = 14 cm.

55.



Suppose the cow was tied at C. Total area it can graze = Area of the sector ECD with central angle 270° and radius 21 m + Area of the sector EAF with central angle 90°

and radius 7 m $= \dfrac{270°}{360°}\pi(21)^2 + \dfrac{90°}{360°}\pi(7)^2$

$= \dfrac{3}{4}\pi(441) + \dfrac{1}{4}\pi(49) = 343\pi$ m^2.

56. Area of the shaded region

= Area of the sector PQS – Area of ΔPQS

$QS = \sqrt{2}$ (side) $= 7\sqrt{2}$ cm.

Side = 7 cm

$\therefore$ Area of sector PQS

$= \dfrac{90°}{360°}\pi\,(7)^2 = \dfrac{49\pi}{4}$ cm^2

Area of $\Delta PQS = \dfrac{1}{2}(PQ)(PS)$

$= \dfrac{1}{2}(7)^2 = \dfrac{49}{2}$ cm^2

$\therefore$ Required area $= \left(\dfrac{49\pi}{4} - \dfrac{49}{2}\right)$ cm^2

$= \dfrac{49}{4}(\pi - 2)$ cm^2.

57. Side of each of the first three cubes $= \dfrac{4\sqrt{2}}{\sqrt{2}} = 4$ cm.

Side of each of the other three cubes $= \dfrac{8\sqrt{2}}{\sqrt{2}}$ = 8 cm.

Let the side of the cube formed be a cm.

Total volume of the six cubes = $3(4^3 + 8^3) = 3(64 + 512) = 1728$ cm^3.

$\therefore\ a^3 = 1728$

$a = 12.$

58. Volume of the pipe = $\pi(14^2 - 7^2)(30)$ cm^3

$\therefore$ Volume of the gold = $\pi(14^2 - 7^2)(30)$ cm^3

Volume of the bronze in the pipe

$= \pi(7)^2\ (30)$ cm^3

Weight of the pipe (in gms) = Weight of gold in it (in gm) + Weight of bronze in it (in gm)

$= [\pi(14^2 - 7^2)(30)]\ [20] + [\pi(7)^2\ (30)]\ [30]$

$= 30\pi[(196 - 49)(20) + (49)(30)]$

$= 30\pi[2940 + 1470] = 132300\pi$ gm.

59. Let the depth of the sump be h m.

Volume of water flowing through the pipe = (180)(0.5) m^3/minute = 90 m^3/minute.

Total volume of water which must flow through to fill the sump = (90)(30) = 2700 cm^3.

$\therefore\ (24)(20)(h) = 2700$

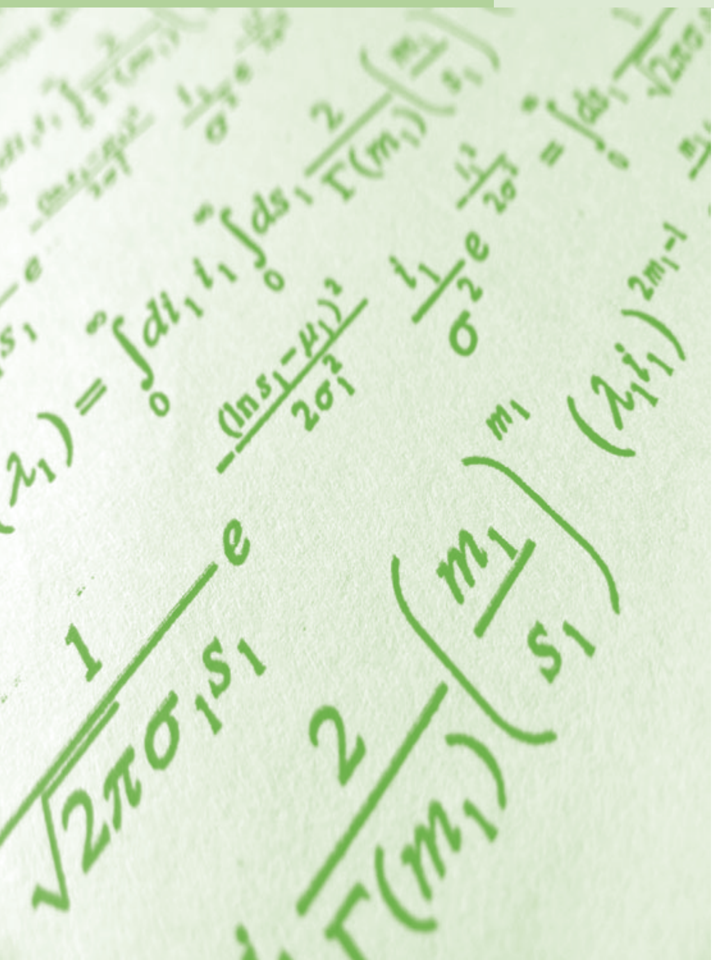
$h = 5.625.$

HINTS AND EXPLANATION

Chapter

15

Coordinate Geometry



REMEMBER

Before beginning this chapter, you should be able to:

- Label coordinates of a point and tell conversion of signs
- Show points on the plane, distance between the points

KEY IDEAS

After completing this chapter, you would be able to:

- Study distance between a point and the axes and apply the distance formula in solving word problems
- Explain signs of coordinates of a point in four quadrants
- Understand inclination and slope of a straight line
- Obtain equation of a line parallel or perpendicular to the given line
- Find out midpoint, centroid, median and altitude of a triangle
- Calculate areas of triangle and quadrilateral

INTRODUCTION

Let $X'OX$ and YOY' be two mutually perpendicular lines intersecting at the point O in a plane.

These two lines are called reference lines or coordinate axes. The horizontal reference line $X'OX$ is called X-axis and the vertical reference line YOY' is called Y-axis.

The point of intersection of these two axes, i.e., O is called the origin. The plane containing the coordinate axes is called coordinate plane or XY-plane.

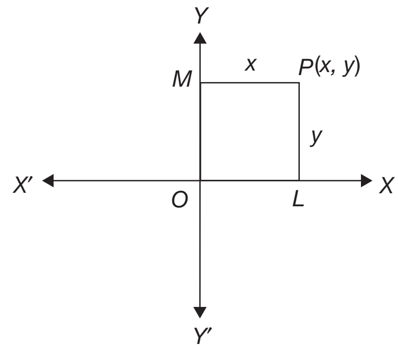


Figure 15.1

COORDINATES OF A POINT

Let P be a point in the XY-plane. Draw perpendiculars PL and PM to X-axis and Y-axis respectively.

Let $PL = y$ and $PM = x$. Then, the point P is taken as (x, y). Here x and y are called the rectangular Cartesian coordinates or simply coordinates of the point P. x is called x-coordinate or abscissa and y is called y-coordinate or ordinate of the point P. $P(x, y)$ is x units away from Y-axis and y units away from X-axis.

Convention of Signs

1. Towards the right side of the Y-axis, x-coordinate of any point on the graph paper is taken positive and towards the left side of the Y-axis, x-coordinate is taken negative.
2. Above the X-axis, the y-coordinate of any point on the graph paper is taken positive and below the X-axis, y-coordinate is taken negative.

If (x, y) is a point in the plane and Q_1, Q_2, Q_3, Q_4 are the four quadrants of rectangular coordinate system, then

1. If $x > 0$ and $y > 0$, then $(x, y) \in Q_1$.
2. If $x < 0$ and $y > 0$, then $(x, y) \in Q_2$.
3. If $x < 0$ and $y < 0$, then $(x, y) \in Q_3$.
4. If $x > 0$ and $y < 0$, then $(x, y) \in Q_4$.

EXAMPLE 15.1

If $x < 0$ and $y > 0$, then $(-x, y)$ lies in which quadrant?

SOLUTION

$x < 0 \Rightarrow -x > 0$

$\therefore$ The point $(-x, y)$ lies in the first quadrant, i.e., Q_1.

EXAMPLE 15.2

If $(a, b) \in Q_3$, then $(-a, -b)$ belongs to which quadrant?

SOLUTION

Given $(a, b) \in Q_3 \Rightarrow a < 0$, $b < 0$ then $-a$ is positive and $-b$ is also positive therefore $(-a, -b) \in Q_1$.

EXAMPLE 15.3

Plot the points $A(2, 3)$, $B(-1, 2)$, $C(-3, -2)$, and $D(4, -2)$ in the XY-plane.

SOLUTION

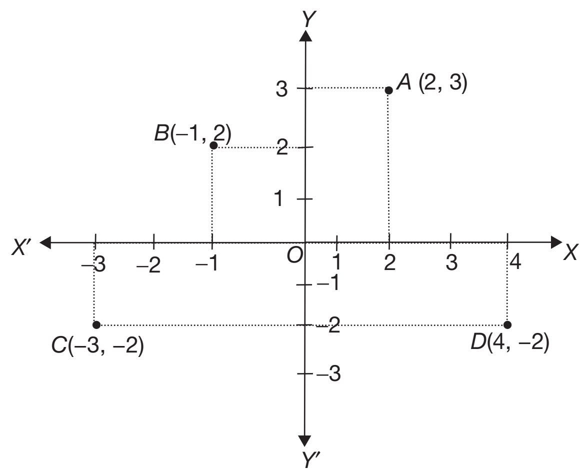


Figure 15.2

POINTS ON THE PLANE

Point on *X*-axis and *Y*-axis

Let P be a point on X-axis, so that its distance from X-axis is zero. Hence, the point P can be taken as $(x, 0)$.

Let P' be a point on Y-axis, so that its distance from Y-axis is zero. Hence, the point P' can be taken as $(0, y)$.

Distance Between Two Points

Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Draw perpendiculars AL and BM from A and B to X-axis, AN is the perpendicular drawn from A on to BM.

From right triangle ABN, $AB = \sqrt{AN^2 + BN^2}$.

Here, $AN = x_2 - x_1$, and $BN = y_2 - y_1$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Hence, the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ units.}$$

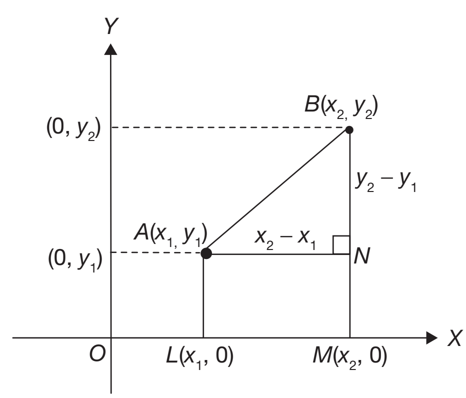


Figure 15.3

Note The distance of a point $A(x_1, y_1)$ from origin $O(0, 0)$ is $OA = \sqrt{x_1^2 + y_1^2}$.

EXAMPLE 15.4

Find the distance between the points (3, −5) and (5, −1).

SOLUTION

Let the given points be $A(3, -5)$ and $B(5, -1)$.

$$AB = \sqrt{(5-3)^2 + (-1-(-5))^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5} \text{ units.}$$

EXAMPLE 15.5

Find a if the distance between the points $P(11, -2)$ and $Q(a, 1)$ is 5 units.

SOLUTION

Given, $PQ = 5$

$$\Rightarrow \sqrt{(a-11)^2 + (1-(-2))^2} = 5$$

Taking square on both sides, we get

$$(a - 11)^2 = 25 - 9 = 16$$
$$a = 11 = \sqrt{16}$$
$$a - 11 = \pm 4$$
$$\therefore\ a = 15 \text{ or } 7.$$

EXAMPLE 15.6

Find the coordinates of a point on Y-axis which is equidistant from the points (13, 2) and (12, −3).

SOLUTION

Let $P(0, y)$ be the required point and the given points be $A(12,-3)$ and $B(13, 2)$.

Then $PA = PB$ (given)

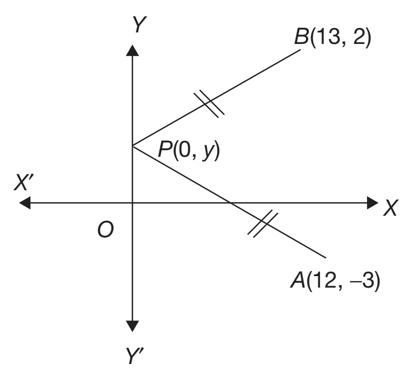


Figure 15.4

$$\sqrt{(12-0)^2 + (-3-y)^2} = \sqrt{(13-0)^2 + (2-y)^2} \Rightarrow \sqrt{144 + (y+3)^2} = \sqrt{169 + (2-y)^2}$$

Taking square on both sides, we get

$$144 + 9 + y^2 + 6y = 169 + 4 + y^2 - 4y$$
$$\Rightarrow 10y = 20 \Rightarrow y = 2.$$

$\therefore$ The required point on Y-axis is (0, 2).

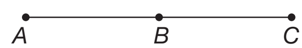
Applications of Distance Formula

Collinearity of Three Points Let A, B and C be three given points. The distances AB, BC and CA can be calculated using distance formula.

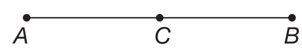
If the sum of any two of these distances is found to be equal to the third distance, then the points A, B and C will be collinear.

Notes

1. If $AB + BC = AC$, then the points A, B and C are collinear.



2. If $AC + CB = AB$, then the points A, C and B are collinear.



3. $BA + AC = BC$, then the points B, A and C are collinear.

By notes (1), (2) and (3), we can find the position of points in collinearity.

EXAMPLE 15.7

Show that the points $A(2, 3)$, $B(3, 4)$ and $C(4, 5)$ are collinear.

SOLUTION

Given, $A = (2, 3)$, $B = (3, 4)$ and $C = (4, 5)$

$$AB = \sqrt{(3-2)^2 + (4-3)^2} = \sqrt{2} \text{ units.}$$
$$BC = \sqrt{(4-3)^2 + (5-4)^2} = \sqrt{2} \text{ units.}$$
$$AC = \sqrt{(4-2)^2 + (5-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units.}$$

Now, $AB + BC = \sqrt{2} + \sqrt{2} = 2\sqrt{2} = AC$

That is, $AB + BC = AC$.

Hence, the points A, B and C are collinear.

EXAMPLE 15.8

Show that the points (2, 4), (6, 8) and (2, 8) form an isosceles right triangle when joined.

SOLUTION

Let $A(2, 4)$, $B(6, 8)$ and $C(2, 8)$ be the given points.

$$AB = \sqrt{(6-2)^2 + (8-4)^2} = \sqrt{32} \text{ units.}$$

$$BC = \sqrt{(2-6)^2 + (8-8)^2} = \sqrt{16} = 4 \text{ units.}$$

$$AC = \sqrt{(2-2)^2 + (8-4)^2} = \sqrt{16} = 4 \text{ units.}$$

Clearly, $BC^2 + AC^2 = AB^2$

$\Rightarrow \angle C = 90°$

Also, $BC = AC$.

Hence, the given points form the vertices of a isosceles right triangle.

EXAMPLE 15.9

Show that the points $(1, -1)$, $(-1, 1)$ and $(\sqrt{3}, \sqrt{3})$ when joined, form an equilateral triangle.

SOLUTION

Let $A(1, -1)$, $B(-1, 1)$ and $C(\sqrt{3}, \sqrt{3})$ be the given points.

Then, $AB = \sqrt{(-1-1)^2 + (1+1)^2} = \sqrt{4+4} = \sqrt{8}$ units

$$BC = \sqrt{(\sqrt{3}-(-1))^2 + (\sqrt{3}-1)^2} = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$$

$$= \sqrt{2(3+1)}\ [\because (a+b)^2 + (a-b)^2 = 2(a^2+b^2)] = \sqrt{8} \text{ units}$$

$$CA = \sqrt{(1-\sqrt{3})^2 + (-1-\sqrt{3})^2} = \sqrt{(1-\sqrt{3})^2 + (1+\sqrt{3})^2} = \sqrt{8} \text{ units}$$

$\therefore AB = BC = CA = \sqrt{8}$

Hence, the points $(1, -1)$, $(-1, 1)$ and $(\sqrt{3}, \sqrt{3})$ when joined, form an equilateral triangle.

EXAMPLE 15.10

Show that the points $(2, 0)$, $(-6, -2)$, $(-4, -4)$ and $(4, -2)$ form a parallelogram.

SOLUTION

Let the given points be $A(2, 0)$, $B(-6, -2)$, $C(-4, -4)$ and $D(4, -2)$

Then, $AB = \sqrt{(-6-2)^2 + (-2-0)^2} = \sqrt{68}$ units.

$$BC = \sqrt{(-4+6)^2 + (-4+2)^2} = \sqrt{8} \text{ units}$$

$$CD = \sqrt{(4-4)^2 + (-4+2)^2} = \sqrt{68} \text{ units}$$

$$DA = \sqrt{(4-2)^2 + (-2-0)^2} = \sqrt{8} \text{ units}$$

$$AC = \sqrt{(-4-2)^2 + (-4-0)^2} = \sqrt{52} \text{ units}$$

$$BD = \sqrt{(4+6)^2 + (-2+2)^2} = \sqrt{10} \text{ units}$$

Clearly,

$AB = CD$, $BC = DA$ and $AC \neq BD$.

That is, the opposite sides of the quadrilateral are equal and diagonals are not equal.

Hence, the given points form a parallelogram.

EXAMPLE 15.11

The vertices of a ΔABC are $A(1, 2)$, $B(3, -4)$ and $C(5, -6)$. Find its circum-centre and circum-radius.

SOLUTION

Let $S(x, y)$ be the circum-centre of ΔABC.

$\therefore\ SA^2 = SB^2 = SC^2$.

Consider,

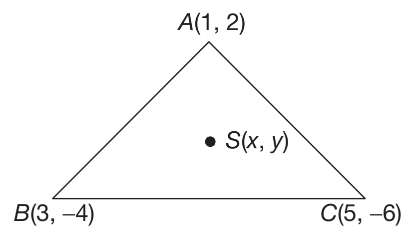


Figure 15.5

$$SA^2 = SB^2$$

$\Rightarrow (x-1)^2 + (y-2)^2 = (x-3)^2 + (y+4)^2$

$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 + 8y + 16$

$\Rightarrow -2x - 4y + 1 + 4 = -6x + 9 + 8y + 16$

$\Rightarrow 4x - 12y - 20 = 0$

$$x - 3y = 5 \qquad (1)$$

$SB^2 = SC^2$

$\Rightarrow (x-3)^2 + (y+4)^2 = (x-5)^2 + (y+6)^2$

$$x - y = 9 \qquad (2)$$

Solving Eqs. (1) and (2), we have

$x = 11$ and $y = 2$

$\therefore$ The required circum-centre of ΔABC is (11, 2).

Circum-radius $= SA = \sqrt{(11-1)^2 + (2-2)^2} = 10$ units.

EXAMPLE 15.12

Find the area of the circle whose centre is (–3, 2) and (2, 5) is a point on the circle.

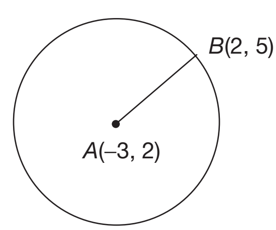


Figure 15.6

SOLUTION

Let the centre of the circle be $A(-3, 2)$ and point of the circumference be $B(2, 5)$.

Radius of the circle $= AB = \sqrt{(2+3)^2 + (5-2)^2} = \sqrt{25+9}$

$r = \sqrt{34}$ units.

$\therefore$ The area of circle $= \pi r^2$

$= \pi(\sqrt{34})^2 = 34\pi$ sq. units.

EXAMPLE 15.13

Find the area of a square whose one pair of the opposite vertices are (3, 4) and (5, 6).

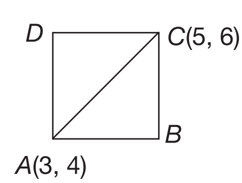


Figure 15.7

SOLUTION

Let the given vertices be $A(3, 4)$ and $C(5, 6)$.

Length of $AC = \sqrt{(3-5)^2 + (4-6)^2} = \sqrt{8}$ units.

Area of the square $= \dfrac{AC^2}{2} = \dfrac{(\sqrt{8})^2}{2} = 4$ sq. units.

EXAMPLE 15.14

Find the ortho-centre of the ΔABC formed by the vertices $A(2, 2)$, $B(6, 3)$ and $C(4, 11)$.

SOLUTION

The given vertices of ΔABC are $A(2, 2)$, $B(6, 3)$ and $C(4, 11)$.

$$\text{Length of } AB = \sqrt{(6-2)^2 + (3-2)^2} = \sqrt{17} \text{ units}$$

$$\text{Length of } BC = \sqrt{(6-4)^2 + (3-11)^2} = \sqrt{68} \text{ units}$$

$$\text{Length of } AC = \sqrt{(4-2)^2 + (11-2)^2} = \sqrt{85} \text{ units}$$

Clearly, $AC^2 = AB^2 + BC^2$

ABC is a right triangle, right angle at B.

Hence, ortho-centre is the vertex containing right angle, i.e., $B(6, 3)$.

STRAIGHT LINES

Inclination of a Line

The angle made by a straight line with positive direction of X-axis in the anti-clockwise direction is called its inclination.

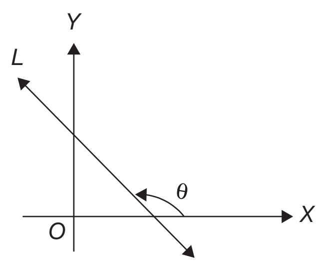


Figure 15.8

Slope or Gradient of a Line

If θ is the inclination of a line L, then its slope is denoted by m and is given by $m = \tan\theta$.

Example: The inclination of the line l in the Fig. 15.9 is 45°.

∴ The slope of the line is $m = \tan 45° = 1$.

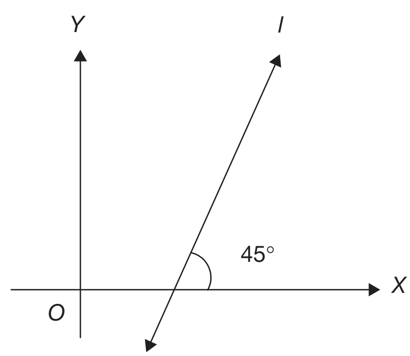


Figure 15.9

Example: The line L in the following figure makes an angle of 45° in clockwise direction with X-axis. So, the inclination of the line L is $180° - 45° = 135°$.

∴ The slope of the line L is $m = \tan 135° = -1$.

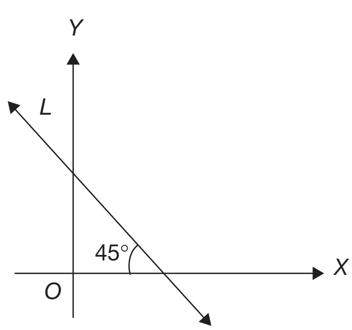


Figure 15.10

Some Results on the Slope of a Line

1. The slope of a horizontal line is zero. Hence:
   (i) Slope of X-axis is zero.
   (ii) Slope of any line parallel to X-axis is also zero.
2. The slope of a vertical line is not defined. Hence:
   (i) Slope of Y-axis is undefined.
   (ii) Slope of any line parallel to Y-axis is also undefined.

Theorem 1

Two non-vertical lines are parallel if and only if their slopes are equal.

Proof: Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively.

If θ_1 and θ_2 are the inclinations of the lines, L_1 and L_2 respectively, then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Now, since $L_1 \parallel L_2$. Then, $\theta_1 = \theta_2$.

($\because$ They form a pair of corresponding angles)

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

Conversely, let $m_1 = m_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2$

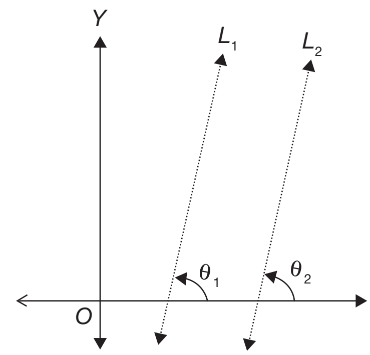


Figure 15.11

$\Rightarrow L_1 \parallel L_2$ ($\because$ θ_1 and θ_2 form a pair of corresponding angles)

Hence, two non-vertical lines are parallel if and only if their slopes are equal.

Theorem 2

Two non-vertical lines are perpendicular to each other if and only if the product of their slopes is -1.

Proof: Let L_1 and L_2 be two non-vertical lines with slopes, m_1 and m_2.

If θ_1 and θ_2 are the inclinations of the lines L_1 and L_2 respectively, then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

If $L_1 \perp L_2$, then

$\theta_2 = 90° + \theta_1$

($\because$ The exterior angle of a triangle is equal to the sum of two opposite interior angles)

$\Rightarrow \tan\theta_2 = \tan(90° + \theta_1)$

$\Rightarrow \tan\theta_2 = -\cot\theta_1$

$\Rightarrow \tan\theta_2 = \dfrac{-1}{\tan\theta_1}$ $[\because \theta_1 \neq 0]$

$\Rightarrow \tan\theta_1 \cdot \tan\theta_2 = -1$

$\therefore\ m_1 m_2 = -1$

Conversely, let $m_1 m_2 = -1$

$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$

$\Rightarrow \tan\theta_2 = \dfrac{-1}{\tan\theta_1} [\because \theta_1 \neq 0]$

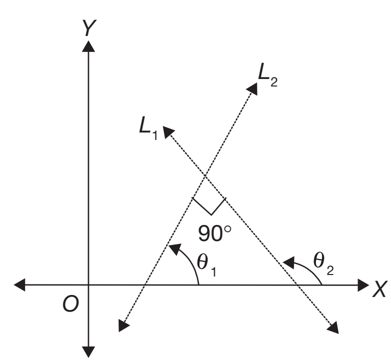


Figure 15.12

$\Rightarrow \tan\theta_2 = -\cot\theta_1$

$\Rightarrow \tan\theta_2 = \tan\,(90° + \theta_1)$

$\Rightarrow \theta_2 = 90° + \theta_1$

$\Rightarrow L_1 \perp L_2.$

Hence, two non-vertical lines are perpendicular to each other if and only if the product of their slopes is -1.

The Slope of a Line Passing Through the Points (x_1, y_1) and (x_2, y_2)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points.

Let AB be the straight line passing through the points A and B.

Let θ be the inclination of the line $\overleftrightarrow{AB}$.

Draw the perpendiculars AL and BM on to X-axis from A and B respectively. Also draw $AN \perp BM$.

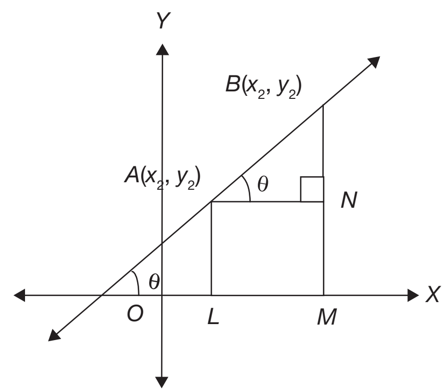


Figure 15.13

Then, $\angle NAB = \theta$

Also, $BN = BM - MN = BM - AL = y_2 - y_1$

$AN = LM = OM - OL = x_2 - x_1$

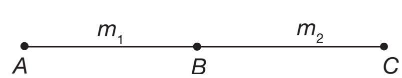
$\therefore$ The slope of the line L is, $m = \tan\theta = \dfrac{BN}{AN} = \dfrac{y_2 - y_1}{x_2 - x_1}.$

Hence, the slope of a line passing through the points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}.$

The following table gives the inclination (θ) of the line and its corresponding slope (m) for some particular values of θ.

θ	0°	30°	45°	60°	90°	120°	135°	150°
$m = \tan\theta$	0	$\dfrac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$\dfrac{-1}{\sqrt{3}}$

Note If the points A, B and C are collinear, then the slope of AB = the slope of BC.



That is, if $m_1 = m_2$, then A, B and C are collinear.

EXAMPLE 15.15

Find the slope of line joining the points $(5, -3)$ and $(7, -4)$.

SOLUTION

Let $A(5, -3)$ and $B(7, -4)$ be the given points.

Then, the slope of $\overleftrightarrow{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-4 - (-3)}{7 - 5} = \frac{-1}{2}.$$

EXAMPLE 15.16

Find the value of k if the slope of the line joining the points $(k, 4)$ and $(-3, -2)$ is $\frac{1}{2}$.

SOLUTION

Let the given points be $A(k, 4)$ and $B(-3, -2)$.

Given, the slope of $\overleftrightarrow{AB} = \frac{1}{2}$

$$\frac{-2-4}{-3-k} = \frac{1}{2}$$
$$\Rightarrow -12 = -3 - k$$
$$\Rightarrow k = -3 + 12$$
$$\Rightarrow k = 9.$$

EXAMPLE 15.17

Find the value of m, if the line passing through the points $A(2, -3)$ and $B(3, m + 5)$ is perpendicular to the line passing through the points $P(-2, 3)$ and $Q(-4, -5)$.

SOLUTION

The slope of $\overleftrightarrow{AB}$, i.e., $m_1 = \frac{y_2 - y_1}{x_2 - x_1}$.

$m_1 = m + 8$.

The slope of PQ, i.e., $m_2 = \frac{y_2 - y_1}{x_2 - x_1} = 4$.

Since, $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are perpendicular to each other $\Rightarrow m_1 m_2 = -1$, i.e.,

$$(m + 8) \times (4) = -1$$

$\Rightarrow m + 8 = \frac{-1}{4}$.

Hence, $m = \frac{-33}{4}$.

EXAMPLE 15.18

If the points $(-3, 6)$, $(-9, a)$ and $(0, 15)$ are collinear, then find a.

SOLUTION

Let the given points be $A(-3, 6)$, $B(-9, a)$ and $C(0, 15)$.

The slope of $AB = \frac{a-6}{-9+3} = \frac{6-a}{6}$

The slope of $BC = \frac{a-15}{-9-0} = \frac{15-a}{9}$

Since the points A, B and C are collinear.

The slope of $\overleftrightarrow{AB} =$ the slope of $\overleftrightarrow{BC}$

$$\Rightarrow \frac{6-a}{6} = \frac{15-a}{9}$$
$$\Rightarrow 18 - 3a = 30 - 2a$$
$$\Rightarrow a = -12$$

Hence, $a = -12$.

Intercepts of a Straight Line

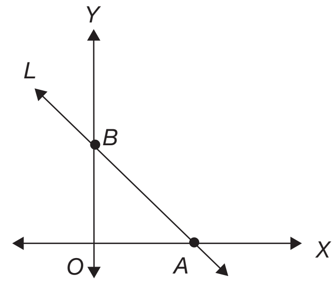


Figure 15.14

Say a straight line L meets X-axis in A and Y-axis in B.
Then, OA is called the x-intercept and OB is called the y-intercept.

Note OA and OB are taken as positive or negative based on whether the line meets positive or negative axes.

Example: The line L in the given figure meets X-axis at $A(4, 0)$ and Y-axis at $B(0, -5)$.

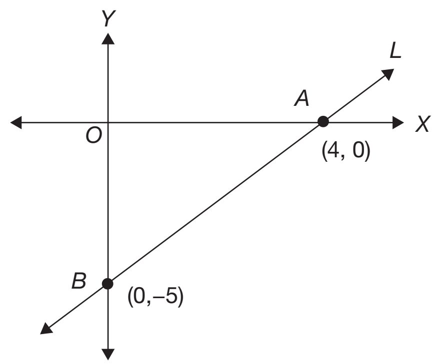


Figure 15.15

Hence, the x-intercept = 4 and y-intercept = -5.

Equation of a Line in General Form

An equation of the form, $ax + by + c = 0$. (where $a + b \neq 0$, i.e., a and b are not simultaneously equal to zero), which is satisfied by every point on a line and not by any point outside the line, is called the equation of a line.

Equations of Some Standard Lines

Equation of *X*-axis We know that the y-coordinate of every point on X-axis is zero so, if $P(x, y)$ is any point on X-axis, then $y = 0$.

Hence, the equation of X-axis is $y = 0$.

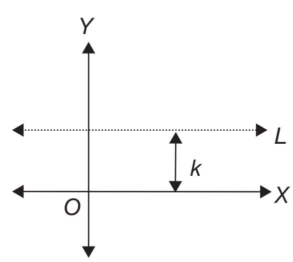


Figure 15.16

Equation of *Y*-axis We know that the x-coordinate of every point on Y-axis is zero. So, if $P(x, y)$ is any point on Y-axis, then $x = 0$, hence, the equation of Y-axis is $x = 0$.

Equation of a Line Parallel to *X*-axis Let L be a line parallel to X-axis and at a distance of k units away from X-axis.

Then, the y-coordinate of every point on the line L is k.

So, if $P(x, y)$ is any point on the line L, then $y = k$.

Hence, the equation of a line parallel to X-axis at a distance of k units from it is $y = k$.

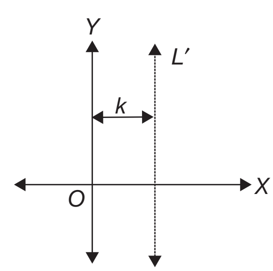


Figure 15.17

Note For the lines lying below X-axis, k is taken as negative.

Equation of a Line Parallel to *Y*-axis Let L' be a line parallel to Y-axis and at a distance of k units away from it.

Then the x-coordinate of every point on the line L' is k.

So, if $P(x, y)$ is any point on the line L', then $x = k$.

Hence, the equation of a line parallel to Y-axis and at a distance of k units from it is $x = k$.

Note For the lines lying towards the left side of Y-axis, k is taken as negative.

Oblique Line

A straight line which is neither parallel to X-axis nor parallel to Y-axis is called an oblique line or an inclined line.

Different Forms of Equations of Oblique Lines

Gradiant Form (or) Slope Form The equation of a straight line with slope m and passing through origin is given by $y = mx$.

Point-Slope Form The equation of a straight line passing through the point (x_1, y_1) and with slope m is given by $y - y_1 = m(x - x_1)$.

Slope-intercept Form The equation of a straight line with slope m and having y-intercept c is given by $y = mx + c$.

Two-point Form The equation of a straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Intercept Form The equation of a straight line with x-intercept as a and y-intercept as b is given by $\frac{x}{a} + \frac{y}{b} = 1$.

Note Area of triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is $\frac{1}{2}|ab|$ sq. units.

EXAMPLE 15.19

Find the equation of a line parallel to X-axis and passing through the point $(3, -4)$.

SOLUTION

We know that the equation of a line parallel to X-axis can be taken as $y = k$.

Given, the line passes through the point $(3, -4)$

$$\Rightarrow k = -4$$

Hence, the equation of the required line is $y = -4$, i.e., $y + 4 = 0$.

EXAMPLE 15.20

Find the equation of a line having a slope of $-\frac{3}{4}$ and passing through the point $(3, -4)$.

SOLUTION

We know that, the equation of a line passing through the point (x_1, y_1) and having a slope m is given by $y - y_1 = m(x - x_1)$.

Hence, the equation of the required line is

$$y-(-4)=-\frac{3}{4}(x-3)$$

$$\Rightarrow 4(y+4)=-3(x-3)$$

$$\Rightarrow 3x+4y+7=0.$$

EXAMPLE 15.21

Find the equation of a line making intercepts 3 and −4 on the coordinate axes respectively.

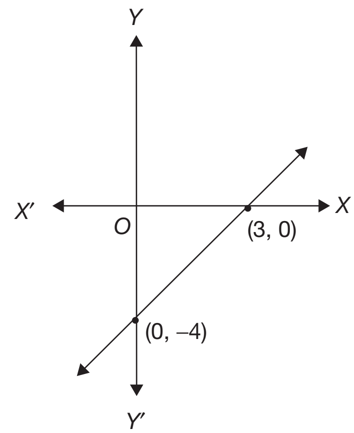


Figure 15.18

SOLUTION

Given,

x-intercept $(a)=3$

y-intercept $(b)=-4$

∴ The equation of the required line is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\text{i.e., } \frac{x}{3}+\frac{y}{-4}=1$$

$$4x-3y=12 \quad \text{(or)} \quad 4x-3y-12=0.$$

Area of Triangle

Consider $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ as the vertices of ΔABC.

Draw perpendiculars AP, BQ and CR on to X-axis.

Area of ΔABC = Area of trapezium $APQB$ + Area of trapezium $APRC$ − Area of trapezium $BCRQ$

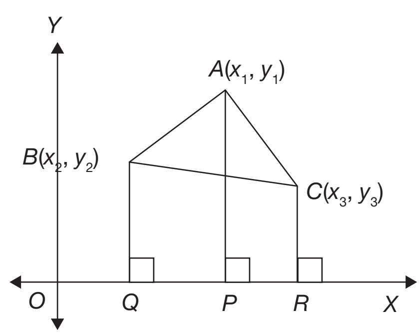


Figure 15.19

$$=\frac{1}{2}QP(AP+BQ)+\frac{1}{2}PR(AP+CR)-\frac{1}{2}QR(BQ+CR)$$

Here $QP=x_1-x_2$, $PR=x_3-x_1$, $QR=x_3-x_2$,

$AP=y_1$, $BQ=y_2$, $CR=y_3$

$$=\frac{1}{2}(x_1-x_2)(y_1+y_2)+\frac{1}{2}(x_3-x_1)(y_1+y_3)-\frac{1}{2}(x_3-x_2)(y_2+y_3)$$

$$=\frac{1}{2}(x_1y_2-x_1y_3-x_2y_1+x_2y_3+x_3y_1-x_3y_2)$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

As the area is always positive

∴ Area of ΔABC, $\Delta=\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$ sq. units

or

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1-x_2 & y_1-y_2 \\ x_2-x_3 & y_2-y_3 \end{vmatrix} \text{ sq. units.}$$

Notes

1. Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and $(0, 0)$ is, $\Delta = \frac{1}{2}|x_1y_2 - x_2y_1|$.
2. Area of ΔABC is zero, if the points A, B and C are collinear.
3. Area of triangle DEF formed by the mid-points of the sides of the ΔABC is $\frac{1}{4}$ of the area of ΔABC, i.e., Area of $\Delta ABC = 4$(Area of ΔDEF).
4. If G is the centroid of ΔABC, then Area of $\Delta ABC = 3$(Area of ΔAGB) $= 3$(Area of ΔBGC) $= 3$(Area of ΔACG).

Area of a Quadrilateral

Area of a quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is given by $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$.

EXAMPLE 15.22

Find the area of the triangle whose vertices are $A(1, -2)$, $B(3, 4)$ and $C(2, 3)$.

SOLUTION

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 1-3 & -2-4 \\ 3-2 & 4-3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -6 \\ 1 & 1 \end{vmatrix} = \frac{1}{2}|-2-(-6)| = 2$ sq. units.

EXAMPLE 15.23

Find the value of p, if the points $A(2, 3)$, $B(-1, 6)$ and $C(p, 4)$ are collinear.

SOLUTION

Given, the points $A(2, 3)$, $B(-1, 6)$ and $C(p, 4)$ are collinear.

$\therefore$ Area of $\Delta ABC = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2-(-1) & 3-6 \\ -1-p & 6-4 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3 & -3 \\ -1-p & 2 \end{vmatrix} = 0$$

$\Rightarrow 6 - 3(1 + p) = 0 \Rightarrow p = 1$

Hence, $p = 1$.

Section Formulae

Section Formula

Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $C(x, y)$ be any point on AB that divide AB in the ratio $m : n$.

Draw perpendiculars AL, CN and BM to X-axis.

AP and CQ are perpendiculars drawn to CN and BM.

Now it is clear that ΔAPC and ΔCQB are similar.

$$\therefore \frac{AC}{CB} = \frac{AP}{CQ} = \frac{CP}{BQ} \qquad (1)$$

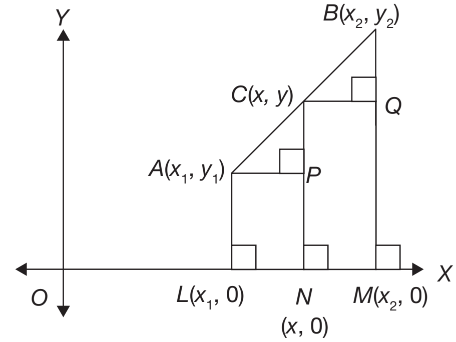


Figure 15.20

Here, $LN = x - x_1$ and $NM = x_2 - x$.

From Eq. (1), we have

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}, \quad (AP = LN \text{ and } CQ = NM)$$

$$\Rightarrow mx_2 - mx = nx - nx_1.$$

$$\Rightarrow mx_2 + nx_1 = mx + nx.$$

$$\therefore x = \frac{mx_2 + nx_1}{m + n}.$$

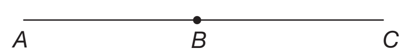
Similarly we can obtain, $\dfrac{m}{n} = \dfrac{y - y_1}{y_2 - y} \Rightarrow y = \dfrac{my_2 + ny_1}{m + n}$.

Hence, the coordinates of 'C' are $\left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}\right)$.

In this case we notice that point C lies in between A and B. So we say that C divides AB in the ratio $m : n$ internally.

Notes

1. When C does not lie between A and B, i.e., as shown below, then we say that C divides AB in $m : n$ ratio externally, then the coordinates of C are $\left(\dfrac{mx_2 - nx_1}{m - n}, \dfrac{my_2 - ny_1}{m - n}\right)$.



2. Let $P(x, y)$ divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$, then, $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$ (or) $\dfrac{y - y_1}{y_2 - y}$.
3. X-axis divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $-y_1 : y_2$ (or) $y_1 : -y_2$.
5. Y-axis divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $-x_1 : x_2$.

EXAMPLE 15.24

Find the coordinates of the point P which divides the line segment joining the points $A(3, -2)$ and $B(2, 6)$ internally in the ratio 2 : 3.

SOLUTION

Given, $P(x, y)$ divides AB internally in the ratio 2 : 3.

So, $P = \left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}\right)$.

Here, $(x_1, y_1) = (3, -2)$, $(x_2, y_2) = (2, 6)$ and $m : n = 2 : 3$

$$\therefore P = \left(\frac{2(2) + 3(3)}{2 + 3}, \frac{2(6) + 3(-2)}{2 + 3}\right) = \left(\frac{13}{5}, \frac{6}{5}\right).$$

Hence, $P\left(\dfrac{13}{5}, \dfrac{6}{5}\right)$ is the required point.

EXAMPLE 15.25

Find the coordinates of a point P which divides the line segment joining the points $A(1, 3)$ and $B(3, 4)$ externally in the ratio 3 : 4.

SOLUTION

Given, P divides AB externally in the ratio 3 : 4.

So, $P = \left(\dfrac{mx_2 - nx_1}{m - n}, \dfrac{my_2 - ny_1}{m - n}\right)$

Here, $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (3, 4)$ and $m : n = 3 : 4$.

$\therefore P = \left(\dfrac{3(3) - 4(1)}{3 - 4}, \dfrac{3(4) - 4(3)}{3 - 4}\right) = (-5, 0)$

Hence, $P = (-5, 0)$.

EXAMPLE 15.26

Find the ratio in which the point $P(3, -2)$ divides the line segment joining the points $A(1, 2)$ and $B(-1, 6)$.

SOLUTION

The ratio in which P divides AB is $AP : PB = (3 - 1) : (-1 - 3) = 2 : -4 = -1 : 2$.

Hence, P divides AB externally in the ratio 1 : 2.

EXAMPLE 15.27

Find the ratio in which the line joining the points $(2, -3)$ and $(3, 1)$ is divided by X-axis and Y-axis.

SOLUTION

The ratio in which X-axis divides is $-y_1 : y_2$,

i.e., $-(-3) : 1 = 3 : 1$

The ratio in which Y-axis divides is $-x_1 : x_2$

$= -2 : 3$

That is, 2 : 3 externally.

Mid-point

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points and M be the mid-point of AB.

Then, M divides AB in the ratio 1 : 1 internally.

So, $M = \left(\dfrac{1 \cdot x_2 + 1 \cdot x_1}{1 + 1}, \dfrac{1 \cdot y_2 + 1 \cdot y_1}{1 + 1}\right) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Hence, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are given by $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Points of Trisection

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points. Then, the two points which divide AB in the ratio 1 : 2 and 2 : 1 internally are called the points of trisection of AB.

Further, if P and Q are the points of trisection of AB respectively,

Then $P = \left(\dfrac{2x_1 + x_2}{3}, \dfrac{2y_1 + y_2}{3}\right)$ and $Q = \left(\dfrac{x_1 + 2x_2}{3}, \dfrac{y_1 + 2y_2}{3}\right)$.

Notes

1. If the mid-points of ΔABC are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$, then its vertices are $A(-x_1 + x_2 + x_3, -y_1 + y_2 + y_3)$, $B(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$ and $C(x_1 + x_2 - x_3, y_1 + y_2 - y_3)$.
2. The fourth vertex of a parallelogram whose three vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) in order is $(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$.

EXAMPLE 15.28

Find the mid-point of the line segment joining the points (1, −3) and (6, 5).

SOLUTION

Let $A(1, -3)$ and $B(6, 5)$ be the given points and M be the mid-point of AB.

Then, $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right) = \left(\dfrac{1+6}{2}, \dfrac{-3+5}{2}\right) = \left(\dfrac{7}{2}, 1\right)$

Hence, the mid-point of AB is $\left(\dfrac{7}{2}, 1\right)$.

EXAMPLE 15.29

Find the points of trisection of the line segment joining the points (3, −2) and (4, 1).

SOLUTION

Let $A(3, -2)$ and $B(4, 1)$ be the given points.

Let P and Q be the points of trisection of AB respectively.

Then, $P = \left(\dfrac{2(3)+4}{3}, \dfrac{2(-2)+1}{3}\right)$ and $Q = \left(\dfrac{3+2(4)}{3}, \dfrac{-2+2(1)}{3}\right)$

$\Rightarrow P = \left(\dfrac{10}{3}, -1\right)$ and $Q = \left(\dfrac{11}{3}, 0\right)$.

Hence, the points of trisection are $\left(\dfrac{10}{3}, -1\right)$ and $\left(\dfrac{11}{3}, 0\right)$.

EXAMPLE 15.30

Find the fourth vertex of the rhombus formed by (−1, −1), (6, 1) and (8, 8).

SOLUTION

Let the three vertices of rhombus be $A(-1, -1)$, $B(6, 1)$ and $C(8, 8)$, then fourth vertex $D(x, y)$ is given by,

$$\begin{aligned} D(x, y) &= (x_1 - x_2 + x_3, y_1 - y_2 + y_3) \\ &= (-1 - 6 + 8, -1 - 1 + 8) \\ &= (1, 6) \end{aligned}$$

Hence, the required fourth vertex is $D(1, 6)$.

EXAMPLE 15.31

Find the area of the triangle formed by the mid-points of the sides of ΔABC, where $A(3, 2)$, $B(-5, 6)$ and $C = (8, 3)$.

SOLUTION

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 3-(-5) & 2-6 \\ -5-8 & 6-3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 8 & -4 \\ -13 & 3 \end{vmatrix} = \frac{1}{2}|8(3) - 4(13)|$

$= \frac{1}{2}|24 - 52|$

$= \frac{1}{2}|-28| = 14$ sq. units.

Hence, the area of triangle formed by the mid-points of the sides of $\Delta ABC = \frac{1}{4}$(Area of ΔABC)

$= \frac{1}{4}(14) = 3.5$ sq. units.

Centroid

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and G be its centroid. Then, the coordinates of G are given by, $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

EXAMPLE 15.32

Find the centroid of ΔABC whose vertices, are $A(1, -3)$, $B(-3, 6)$ and $C(-4, 3)$.

SOLUTION

Given, $A(1, -3)$, $B(-3, 6)$ and $C(-4, 3)$.

So, centroid of $\Delta ABC = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{1 - 3 - 4}{3}, \frac{-3 + 6 + 3}{3}\right) = (-2, 2)$.

Hence, $(-2, 2)$ is the centroid of ΔABC.

EXAMPLE 15.33

Find the centroid of the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 10$ as sides.

SOLUTION

Let OAB be the triangle formed by the given lines.

O is the point of intersections of $x = 0$ and $y = 0$, i.e., origin $\Rightarrow O(0, 0)$.

A is the point of intersection of $x = 0$ and $x + y = 10$, i.e., $A(0, 10)$ and B is the point of intersection of $y = 0$ and $x + y = 10$, i.e., $B(10, 0)$.

$\therefore$ Centroid of ΔOAB is

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$= \left(\frac{0+0+10}{3}, \frac{0+10+0}{3}\right)$$

$$= \left(\frac{10}{3}, \frac{10}{3}\right).$$

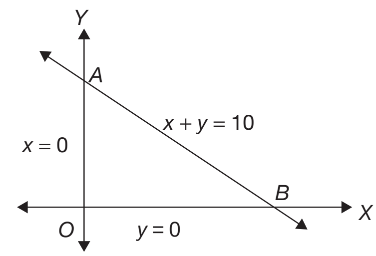


Figure 15.21

EXAMPLE 15.34

Find the third vertex of ΔABC if two of its vertices are $A(-3, 2)$, $B(1, 5)$ and its centroid is $G(3, -4)$.

SOLUTION

Let $C(x, y)$ be the third vertex.

Given, centroid of $\Delta ABC = (3, -4)$

$$\Rightarrow \left(\frac{-3+1+x}{3}, \frac{2+5+y}{3}\right) = (3, -4)$$

$$\Rightarrow \frac{-2+x}{3} = 3, \frac{7+y}{3} = -4$$

$$\Rightarrow x = 11, y = -19$$

$\therefore$ The third vertex is $(11, -19)$.

Equation of a Line Parallel or Perpendicular to the Given Line

Let $ax + by + c = 0$ be the equation of a straight line, then,

1. The equation of a line passing through the point (x_1, y_1) and parallel to the given line is given by $a(x - x_1) + b(y - y_1) = 0$.
2. The equation of a line passing through the point (x_1, y_1) and perpendicular to the given line is given by $b(x - x_1) - a(y - y_1) = 0$.

EXAMPLE 15.35

Find the equation of a line passing through the point $A(2, -3)$ and parallel to the line $2x - 3y + 6 = 0$.

SOLUTION

Here, $(x_1, y_1) = (2, -3)$, $a = 2$ and $b = -3$.

∴ Equation of the line passing through $A(2, -3)$ and parallel to the line $2x - 3y + 6 = 0$ is $a(x - x_1) + b(y - y_1) = 0$, i.e.,

$$2(x - 2) - 3(y + 3) = 0$$
$$\Rightarrow 2x - 3y - 13 = 0$$

Hence, the equation of the required line is $2x - 3y - 13 = 0$.

EXAMPLE 15.36

Find the equation of a line passing through the point (5, 2) and perpendicular to the line $3x - y + 6 = 0$.

SOLUTION

Here, $(x_1, y_1) = (5, 2)$, $a = 3$ and $b = -1$.

∴ Equation of the line perpendicular to $3x - y + 6 = 0$ and passing through the point (5, 2) is $b(x - x_1) - a(y - y_1) = 0$, i.e.,

$$-1(x - 5) - 3(y - 2) = 0$$
$$\Rightarrow (x - 5) + 3(y - 2) = 0$$
$$\Rightarrow x + 3y - 11 = 0$$

Hence, the equation of the required line is $x + 3y - 11 = 0$.

EXAMPLE 15.37

The line $(5x - 8y + 11)\lambda + (8x - 3y + 4) = 0$ is parallel to X-axis. Find λ.

SOLUTION

The given line is,

$(5x - 8y + 11)\lambda + (8x - 3y + 4) = 0$

That is, $x(5\lambda + 8) - y(8\lambda + 3) + (11\lambda + 4) = 0$.

As the given line is parallel to X-axis, its slope = 0, i.e.,

$$\frac{-(5\lambda + 8)}{-(8\lambda + 3)} = 0$$
$$\Rightarrow 5\lambda + 8 = 0.$$

Hence, $\lambda = \frac{-8}{5}$.

Median of the Triangle

A line drawn from the vertex, which bisects the opposite side is called a median of the triangle.

EXAMPLE 15.38

Find the median to the side BC of the triangle whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

SOLUTION

Let E be the mid-point of side BC.

$\therefore E = \left(\frac{2+4}{2}, \frac{3+5}{2}\right) = (3, 4)$

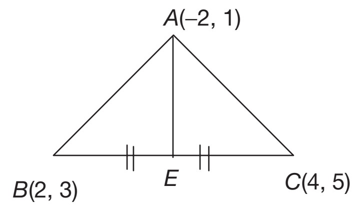


Figure 15.22

Equation of line AE is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y - 1 = \frac{4-1}{3-(-2)}(x+2)$$

$$y - 1 = \frac{3}{5}(x+2)$$

$$5y - 5 = 3x + 6$$

$\therefore$ The required equation of the median is $3x - 5y + 11 = 0$.

Altitude of the Triangle

A perpendicular dropped from the vertex to the opposite side in a triangle is called an altitude.

EXAMPLE 15.39

Find the equation of the altitude drawn to side BC of ΔABC, whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

SOLUTION

Let AD be the altitude drawn to side BC.

Slope of $BC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = 1$

$\therefore$ Slope of $AD = -1$ ($\because AD \perp BC$)

Equation of AD is,

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -1(x+2)$$

$$y - 1 = -x - 2$$

$\Rightarrow x + y + 1 = 0.$

$\therefore$ The required equation of altitude is $x + y + 1 = 0$.

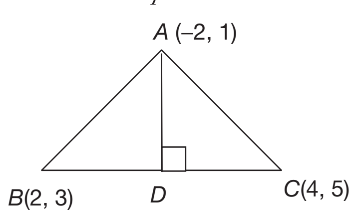


Figure 15.23

EXAMPLE 15.40

Find the equation of the perpendicular bisector drawn to the side BC of ΔABC, whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

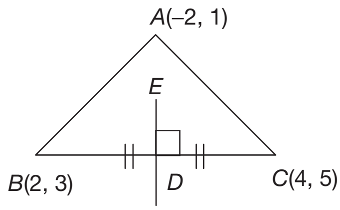


Figure 15.24

SOLUTION

$\therefore D = \left(\dfrac{2+4}{2}, \dfrac{3+5}{2}\right)$

$D = (3, 4)$

Slope of $BC = \dfrac{5-3}{4-2} = 1$

$\Rightarrow$ Slope of $ED = -1$ ($\because ED \perp BC$)

Equation of ED is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 4 = -1(x - 3)$$

$$\Rightarrow y - 4 = -x + 3$$

$$\Rightarrow x + y - 7 = 0$$

$\therefore$ The required equation of the perpendicular bisector is $x + y - 7 = 0$.

TEST YOUR CONCEPT

Very Short Answer Type Questions

1. If the inclination of a line is 45°, then the slope of the line is _______.
2. If the point (x, y) lies in the third quadrant, then x is _______ and y is _______.
3. The point of intersection of the lines $x = 2$ and $y = 3$ is _______.
4. The point of intersection of medians of triangle is called its _______.
5. The line $y + 7 = 0$ is parallel to _______ axis.
6. The distance of a point (2, 3) from Y-axis is _______.
7. The slope of the line perpendicular to $5x + 3y + 1 = 0$ is _______.
8. If (x, y) represents a point and $|x| > 0$ and $y > 0$ then in which quadrants can the point lie?
9. The line $4x + 7y + 9 = 0$ meet the X-axis at _______ and Y-axis at _______.
10. If (x, y) represents a point and $xy < 0$, then the point may lie in _______ or _______ quadrant.
11. The area of the triangle, with vertices $A(2, 0)$, $B(0, -4)$ and origin is _______.
12. The area of the triangle formed by the points (0, 0), (0, a), (b, 0) is _______.
13. The area of the triangle formed by the line $ax + by + c = 0$ with coordinate axes is _______.
14. If the points (5, 5), (7, 7) and (a, 8) are collinear, then the value of a is _______.
15. Area of a triangle whose vertices are (0, 0), (x_1, y_1), (x_2, y_2) is _______.
16. The end vertices of one diagonal of a parallelogram are (1, 3) and (5, 7), then the mid-point of the other diagonal is _______.
17. The ortho-centre of the triangle formed by the points (0, 1), (1, 2) and (0, 2) is _______.
18. The two straight lines, $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular to each other, then $m_1m_2 =$ _______.
19. If $A(4, 0)$, $B(0, -6)$ are the two vertices of a triangle OAB where O is origin, then the circumcentre of the triangle OAB is _______.
20. If the centre and radius of a circle is (3, 4) and 7, then the position of the point (5, 3) wrt the circle is _______.
21. If $A(2, 3)$, $B(x, y)$ and $C(4, 3)$ are the vertices of a right triangle, right angled at A, then $x =$ _______.
22. The coordinates of the points P which divides (1, 0) and (0, 0) in 1 : 2 ratio are _______.
23. The points A, B and C represents the vertices of a ΔABC. AD is the median drawn from A to BC, then the centroid of the triangle divides AD in _______ ratio.
24. If (5, 7) and (9, 3) are the ends of the diameter of a circle, then the centre of the circle is _______.
25. What is the formula for calculating the area of a quadrilateral whose vertices are (x_1, y_1),(x_2, y_2), (x_3, y_3), (x_4, y_4)?
26. The centroid of the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 6$ is _______.
27. The line $ax + by + c = 0$ is such that $a = 0$ and $bc \neq 0$, then the line is perpendicular to _______ axis.
28. If A is one of the points of trisection of the line joining B and C, then A divides BC in the ratio _______ (or) _______.
29. The ratio that the line joining points (3, −6) and (4, 9) is divided by X-axis is _______.
30. If (1, 2), (3, 4) and (0, 6) are the three vertices of a parallelogram taken in that order, then the fourth vertex is _______.

Short Answer Type Questions

31. Find λ, if the line $(3x - 2y + 5) + \lambda(3x - y + 4) = 0$ passes through the mid-point of the line joining the points $A(2, 3)$ and $B(4, 9)$.
32. Find the distance between the points (3, −5) and (−4, 7).

33. Find the slope of the line perpendicular to $\overleftrightarrow{AB}$ where $A(5, -6)$ and $B(2, -7)$.

34. If $A(-2, -1)$, $B(-4, 5)$ and $C(2, 3)$ are three vertices of the parallelogram $ABCD$, then find the vertex D.

35. Find the equation of line parallel to Y-axis and passing through the point $(3, -5)$.

36. Find the equation of a line, whose inclination is $30°$ and making an intercept of $\frac{-3}{5}$ on Y-axis.

37. Find the equation of a straight line whose slope is -5 and making an intercept 3 on the Y-axis.

38. Find the centroid of a triangle whose vertices are $(3, -1)$, $(2, 4)$ and $(-8, 6)$.

39. Find the equation of a line passing through the point $(2, -3)$ and parallel to the line $2x - 3y + 8 = 0$.

40. If $A(2, -2)$, $B(3, 4)$ and $C(7, 2)$ are the mid points of the sides PQ, QR and RP respectively of ΔPQR, then find its vertices.

41. Let $(-3, 2)$ be one end of a diameter of a circle with centre $(4, 6)$. Find the other end of the diameter.

42. Let $A(-1, 2)$ and $D(3, 4)$ be the ends points of the median AD of ΔABC. Find the centroid of ΔABC.

43. Find the point of intersection of the lines $3x + 5y + 2 = 0$ and $4x + 7y + 3 = 0$.

44. Find the coordinates of the point which divides the line joining the points $A(-3, 2)$ and $B(2, 6)$ internally in the ratio $3 : 2$.

45. Find the point on Y-axis which is equidistant from $A(3, -6)$ and $B(-2, 5)$.

Essay Type Questions

46. Find the equation of a line parallel to the line $2x + 3y - 6 = 0$ and where sum of intercepts is 10.

47. Find the equation of a line passing through the point of intersection of the lines $5x - y - 7 = 0$ and $3x - 2y - 7 = 0$ and parallel to the $x + 3y - 5 = 0$.

48. Find the equation of a line with y-intercept -4 and perpendicular to a line passing through the points $A(1, -2)$ and $B(-3, 2)$.

49. Find the circum-centre of the triangle formed by the points $(2, 3)$, $(1, -5)$ and $(-1, 4)$.

50. Let $A(3, 2)$, $B(-4, 1)$, $C(-3, 1)$ and $D(2, -4)$ be the vertices of a quadrilateral $ABCD$. Find the area of the quadrilateral formed by the mid-points of the sides of the quadrilateral $ABCD$.

CONCEPT APPLICATION

Level 1

1. The lines, $x = 2$ and $y = 3$ are
   (a) parallel to each other.
   (b) perpendicular to each other.
   (c) neither parallel nor perpendicular to each other.
   (d) None of these

2. The lines, $x = -2$ and $y = 3$ intersect at the point ______.
   (a) $(-2, 3)$ (b) $(2, -3)$
   (c) $(3, -2)$ (d) $(-3, 2)$

3. The slope of the line joining the points $(2, k - 3)$ and $(4, -7)$ is 3. Find k.
   (a) -10 (b) -6
   (c) -2 (d) 10

4. The centre of a circle is $C(2, -3)$ and one end of the diameter AB is $A(3, 5)$. Find the coordinates of the other end B.
   (a) $(1, -11)$ (b) $(5, 2)$
   (c) $(1, 8)$ (d) $(1, 11)$

5. The angle made by the line $\sqrt{3}x - y + 3 = 0$ with the positive direction of X-axis is ______.

(a) 30° (b) 45°

(c) 60° (d) 90°

6. The points on X-axis which are at a distance of $\sqrt{13}$ units from (–2, 3) is _______.

(a) (0, 0),(–2, –3) (b) (0, 0), (–4, 0)

(c) (0, 0), (2, 3) (d) (0, 0), (4, 0)

7. The point P lying in the fourth quadrant which is at a distance of 4 units from X-axis and 3 units from Y-axis is _______.

(a) (4, –3) (b) (4, 3)

(c) (3, –4) (d) (–3, 4)

8. The radius of a circle with centre (–2, 3) is 5 units. The point (2, 5) lies

(a) on the circle.

(b) inside the circle.

(c) outside the circle.

(d) None of these

9. The points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$

(a) are collinear.

(b) form a scalene triangle.

(c) form an equilateral triangle.

(d) None of these

10. Find λ, if the line $3x - \lambda y + 6 = 0$ passes through the point (–3, 4).

(a) $\frac{3}{4}$ (b) $\frac{-3}{4}$

(c) $\frac{4}{3}$ (d) $\frac{-4}{3}$

11. If $A(-2, 3)$ and $B(2, 3)$ are two vertices of ΔABC and $G(0, 0)$ is its centroid, then the coordinates of C are _______.

(a) (0, –6) (b) (–4, 0)

(c) (4, 0) (d) (0, 6)

12. Let ΔABC be a right triangle in which $A(0, 2)$ and $B(2, 0)$. Then the coordinates of C can be _______.

(a) (0, 0)

(b) (2, 2)

(c) Either (a) or (b)

(d) None of these

13. If $A(4, 7)$, $B(2, 5)$, $C(1, 3)$ and $D(-1, 1)$ are the four points, then the lines AC and BD are _______.

(a) perpendicular to each other

(b) parallel to each other

(c) neither parallel nor perpendicular to each other

(d) None of these

14. Find the area of the triangle formed by the line $5x - 3y + 15 = 0$ with coordinate axes.

(a) 15 cm^2 (b) 5 cm^2

(c) 8 cm^2 (d) $\frac{15}{2}$ cm^2

15. Equation of a line whose inclination is 45° and making an intercept of 3 units on X-axis is

(a) $x + y - 3 = 0$.

(b) $x - y - 3 = 0$.

(c) $x - y + 3 = 0$.

(d) $x + y + 3 = 0$.

16. The centre of a circle is $C(2, k)$. If $A(2, 1)$ and $B(5, 2)$ are two points on its circumference, then the value of k is _______.

(a) 6 (b) 2

(c) –6 (d) –2

17. The lines $x = -1$ and $y = 4$ are _______.

(a) perpendicular to each other

(b) parallel to each other

(c) neither parallel nor perpendicular to each other

(d) None of these

18. The distance between the points $(2k + 4, 5k)$ and $(2k, -3 + 5k)$ in units is _______.

(a) 1 (b) 2

(c) 4 (d) 5

19. The equation of the line with inclination 45° and passing through the point (–1, 2) is _______.

(a) $x + y + 3 = 0$ (b) $x - y + 3 = 0$

(c) $x - y - 3 = 0$ (d) $x + y - 3 = 0$

20. The distance between the points $(3k + 1, -3k)$ and $(3k - 2, -4 - 3k)$ (in units) is _______.

(a) $3k$ (b) $5k$

(c) 5 (d) 3

PRACTICE QUESTIONS

21. The angle made by the line $x-\sqrt{3}y+1=0$ with the positive Y-axis is _______.

(a) 60° (b) 30°

(c) 45° (d) 90°

22. If ΔABC is a right triangle in which $A(3, 0)$ and $B(0, 5)$, then the coordinates of C can be _______.

(a) (5, 3) (b) (3, 5)

(c) (0, 0) (d) Both (b) and (c)

23. If the roots of the quadratic equation $2x^2 - 5x + 2 = 0$ are the intercepts made by a line on the coordinate axes, then the equation of the line can be

(a) $4x + y = 2.$ (b) $2x + 5y + 2.$

(c) $x + 4y = 2.$ (d) Both (a) and (c)

24. The inclination of the line $\sqrt{3}x-y+5=0$ with X-axis is _______.

(a) 90° (b) 45°

(c) 60° (d) 30°

25. The equation of the line parallel to $3x - 2y + 7 = 0$ and making an intercept -4 on X-axis is

(a) $3x - 2y + 12 = 0.$

(b) $3x - 2y - 12 = 0.$

(c) $3x + 2y - 12 = 0.$

(d) $3x + 2y + 12 = 0.$

26. A triangle is formed by the lines $x + y = 8$, X-axis and Y-axis. Find its centroid.

(a) $\left(\frac{8}{3}, \frac{8}{3}\right)$ (b) (8, 8)

(c) (4, 4) (d) (0, 0)

27. The point which divides the line joining the points $A(1, 2)$ and $B(-1, 1)$ internally in the ratio 1 : 2 is _______.

(a) $\left(\frac{-1}{3}, \frac{5}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{5}{3}\right)$

(c) $(-1, 5)$ (d) (1, 5)

28. Find the area of the triangle formed by the line $3x - 4y + 12 = 0$ with the coordinate axes.

(a) 6 units^2 (b) 12 units^2

(c) 1 units^2 (d) 36 units^2

29. The equation of a line whose sum of intercepts is 5 and the area of the triangle formed by the line with positive coordinate axis is 2 sq. units can be

(a) $x + y = 4.$ (b) $x + 4y = 4.$

(c) $y + 4x + 4 = 0.$ (d) $y = x + 4.$

30. Find the equation of a line which divides the line segment joining the points (1, 1) and (2, 3) in the ratio 2 : 3 perpendicularly.

(a) $5x - 5y + 2 = 0$

(b) $5x + 5y + 2 = 0$

(c) $x + 2y - 5 = 0$

(d) $x + 2y + 7 = 0$

Level 2

31. The equation of the line making an angle of 45° with X-axis in positive direction and having y-intercept as -3 is _______.

(a) $3x - y + 1 = 0$

(b) $3x + y - 1 = 0$

(c) $x - y + 3 = 0$

(d) $x - y = 3$

32. The ratio in which the line joining points $(a + b, b + a)$ and $(a - b, b - a)$ is divided by the point(a, b) is _______.

(a) $b : a$ internally (b) 1 : 1 internally

(c) $a : b$ externally (d) 2 : 1 externally

33. Which of the following lines is perpendicular to $x + 2y + 3 = 0$?

(a) $2\sqrt{2}x-y+3=0$

(b) $\sqrt{2}x+\sqrt{2}y-5=0$

(c) $2\sqrt{2}x-\sqrt{2}y+3=0$

(d) $x+\sqrt{2}y+4=0$

PRACTICE QUESTIONS

34. The ortho-centre of the triangle formed by the points (0, 3), (0, 0) and (1, 0) is _______.

(a) $\left(\frac{1}{3}, 1\right)$ (b) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(c) (0, 0) (d) $\left(0, \frac{3}{2}\right)$

35. The equation of the line passing through the point of intersection of the lines $x + 2y + 3 = 0$ and $2x - y + 5 = 0$ and parallel to X-axis is _______.

(a) $5y + 1 = 0$ (b) $5x - 13 = 0$

(c) $5x + 13 = 0$ (d) $5y - 1 = 0$

36. Find the equation of a line which divides the line segment joining the points (1, –2) and (3, –1) in the ratio 3 : 1 perpendicularly.

(a) $x - 2y - 5 = 0$

(b) $6x + 4y - 5 = 0$

(c) $3x + 2y - 5 = 0$

(d) $8x + 4y - 15 = 0$

37. If $x + p = 0$, $y + 4 = 0$ and $x + 2y + 4 = 0$ are concurrent, then $p =$ _______.

(a) 4 (b) –2

(c) –4 (d) 2

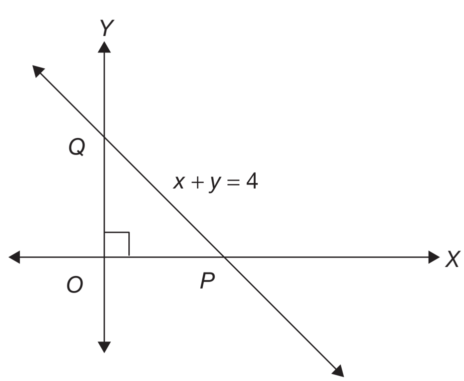
38. The perpendicular bisector of the side PQ is

(a) $x - y = 0$.

(b) $x + y - 2 = 0$.

(c) $3x - 2y - 2 = 0$.

(d) $x + 2y - 6 = 0$.



39. The ortho-centre of the triangle formed by the vertices $A(4, 6)$, $B(4, 3)$ and $C(2, 3)$ is _______.

(a) (2, 3) (b) (4, 3)

(c) (4, 6) (d) (3, 4)

40. The circum-centre and the ortho-centre of the triangle formed by the sides $y = 0$, $x = 0$ and $2x + 3y = 6$ are respectively _______.

(a) (3, 2), (0, 0) (b) $\left(\frac{3}{2}, 1\right)$, (0, 0)

(c) (–3, 1), (0, 0) (d) $\left(-\frac{3}{2}, 1\right)$, (0, 0)

41. The equation of median drawn to the side BC of ΔABC whose vertices are $A(1, -2)$, $B(3, 6)$ and $C(5, 0)$ is _______.

(a) $5x - 3y - 11 = 0$

(b) $5x + 3y - 11 = 0$

(c) $3x - 5y + 11 = 0$

(d) $3x - 5y - 11 = 0$

42. Find the equation of the line passing through (1, 1) and forming an area of 2 sq. units with positive coordinate axis.

(a) $2x + 3y = 5$ (b) $x - y + 2 = 0$

(c) $x + y - 2 = 0$ (d) $x - y + 1 = 0$

43. If the vertices of a triangle are $A(3, -3)$, $B(-3, 3)$ and $C(-3\sqrt{3}, -3\sqrt{3})$, then the distance between the ortho-centre and the circum-centre is _______.

(a) $6\sqrt{2}$ units (b) $6\sqrt{3}$ units

(c) 0 unit (d) None of these

44. The circum-centre of the triangle formed by the lines $x + 4y = 7$, $5x + 3y = 1$ and $3x - 5y = 21$ is _______.

(a) (–3, 2) (b) (3, 1)

(c) (3, –1) (d) (–3, –2)

45. If the line $(3x - 8y + 5) + a(5x - 3y + 10) = 0$ is parallel to X-axis, then a is _______.

(a) $-\frac{8}{3}$ (b) $-\frac{3}{5}$

(c) –2 (d) $-\frac{1}{2}$

46. Find the equation of the median of the triangle, formed by the line $8x + 5y = 40$ with the coordinate axes. Given that the median is passing through the origin.

(a) $5x - 8y = 0$ (b) $5x + 8y = 0$

(c) $8x - 5y = 0$ (d) $8x + 5y = 0$

PRACTICE QUESTIONS

47. Find the area of a triangle formed by the lines $4x - y - 8 = 0$, $2x + y - 10 = 0$ and $y = 0$ (in sq units).

(a) 5 (b) 6

(c) 4 (d) 3

48. Find the length of the longest side of the triangle formed by the line $3x + 4y = 12$ with the coordinate axes.

(a) 9 (b) 16

(c) 5 (d) 7

49. The following are the steps involved in finding the area of the triangle with vertices (2, 3), (4, 7) and (8, 5). Arrange them in the sequential order from first to last.

(A) $\frac{1}{2}|(-2)2-(-4)(-4)| = \frac{1}{2}|-20| = 10$ sq. units

(B) Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$

(C) Let $(x_1, y_1) = (2, 3)$, $(x_2, y_2) = (4, 7)$ and $(x_3, y_3) = (8, 5)$

(D) Area of triangle $= \frac{1}{2}\begin{vmatrix} 2-4 & 4-8 \\ 3-7 & 7-5 \end{vmatrix}$

(a) ABCD (b) CBDA

(c) CBAD (d) None of these

50. The following are the steps involved in finding the centroid of the triangle with vertices (5, 4), (6,7) and (1, 1). Arrange them in sequential order from first to last.

(A) The required centroid (4, 4).

(B) The centroid of the triangle with the vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

(C) The centroid is $\left(\frac{5+6+1}{3}, \frac{4+7+1}{3}\right)$.

(a) ABC (b) CBA

(c) BAC (d) BCA

Level 3

51. In what ratio does the line $4x + 3y - 13 = 0$ divide the line segment joining the points (2, 1) and (1, 4)?

(a) 3 : 2 internally

(b) 2 : 3 externally

(c) 2 : 3 internally

(d) 3 : 2 externally

52. If $A(3, 4)$, $B(1, -2)$ are the two vertices of triangle ABC and $G(3, 5)$ is the centroid of the triangle, then the equation of AC is ______.

(a) $4x - 5y - 7 = 0$

(b) $4x - 5y + 8 = 0$

(c) $9x - 2y - 23 = 0$

(d) $9x - 2y - 19 = 0$

53. If $ax + 4y + 3 = 0$, $bx + 5y + 3 = 0$ and $cx + 6y + 3 = 0$ are concurrent lines, then $a + c =$ ______.

(a) $3b$ (b) $2b$

(c) b (d) $4b$

54. If (5, 3), (4, 2) and (1, –2) are the mid points of sides of triangle ABC, then the area of ΔABC is

(a) 2 sq. units (b) 3 sq. units

(c) 1 sq. unit (d) 4 sq. units

55. Find the distance between the ortho-centre and circum-centre of the triangle formed by joining the points (5, 7), (4, 10) and (6, 9).

(a) $\sqrt{\frac{5}{4}}$ units (b) $\sqrt{\frac{5}{2}}$ units

(c) $\sqrt{10}$ units (d) $\sqrt{5}$ units

56. Area of the region formed by $4|x| + 3|y| = 12$ is ______.

(a) 18 sq. units (b) 20 sq. units

(c) 24 sq. units (d) 36 sq. units

57. Find the radius of the circle which passes through the origin, (0, 4) and (4, 0).

(a) 2 (b) $4\sqrt{2}$

(c) $\sqrt{8}$ (d) $3\sqrt{2}$

58. Find the circum-centre of the triangle whose vertices are $(0, 0), (3, \sqrt{3})$ and $(0, 2\sqrt{3})$.

(a) $(1, \sqrt{3})$ (b) $(\sqrt{3}, \sqrt{3})$

(c) $(2\sqrt{3}, 1)$ (d) $(2, \sqrt{3})$

59. Find the ortho-centre of the triangle formed by the lines $3x - 4y = 10$, $8x + 6y = 15$ and Y-axis.

(a) $\left(\frac{3}{5}, \frac{7}{5}\right)$ (b) $\left(\frac{12}{5}, \frac{-7}{10}\right)$

(c) $\left(\frac{3}{5}, \frac{12}{5}\right)$ (d) $\left(\frac{12}{5}, \frac{-3}{10}\right)$

60. The equation of a line which passes through (2, 3) and the product of whose intercepts on the coordinate axis is 27, can be _______.

(a) $5x + 4y = 22$

(b) $3x - y = 3$

(c) $3x + 4y = 18$

(d) $2x + 3y = 13$

PRACTICE QUESTIONS

TEST YOUR CONCEPT

Very Short Answer Type Questions

1. 1
2. negative, negative
3. (2, 3)
4. centroid
5. X-axis
6. 2 units
7. $\frac{3}{5}$
8. first quadrant or second quadrant
9. $\left(-\frac{9}{4}, 0\right)$ and $\left(0, -\frac{9}{7}\right)$
10. second quadrant or fourth quadrant
11. 4 sq. units
12. $\frac{1}{2}|ab|$ sq. units
13. $\frac{1}{2}\left|\frac{c^2}{ab}\right|$
14. 8
15. $\frac{1}{2}|x_1y_2 - x_2y_1|$
16. (3, 5)
17. (0, 2)
18. –1
19. (2, –3)
20. inside the circle
21. 2
22. $\left(\frac{2}{3}, 0\right)$
23. 2 : 1
24. (7, 5)
25. $\frac{1}{2}|(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)|(y_1 - y_3)|$ sq. units
26. (2, 2)
27. Y-axis
28. 2 : 1; 1 : 2
29. 2 : 3
30. (–2, 4)

Short Answer Type Questions

31. $\left(\frac{-9}{10}, -\frac{7}{10}\right)$
32. $\sqrt{193}$ units
33. –3
34. (4, –3)
35. $x = 3$
36. $5\sqrt{3}y - 5x + 3\sqrt{3} = 0.$
37. $5x + y - 3 = 0$
38. (–1, 3)
39. $2x - 3y - 13 = 0$
40. $P(6, -4)$, $Q(-2, 0)$ and $R(8, 8)$
41. (11, 10)
42. $\left(\frac{5}{3}, \frac{10}{3}\right)$
43. (1 – 1)
44. $\left(0, \frac{22}{5}\right)$
45. $\left(0, \frac{-8}{11}\right)$

Essay Type Questions

46. $2x + 3y - 12 = 0$
47. $x + 3y + 5 = 0$
48. $x - y - 4 = 0$
49. $\lambda = -\frac{2}{7}$
50. 9 sq. units

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (a)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (b)
11. (a)	**12.** (c)	**13.** (b)	**14.** (d)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (c)
21. (a)	**22.** (d)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (a)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (c)

Level 2

31. (d)	**32.** (b)	**33.** (c)	**34.** (c)	**35.** (a)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (b)	**40.** (b)
41. (a)	**42.** (c)	**43.** (c)	**44.** (b)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (b)	**50.** (d)

Level 3

51. (c)	**52.** (d)	**53.** (b)	**54.** (a)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (a)	**59.** (b)	**60.** (c)

CONCEPT APPLICATION

Level 1

3. Use slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
4. Use the mid-point formula.
5. Slope $(m) = \tan\theta$.
6. Assume the point on X-axis as $(a, 0)$ and use the distance formula.
7. X-coordinate of the point represents the distance of the point from Y-axis.
8. Use the distance formula.
9. Use the condition for collinearity.
10. Substitute the given point in the line.
11. Use the centroid formula.
12. Use right triangle properties.
13. Find the slopes of AC and BD.
14. Find the intercepts made on X-axis and Y-axis. Then, area $= \dfrac{1}{2}|ab|$.
15. Slope $= \tan\theta$ and intercept on X-axis $= 3$.
16. Radius of any circle is constant.
17. $x = -1$ is a vertical line and $y = 4$ is a horizontal line.
18. Use the distance formula.
19. Find slope and use point slope form of the line.
20. Use distance formula.
21. Slope $m = \tan\theta$ and the angle between positive X-axis and Y-axis is 90°.
22. Use $AC^2 = AB^2 + BC^2$.
23. Find the roots and substitute in $\dfrac{x}{a} + \dfrac{y}{b} = 1$.
24. The inclination of a line with the positive X-axis is called the slope.
25. (i) If two lines are parallel, then their slopes are equal.
    (ii) The equation of any line parallel to $ax + by + c = 0$ can be taken as $ax + by + k = 0$.
    (iii) Find the X-intercept of the line and equate it to the given x-intercept then get the value of k.
26. (i) Evaluate the vertices of the triangle and proceed.
    (ii) Find the point, where the line cuts X-axis and Y-axis.
    (iii) These two points and the origin are the vertices of the triangle.
    (iv) Then use the formula centroid
    $$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$
27. Use the section formula.
28. Area of the triangle formed by the line $\dfrac{1}{2}|ab|$, a, b are intercepts of the line.
29. (i) $a + b = 5$ and $\dfrac{1}{2}|ab| = 2$.
    (ii) The equation of a line in the intercept form is
    $$\frac{x}{a} + \frac{y}{b} = 1 \quad \Rightarrow \quad a + b = 5.$$
    (iii) Then use the formula to find the area of the triangle formed by a line with coordinate axis, is $A = \dfrac{1}{2}|ab| \Rightarrow |ab| = 4$.
30. (i) Evaluate the point using section formula then proceed.
    (ii) Find the point on the required line that divides the line joining the points in the ratio 2 : 3.
    (iii) If two lines are perpendicular then $m_1 \times m_2 = -1$.
    (iv) Find the slope of the required line.
    (v) Hence find the equation of the line by using slope-point formula.

Level 2

31. (i) Use $y = mx + c$ where $m = \tan\theta$.
    (ii) Slope of the required line is tan 45°.
    (iii) Use the slope and y-intercept form.

32. (i) Use the section formula.

(ii) The ratio in which (x, y) divides the line joining the points (x_1, y_1) and (x_2, y_2) is $-(x - x_1):(x - x_2)$ or $-(y - y_1):(y - y_2)$. If the ratio is negative, it divides externally otherwise divides internally.

33. (i) Find the slope of the given line.

(ii) Use the concept, if two lines are perpendicular to each other, then $m_1 \times m_2 = -1$.

34. First prove the given triangle is a right triangle. In a right triangle, ortho-centre is the vertex containing right angle.

35. First find the point of intersection(x_1, y_1) of first two lines. The equation of the line parallel to X-axis is $y = y_1$.

36. (i) Find the point diving in the ratio 3 : 1.

(ii) Find the equation of line by using point-slope form.

37. (i) Solve Eqs. (2) and (3).

(ii) If three lines are concurrent then the point of intersection of any two lines, always lie on the third line.

38. (i) Find the coordinates of the points P and Q.

(ii) Then find the mid-point of PQ say M.

(iii) Now the required line is perpendicular to PQ and passes through M.

39. (i) Prove the given vertices form a right triangle.

(ii) In a right angled triangle, the vertex containing right angle is the ortho-centre.

40. Clearly the given lines forms a right triangle. In a right triangle circum-centre is the mid-point of hypotenuse and ortho-centre is the vertex containing 90°.

41. Find the equation of the line passing through A and mid-point of BC.

42. The area of triangle formed by $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is $\frac{1}{2}|ab|$.

43. (i) ABC forms an equilateral triangle.

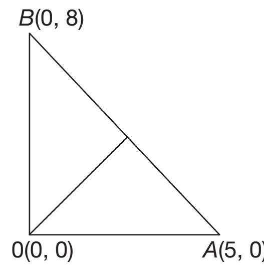
(ii) In an equilateral triangle all the geometric centres except ex-centre coincide.

44. Prove that the given lines form a right triangle, then find the ends of the hypotenuse there by find the mid-point of them.

45. If a line is parallel to X-axis, then its x-coefficient is zero.

46. The vertices of the triangle formed by $8x + 5y = 40$ with the coordinate axes are $0(0, 0)$, $A(5, 0)$, $B(0, 8)$. Mid-point of AB is $\left(\frac{5}{2}, 4\right)$.

Required median is in the form of $y = mx$ and it is passing through $\left(\frac{5}{2}, 4\right)$.



$\Rightarrow 4 = m\left(\frac{5}{2}\right) \Rightarrow m = \frac{8}{5}$

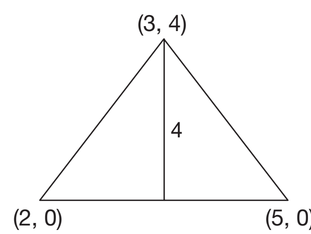
$\therefore y = \frac{8}{5}x \Rightarrow 8x - 5y = 0.$

47. $4x - y - 8 = 0$ and $2x + y - 10 = 0$ intersect at (3, 4) and intersect $y = 0$ at (2, 0) and (5, 0) respectively

Since the base is X-axis

Its height = 4 units

$\Rightarrow \text{Area} = \frac{1}{2} \times (5 - 2) \times 4 = 6$ sq. units



48. $3x + 4y = 12$

$\Rightarrow \frac{x}{4} + \frac{y}{3} = 1$

Vertices of the triangle formed are (4, 0), (0, 3) and (0, 0).

$\Rightarrow$ The longest side is the hypotenuse joining (4, 0) and (0, 3)

$\therefore$ Its length = 5 units.

49. CBDA are in sequential order from first to last.

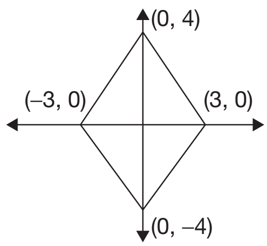
50. BCA are in sequential order from first to last.

Level 3

56. $4\,|x| + 3\,|y| = 12 \quad \Rightarrow 4x - 3y = 12$

$4x + 3y = 12 - 4x - 3y = 12 - 4x + 3y = 12$

These lines form a rhombus.



$$\text{Area} = \frac{1}{2}d_1 d_2 = \frac{1}{2} \times 6 \times 8 = 24 \text{ sq. units.}$$

57. The circle is passing through (0, 0), (0, 4), (4, 0)

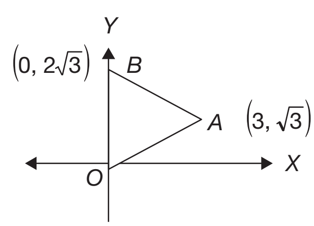
$\Rightarrow$ The circle is the circum-circle of a right triangle whose vertices are (0, 4), (4, 0) and (0, 0)

$\therefore$ Length of the diameter

$$\Rightarrow \sqrt{4^2 + 4^2} = \sqrt{32}$$

$$\text{Radius} = \frac{\sqrt{32}}{2} = \sqrt{8}.$$

58. Let $O(0, 0)$, $A(3, \sqrt{3})$, $B(0, 2\sqrt{3})$



$$OB = \sqrt{(2\sqrt{3})^2} = \sqrt{12}$$

$$AO = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12}$$

$$AB = \sqrt{3^2 + (\sqrt{3} - 2\sqrt{3})^2} = \sqrt{12}$$

$\therefore$ ΔABC is an equilateral triangle

$\Rightarrow \left(\dfrac{0+3+0}{3}, \dfrac{0+\sqrt{3}+2\sqrt{3}}{3}\right) = (1, \sqrt{3})$ is the centroid as well as circum-centre.

59. $3x - 4y = 10$ (1)

$8x + 6y = 15$

$$4x + 3y = \frac{15}{2}$$

Eqs. (1) and (2) are perpendicular lines, their intersecting point is the ortho-centre.

On solving Eqs. (1) and (2), we get the ortho-centre as $\left(\dfrac{12}{5}, \dfrac{-7}{10}\right)$.

60. Let the equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$$\Rightarrow \frac{2}{a} + \frac{3}{b} = 1 \Rightarrow 2b + 3a = ab$$

$\Rightarrow 3a + 2b = 27$

$ab = 27$

$$b = \frac{27}{a}$$

$$\Rightarrow 3a + 2 \times \frac{27}{a} = 27$$

$3a^2 + 54 = 27a$

$3a^2 - 27a + 54 = 0$

$3a^2 - 27a + 54 = 0$

$a^2 - 9a + 18 = 0$

$a^2 - 6a - 3a + 18 = 0$

$a(a - 6) - 3(a - 6) = 0$

$$\Rightarrow a = 6, 3 \Rightarrow b = \frac{9}{2} \text{ or } 9$$

Required equation is $\dfrac{x}{6} + \dfrac{y}{\left(\dfrac{9}{2}\right)} = 1$

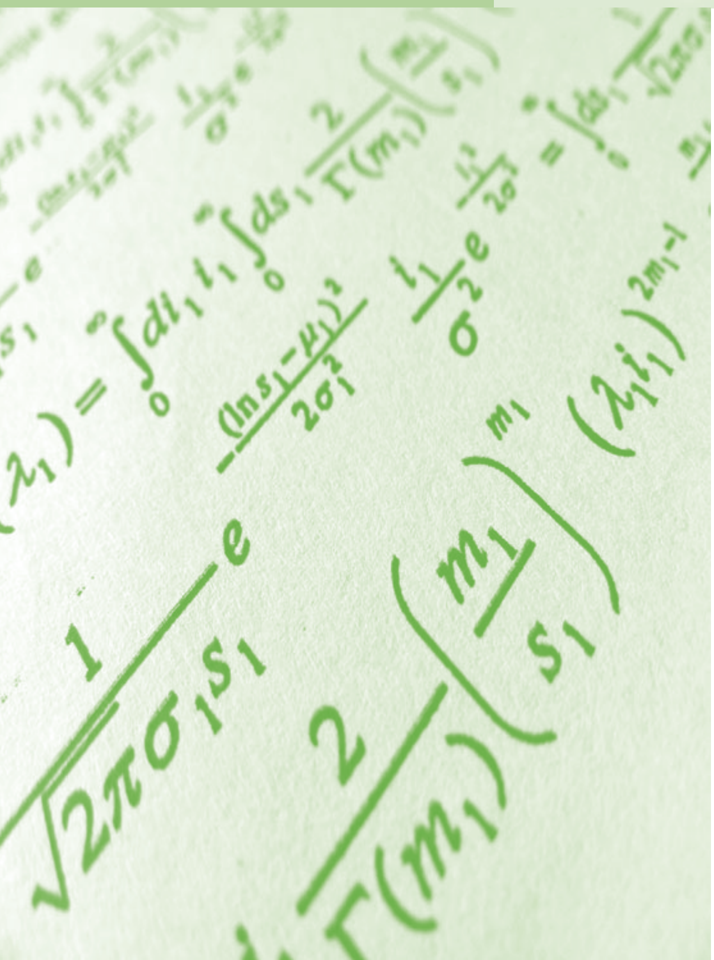
$$\Rightarrow \frac{3x + 4y}{18} = 1 \quad 3x + 4y = 18$$

or $\dfrac{x}{3} + \dfrac{y}{9} = 1$

$\Rightarrow 3x + y = 9$

Chapter 16

Mathematical Induction and Binomial Theorem



REMEMBER

Before beginning this chapter, you should be able to:

- Understand statements, truth tables of different compound statements
- Recall laws of 'algebra of statements'

KEY IDEAS

After completing this chapter, you would be able to:

- State the principle of mathematical induction
- Form a binomial expression and represent it with the help of Pascal triangle
- Apply factorial notation to binomial expressions
- Prove binomial theorem and solve problems related

INTRODUCTION

The process of mathematical induction is an indirect method which helps us to prove complex mathematical formulae that cannot be easily proved by direct methods.

For example, to prove that '$n(n + 1)$ is always divisible by 2' for n being a natural number, we can substitute $n = 1, 2, 3, \ldots$ in $n(n + 1)$, and check in each case if the result is divisible by 2. After checking, for a few of values, we can say that the formula is likely to be correct. Since, we cannot substitute all possible values of n, to prove the formula we use the principle of mathematical induction to prove the given formula.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

If $P(n)$ is a statement such that,

1. $P(n)$ is true for $n = 1$.
2. $P(n)$ is true for $n = k + 1$, when it is true for $n = k$, where k is a natural number then the statement $P(n)$ is true for all natural numbers.

Let us prove some results using this principle.

EXAMPLE 16.1

Prove that $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.

SOLUTION

Let $P(n): 1+2+\cdots+n=\dfrac{n(n+1)}{2}$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = 1 and RHS $=\dfrac{1(1+1)}{2}=1$

$\therefore$ LHS = RHS $\Rightarrow P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$$\therefore 1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

Adding $(k + 1)$ on both sides, we get

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=(k+1)\left(\frac{k}{2}+1\right)=\frac{(k+1)(k+2)}{2}=\frac{(k+1)(\overline{k+1}+1)}{2}$$

$\Rightarrow P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers n.

Hence, $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ for all $n \in N$.

EXAMPLE 16.2

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

SOLUTION

Let $P(n)$: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = 1

RHS = $(1)^2 = 1$

$\therefore$ LHS = RHS

$\Rightarrow P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$\therefore\ 1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

Adding $2k + 1$ on both sides, we get

$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$

$\therefore\ 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$

$\Rightarrow P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers 'n'.

Hence, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all $n \in N$.

EXAMPLE 16.3

Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \dfrac{n(n+1)(n+2)}{3}$.

SOLUTION

Let $P(n): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \dfrac{n(n+1)(n+2)}{3}$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = $1 \cdot 2 = 2$

$$\text{RHS} = \frac{1(1+1)(1+2)}{3} = \frac{2 \times 3}{3} = 2$$

$\therefore$ LHS = RHS

$\Rightarrow P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$$\therefore\ 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k + 1) = \frac{k(k+1)(k+2)}{3}.$$

Adding $(k + 1)(k + 2)$ on both sides, we get

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k + 1) + (k + 1)(k + 2)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$\therefore 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot (k+1) + (k+1)(k+2) = \frac{(k+1)(\overline{k+1}+1)(\overline{k+1}+2)}{3}.$$

$\Rightarrow P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers

Hence, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \dfrac{n(n+1)(n+2)}{3}$, $n \in N$.

EXAMPLE 16.4

Prove that $3^{n+1} > 3(n+1)$.

SOLUTION

Let $P(n)$: $3^{n+1} > 3(n+1)$

Step 1: Put $n = 1$

Then, $3^2 > 3(2)$

$\Rightarrow p(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

Then, $3^{k+1} > 3(k+1)$

Multiplying throughout with '3',

$3^{k+1} \cdot 3 > 3(k+1) \cdot 3 = 9k + 9 = 3(k+2)$

$\Rightarrow 3^{\overline{k+1}+1} > 3(\overline{k+1}+1)$

$P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Hence, $3^{n+1} > 3(n+1)$, $\forall\ n \in N$.

EXAMPLE 16.5

Prove that 7 is a factor of $2^{3n} - 1$ for all natural numbers n.

SOLUTION

Let $P(n)$: 7 is a factor of $2^{3n} - 1$ be the given statement

Step 1: When $n = 1$,

$2^{3(1)} - 1 = 7$ and 7 is a factor of itself.

$\therefore$ $P(n)$ is true for $n = 1$.

Step 2: Let $P(n)$ be true for $n = k$

$\Rightarrow$ 7 is a factor of $2^{3k} - 1$

$\Rightarrow 2^{3k} - 1 = 7M$, where $M \in N$

$\Rightarrow 2^{3K} = 7M + 1$ (1)

Now consider $2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k} \cdot 2^3 - 1$

$= 8(7M + 1) - 1$ (using (1)) $= 56M + 7$ (As $2^{3k} = 7m + 1$)

$\therefore$ $2^{3(k+1)} - 1 = 7(8M + 1)$

$\Rightarrow$ 7 is a factor of $2^{3(k+1)} - 1$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

Hence, 7 is a factor of $2^{3n} - 1$ for all $n \in N$.

Binomial Expression

An algebraic expression containing only two terms is called a binomial expression.

For example, $x + 2y$, $3x + 5y$, $8x - 7y$, etc.

We know that, $(a + b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned}(a + b)^3 &= (a + b)\,(a + b)^2 \\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

Now using a similar approach we can arrive at the expressions for $(a + b)^4$, $(a + b)^5$, etc. However, when the index is large, this process becomes very cumbersome. Hence, we need a simpler method to arrive at the expression for $(a + b)^n$, for $n = 1, 2, 3, \ldots$.

The binomial theorem is the appropriate tool in this case. It helps us arrive at the expression for $(a + b)^n$, for any value of n, by using a few standard coefficients also known as binomial coefficients.

Now, consider the following cases in which we find the expansions when a binomial expression is raised to different powers.

$$\begin{aligned}(x + y)^1 &= x + y \\ (x + y)^2 &= x^2 + 2xy + y^2 \\ (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

In the above examples, the coefficients of the variables in the expansions of the powers of the binomial expression are called binomial coefficients.

When the binomial coefficients are listed, for different values of n, we see a definite pattern being followed.

This pattern is given by the Pascal triangle.

Pascal Triangle

This definite pattern, shown in the following figure, can be used to write the binomial expansions for higher powers such as $n = 6, 7, 8, \ldots$ so on. The binomial theorem gives us a general algebraic formula by means of which any power of a binomial expression can be expanded into a series of simpler terms.

The Exponent in the Binomial	The Coefficients of the Terms in the Expansion
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Before we take up the binomial theorem, let us review the concepts of factorial notation and the ${}^nC_r$ representation.

Factorial Notation and ${}^nC_r$ Representation

The factorial of n is denoted by $n!$ and is defined as, $n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$.

For example, $4! = 1 \times 2 \times 3 \times 4$ and $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$

Also, $0! = 1$ and $n! = n\,(n - 1)!$

For $0 \le r \le n$, we define ${}^nC_r$ as ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

For example, ${}^6C_2 = \dfrac{6!}{(6-2)!2!} = \dfrac{6 \times 5 \times 4!}{4! \times 2!} = 15$

also, ${}^nC_0 = {}^nC_n = 1$; ${}^nC_1 = {}^nC_{n-1} = n$ and ${}^nC_r = {}^nC_{n-r}$

For example, ${}^{10}C_2 = {}^{10}C_8$ and if ${}^nC_3 = {}^nC_5$, then $n = 3 + 5 = 8$.

Binomial Theorem

If n is a positive integer,

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \cdots + {}^nC_r\, x^{n-r}\, y^r + \cdots + {}^nC_n\, y^n.$$

Important Inferences from the above Expansion

1. The number of terms in the expansion is $n + 1$.
2. The exponent of x goes on decreasing by '1' from left to right and the power of y goes on increasing by '1' from left to right.
3. In each term of the expansion the sum of the exponents of x and y is equal to the exponent (n) of the binomial expression.
4. The coefficients of the terms that are equidistant from the beginning and the end have numerically equal, i.e., ${}^nC_0 = {}^nC_n$; ${}^nC_1 = {}^nC_{n-1}$; ${}^nC_2 = {}^nC_{n-2}$ and so on.
5. The general term in the expansion of $(x + y)^n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$.
6. On substituting '$-y$' in place of 'y' in the expansion, we get

   $$(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 - {}^nC_3\, x^{n-3}\, y^3 + \cdots + (-1)^n\, {}^nC_n y^n.$$
   The general term in the expansion $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$.

EXAMPLE 16.6

Expand $(x + 2y)^5$.

SOLUTION

$(x + 2y)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^{5-1}\, (2y) + {}^5C_2\, x^{5-2}\, (2y)^2 + {}^5C_3\, x^{5-3}\, (2y)^3 + {}^5C_4\, x^{5-4}\, (2y)^4 + {}^5C_5\, (2y)^5$.

$\Rightarrow (x + 2y)^5 = x^5 + 5x^4\,(2y) + 10x^3 4y^2 + 10x^2 8y^3 + 5x\, 16y^4 + 2^5 y^5$

$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$.

EXAMPLE 16.7

Find the 3rd term in the expansion of $(3x - 5y)^7$.

SOLUTION

The general term in $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} y^r$

$\therefore\ T_3 = T_{2+1} = (-1)^2\, {}^7C_2\, (3x)^{7-2}\, (5y)^2 = {}^7C_2 (3x)^5\, (5y)^2$.

Middle Terms in the Expansion of $(x + y)^n$

Depending on the nature of n, i.e., whether n is even or odd, there may exist one or two middle terms.

Case 1:

When n is an even number, then there is only one middle term in the expansion $(x + y)^n$, which is $\left(\frac{n}{2}+1\right)$th term.

Case 2:

When n is odd number, there will be two middle terms in the expansion of $(x + y)^n$, which are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

EXAMPLE 16.8

Find the middle term in the expansion of $(2x + 3y)^8$.

SOLUTION

Since n is even number, $\left(\frac{8}{2}+1\right)$th term, i.e., 5th term is the middle term in $(2x + 3y)^8$.

$\therefore\ T_5 = T_{4+1} = {}^8C_4\,(2x)^{8-4}\,(3y)^4 = {}^8C_4\,(2x)^4\,(3y)^4.$

EXAMPLE 16.9

Find the middle terms in the expansion of $(5x - 7y)^7$.

SOLUTION

Since n is an odd number, the expansion contains two middle terms.

$\left(\frac{7+1}{2}\right)$th and $\left(\frac{7+3}{2}\right)$th terms are the two middle terms in the expansion of $(5x - 7y)^7$.

$T_4 = T_{3+1} = (-1)^3\,{}^7C_3\,(5x)^{7-3}\,(7y)^3 = -{}^7C_3\,(5x)^4\,(7y)^3$

$\therefore\ T_5 = T_{4+1} = (-1)^4 \cdot {}^7C_4\,(5x)^{7-4} \cdot (7y)^4 = {}^7C_4\,(5x)^3\,(7y)^4.$

Term Independent of x

In an expansion of form $\left(x^p + \frac{1}{x^q}\right)^n$, the term for which the exponent of x is 0 is said to be the term that is independent of x or a constant term.

For example, in the expansion $\left(x+\frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$, the 2nd term is independent of 'x'.

EXAMPLE 16.10

Find the term independent of x in $\left(x+\dfrac{1}{x}\right)^4$.

SOLUTION

Let T_{r+1} be the independent term of x in the given expansion.

$$\therefore T_{r+1} = {}^4C_r x^{4-r}\left(\frac{1}{x}\right)^r = {}^4C_r \frac{x^{4-r}}{x^r} = {}^4C_r x^{4-2r}.$$

For the term independent of x the power of x should be zero.

$\therefore\ 4 - 2r = 0$ or $r = 2$.

$\Rightarrow T_{2+1} = T_3$ term, is the independent term of the expansion.

Note If r is not a positive integer, then the expansion does not contain constant term.

EXAMPLE 16.11

Find the coefficient of x^2 in $\left(x^2+\dfrac{1}{x^3}\right)^6$.

SOLUTION

Let T_{r+1} be the term containing x^2.

$$T_{r+1} = {}^6C_r (x^2)^{6-r}\left(\frac{1}{x^3}\right)^r = {}^6C_r x^{12-2r}\frac{1}{x^{3r}} = {}^6C_r x^{12-5r}$$

As the coefficient of x is 2.

$12 - 5r = 2 \Rightarrow r = 2$

$\therefore$ Coefficient of $x^2 = {}^6C_2 = 15$.

The Greatest Coefficient in the Expansion of $(1 + x)^n$ (where n is a Positive Integer)

The coefficient of the $(r + 1)$th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

${}^nC_r$ is maximum when $r = \dfrac{n}{2}$ (if n is even) and

$r = \dfrac{n+1}{2}$ or $\dfrac{n-1}{2}$ (if n is odd).

EXAMPLE 16.12

Find the total number of terms in the expansion of $(2 + 3x)^{15} + (2 - 3x)^{15}$.

SOLUTION

$(2 + 3x)^{15} = {}^{15}C_0(2)^{15} + {}^{15}C_1(2)^{14}(3x)^1 + \cdots + {}^{15}C_{14}(2)^1(3x)^{14} + {}^{15}C_{15}(3x)^{15}$ and $(2 - 3x)^{15}$ $= {}^{15}C_0(2)^{15} - {}^{15}C_1(2)^{14}(3x^1) + \cdots + {}^{15}C_{14}(2)^1(3x)^{14} - {}^{15}C_{15}(3x)^{15}$.

Adding the two equations, we see that the terms in even positions get cancelled, and we get $(2 + 3x)^{15} + (2 - 3x)^{15} = 2[{}^{15}C_0(2)^{15} + {}^{15}C_2(2)^{13}(3x)^2 + \cdots + {}^{15}C_{14}(2)^1(3x)^{14}]$

$\therefore$ Total number of terms $= 8$

Alternately, the number of terms in $(a + x)^n + (a - x)^n$, if n is odd, then it is $\dfrac{n+1}{2}$.

Hence, in this case, the number of terms are $\dfrac{15+1}{2} = 8$.

EXAMPLE 16.13

If the expansion $\left(x^2 + \dfrac{1}{x^3}\right)^n$ is to contain an independent term, then what should be the value of n?

SOLUTION

General term, $T_{r+1} = {}^nC_r \cdot x^{n-r}\, y^r$, for $(x + y)^n$.

$\Rightarrow$ General term of $\left(x^2 + \dfrac{1}{x^3}\right)^n$ is ${}^nC_r \cdot x^{2n-2r} \cdot \dfrac{1}{x^{3r}} = {}^nC_r \cdot x^{2n-5r}$.

For a term to be independent of x, $2n - 5r$ should be equal to zero, That is, $2n - 5r = 0$.

$\Rightarrow r = \dfrac{2}{5}n$, since r can take only integral values, n has to be a multiple of 5.

EXAMPLE 16.14

If the coefficient of x^7 in $\left(ax + \dfrac{1}{x}\right)^9$ and x^{-7} in $\left(bx - \dfrac{1}{x}\right)^9$ are equal, find the relation between a and b?

SOLUTION

For $\left(ax + \dfrac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r (ax)^{9-r} \left(\dfrac{1}{x}\right)^r = {}^9C_r (a)^{9-2r} (x)^{9-2r}$ as $9 - 2r = 7$, $r = 1$.

$\therefore$ Coefficient of x^7 is, ${}^9C_1 (a)^{9-1} = 9(a)^8$.

Now, for $\left(bx - \dfrac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r (bx)^{9-r} \left(-\dfrac{1}{x}\right)^r = {}^9C_r (b)^{9-r} (-1)^r x^{9-2r}$ as $9 - 2r = -7$, $r = 8$.

$\therefore$ Coefficient of x^{-7} is, ${}^9C_8\, b^{9-8}\, (-1)^8 = 9b$

$\therefore$ $9a^8 = 9b$, i.e., $a^8 - b = 0$.

EXAMPLE 16.15

Find the term independent of 'x' in the expansion of $(1 + x^2)^4 \left(1 + \dfrac{1}{x^2}\right)^4$.

SOLUTION

$$(1 + x^2)^4 \left(1 + \frac{1}{x^2}\right)^4 = ({}^4C_0 + {}^4C_1 x^2 + \cdots + {}^4C_4 x^8) \times ({}^4C_0 + {}^4C_1 x^{-2} + \cdots + {}^4C_4 x^{-8})$$

The term independent of 'x' is the term containing the coefficient,

$({}^4C_0 \cdot {}^4C_0 \cdot {}^4C_1 \cdot {}^4C_1 + \cdots + {}^4C_4 \cdot {}^4C_4)$

$= ({}^4C_0)^2 + ({}^4C_1)^2 + ({}^4C_2)^2 + ({}^4C_3)^2 + ({}^4C_4)^2$

$= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$.

EXAMPLE 16.16

Find the sum of the coefficients of the terms of the expansion $(1 + x + 2x^2)^6$.

SOLUTION

Substituting $x = 1$, we have $(1 + 1 + 2)^6$, which gives us the sum of the coefficients of the terms of the expansion.

$\therefore$ Sum $= 4^6$.

EXAMPLE 16.17

Find the value of x, if the fourth term in the expansion of $\left(\dfrac{1}{x^2} + x^2 \cdot 2^x\right)^6$ is 160.

SOLUTION

4th term $\Rightarrow T_{3+1} = {}^6C_3 \cdot \left(\dfrac{1}{x^2}\right) \cdot (x^2)^3 \cdot (2^x)^3$

$\therefore$ ${}^6C_3 \cdot (2^x)^3 = 160$

That is, $20 \cdot 2^{3x} = 160$

$\therefore$ $2^{3x} = 8 \Rightarrow 2^{3x} = 2^3$

$\therefore$ $x = 1$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If $p(n)$ is a statement which is true for $n = 1$ and true for $(n + 1)$, then _______.
2. According to the principle of mathematical induction, when can we say that a statement $X(n)$ is true for all natural numbers n?
3. If $p(n) = n(n + 1)(n + 2)$ then highest common factor of $p(n)$, for different values of n where n is any natural number, is _______.
4. Is $2^{3n} - 1$ a prime number for all natural numbers n?
5. The product of $(q - 1)$ consecutive integers where $q > 1$ is divisible by _______.
6. An algebraic expression with two terms is called a _______.
7. In Pascal triangle, each row of coefficients is bounded on both sides by _______.
8. The number of terms in the expansion of $(x + y)^n$ is _______ (where n is a positive integer).
9. In the expansion of $(x + y)^n$, if the exponent of x in second term is 10, what is the exponent of y in 11th term.
10. What is the coefficient of a term in a row of Pascal triangle if in the preceding row, the coefficient on the immediate left is 5 and on the immediate right is 10.
11. In the expansion of various powers of $(x + y)^n$, if the expansion contains 49 terms, then it is the expansion of _______.
12. In the expression of $(x + y)^{123}$, the sum of the exponents of x and y in 63rd term is _______.
13. $(n - r)! =$ _______.
14. The value of ${}^{n+1}C_r =$ _______.
15. If ${}^nC_r = 1$ and $n = 6$, then what may be the value(s) of r be?
16. In the expansion of $(x + y)^n$, $T_{r+1} =$ _______.
17. ${}^7C_2 =$ _______.
18. ${}^{1230}C_0 =$ _______.
19. The coefficient of x in the expansion of $(2x + 3)^5$ is _______ .
20. The coefficient of y^7 in the expansion of $(y + z)^7$ is _______.
21. $(x + y)^3 =$ _______.
22. The term which does not contain 'a' in the expansion of $\left(\frac{x}{a} + 6x\right)^{12}$ is _______.
23. If ${}^{12}C_r\,(4)^{12-r}\,(x)^{12-3r}$ is a constant term in an expansion, then $r =$ _______.
24. Write the first, the middle and the last terms in the expansion of $(x^2 + 1)^3$.
25. Constant term in the expansion of $(x + 3)^{16}$ is _______.
26. The sum of the first n even natural numbers is _______.
27. The sum of the first n odd natural numbers is _______.
28. The elements in the fifth row of Pascal triangle is _______.
29. If ${}^nC_3 = {}^nC_{15}$, then ${}^{20}C_n$ is _______.
30. The inequality $2^n > n$ is true for _______.

Short Answer Type Questions

Directions for questions 31 to 39:
By mathematical Induction prove the following.

31. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^{n-1}$, $n \in N$.
32. $a - b$ divides $a^n - b^n$, $n \in N$.
33. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$, $n \in N$.
34. $2.5 + 3.8 + 4.11 + \cdots +$ upto n terms $= n(n^2 + 4n + 5)$, $n \in N$.
35. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, $n \in N$.
36. $a + (a + d) + (a + 2d) + \cdots$ upto n terms $= \dfrac{n}{2}[2a + (n-1)d]$.

PRACTICE QUESTIONS

37. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1)(3n+2)}$

$=\frac{n}{6n+4}, n \in N.$

38. 9 is a factor of $4^n + 15n - 1$, $n \in N$.

39. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}, n \in N.$

40. Expand $\left(3x^2 + \frac{5}{y^2}\right)^6$.

41. Expand $(5x + 3y)^8$.

42. Find the middle term or terms of the expansion of $(x + 5y)^9$.

43. Find the middle term or terms of the expansion of $\left(x + \frac{1}{x}\right)^6$.

44. Find the 7th term in the expansion of $\left(5x - \frac{1}{7y}\right)^9$.

45. Prove that $2^{n+1} > 2n + 1$; $n \in N$.

Essay Type Questions

46. Find the value of $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$.

47. Find the coefficient of x^{-5} in the expansion of $\left(2x^2 - \frac{1}{5x}\right)^8$.

48. Find the term independent of x in $\left(6x^2 - \frac{1}{7x^3}\right)^{10}$.

49. Find the coefficient of x^3 in the expansion of $\left(x^2 + \frac{1}{3x^3}\right)^4$.

50. Find the term independent of x in $\left(2x^5 + \frac{1}{3x^2}\right)^{21}$.

CONCEPT APPLICATION

Level 1

1. $n^2 + n + 1$ is a/an _____ number for all $n \in N$.
   (a) even (b) odd
   (c) prime (d) co-prime

2. If the expansion $\left(x^3 + \frac{1}{x^2}\right)^n$ contains a term independent of x, then the value of n can be _______.
   (a) 18 (b) 20
   (c) 24 (d) 22

3. $1 + 5 + 9 + \cdots + (4n - 3)$ is equal to _______.
   (a) $n(4n - 3)$ (b) $(2n - 1)$
   (c) $n(2n - 1)$ (d) $(4n - 3)^2$

4. For all $n \in N$, which of the following is a factor of $2^{3n} - 1$?
   (a) 3 (b) 5
   (c) 7 (d) 2

5. For what values of n is $14^n + 11^n$ divisible by 5?
   (a) When n is an even positive integer
   (b) For all values of n
   (c) When n is a prime number
   (d) When n is a odd positive integer

6. The smallest positive integer n for which $n! < \frac{(n-1)^n}{2}$ holds, is _______.
   (a) 4 (b) 3
   (c) 2 (d) 1

7. The third term from the end in the expansion of $\left(\frac{4x}{3y} - \frac{3y}{2x}\right)^9$ is _______.
   (a) ${}^9C_7 \frac{3^5}{2^3}\frac{y^5}{x^5}$ (b) $-{}^9C_7 \frac{3^5}{2^3}\frac{y^5}{x^5}$
   (c) ${}^9C_7 \frac{3^3}{2^5}\frac{y^5}{x^3}$ (d) ${}^9C_7 \frac{3^5}{2^3}\frac{y^5}{x^5}$

8. In the 8th term of $(x + y)^n$, the exponent of x is 3, then the exponent of x in 5th term is _______.
   (a) 5 (b) 4
   (c) 2 (d) 6

PRACTICE QUESTIONS

9. The sum of the elements in the sixth row of Pascal triangle is ______.

(a) 32 (b) 63

(c) 128 (d) 64

10. In $(x + y)^n - (x - y)^n$, if the number of terms is 5, then find n.

(a) 6 (b) 5

(c) 10 (d) 9

11. If the third term in the expansion of $(x + x^{\log_2^x})^6$ is 960, then the value of x is ______.

(a) 2 (b) 3

(c) 4 (d) 8

12. Find the sum of coefficients of all the terms of the expansion $(ax + y)^n$.

(a) ${}^nC_0a^n + {}^nC_1a^{n-1}\ x^{n-1}y + {}^nC_2a^{n-2}\ x^{n-2}y^2 + \cdots + {}^nC_ny^n$

(b) ${}^nC_0a^n + {}^nC_1a^{n-1} + {}^nC_2a^{n-2} + \ldots + {}^nC_n$

(c) 2^n

(d) None of these

13. If the sum of the coefficients in the expansion $(4ax - 1 - 3a^2x^2)^{10}$ is 0, then the value of a can be ______.

(a) 2 (b) 4

(c) 1 (d) 7

14. Find the coefficient of x^4 in the expansion of $\left(2x^2 + \dfrac{3}{x^3}\right)^7$.

(a) ${}^7C_2\ 2^5\ 3^3$ (b) ${}^7C_2\ 2^5\ 3^2$

(c) ${}^7C_2\ 3^5\ 2^2$ (d) ${}^7C_3\ 2^5\ 3^2$

15. $n^2 - n + 1$ is an odd number for all ______.

(a) $n > 1$ (b) $n > 2$

(c) $n \geq 1$ (d) $n \geq 5$

16. $7^{n+1} + 3^{n+1}$ is divisible by ______.

(a) 10 for all natural numbers n

(b) 10 for odd natural numbers n

(c) 10 for even natural numbers n

(d) None of these

17. For $n \in N$, $2^{3n} + 1$ is divisible by ______.

(a) 3^{n+11} (b) 3^{n-11}

(c) 3^{n+1} (d) 3^{n+111}

18. $2^n - 1$ gives the set of all odd natural numbers for all $n \in N$. Comment on the given statement.

(a) True for all values of n

(b) False

(c) True for only odd values of n

(d) True for only prime values of n

19. In the 5th term of $(x + y)^n$, the exponent of y is 4, then the exponent of y in the 8th term is ______.

(a) 1 (b) 7

(c) 5 (d) 9

20. If the coefficients of 6th and 5th terms of expansion $(1 + x)^n$ are in the ratio 7 : 5, then find the value of n.

(a) 11 (b) 12

(c) 10 (d) 9

21. The third term from the end in the expansion of $(3x - 2y)^{15}$ is ______.

(a) $-{}^{15}C_5 3^{13}2^2x^{13}y^2$ (b) ${}^{15}C_5 3^{13}2^2x^{13}y^2$

(c) ${}^{15}C_5 3^2 2^{13}x^2y^{13}$ (d) $-{}^{15}C_5 3^2 2^{13}x^2y^{13}$

22. Find the sixth term in the expansion of $\left(2x^2 - \dfrac{3}{7x^3}\right)^{11}$.

(a) $-{}^{11}C_5\dfrac{2^6 3^5}{7^5}x^3$ (b) ${}^{11}C_5\dfrac{2^6 3^5}{7^5}x^{-3}$

(c) $-{}^{11}C_5\dfrac{2^6 3^5}{7^5}x^{-3}$ (d) $-{}^{11}C_5\dfrac{2^3 3^5}{7^5}x^{-3}$

23. The term independent of x in the expansion of $\left(x^3 - \dfrac{1}{x^2}\right)^{10}$ is ______.

(a) ${}^{10}C_6$ (b) ${}^{10}C_7$

(c) ${}^{10}C_9$ (d) ${}^{-10}C_6$

24. Which term is the constant term in the expansion of $\left(2x - \dfrac{1}{3x}\right)^6$?

(a) 2nd term (b) 3rd term

(c) 4th term (d) 5th term

25. The sum of the coefficients in the expansion of $(x + y)^7$ is ______.

(a) 119 (b) 64

(c) 256 (d) 128

PRACTICE QUESTIONS

26. The number of terms which are not radicals in the expansion $(\sqrt{7}+4)^6+(\sqrt{7}-4)^6$ after simplification, is ______.
(a) 6 (b) 5
(c) 4 (d) 3

27. The coefficient of x^4 in the expansion of $\left(4x^2+\frac{3}{x}\right)^8$ is ______.
(a) ${}^8C_5 12^5$ (b) ${}^8C_4 12^4$
(c) ${}^8C_3 12^3$ (d) ${}^8C_6 12^6$

28. In the expansion of $(a+b)^n$, the coefficients of 15th and 11th terms are equal. Find the number of terms in the expansion.
(a) 26 (b) 25
(c) 20 (d) 24

29. The number of terms in the expansion of $[(2x+3y)^4(4x-6y)^4]^9$ is ______.
(a) 36 (b) 37
(c) 10 (d) 40

30. If sum of the coefficients of the first two odd terms of the expansion $(x+y)^n$ is 16, then find n.
(a) 10 (b) 8
(c) 7 (d) 6

Level 2

31. The number of rational terms in the expansion of $(x^{1/5}+y^{1/10})^{45}$ is ______.
(a) 5 (b) 6
(c) 4 (d) 7

32. The remainder when $9^{49}+7^{49}$ is divided by 64 is ______.
(a) 24 (b) 8
(c) 16 (d) 38

33. If $p(n)=(n-2)(n-1)n(n+1)(n+2)$, then greatest number which divides $p(n)$ for all $n \in N$ is ______.
(a) 12 (b) 24
(c) 120 (d) None of these

34. For $n \in N$, $a^{2n-1}+b^{2n-1}$ is divisible by ______.
(a) $a+b$ (b) $(a+b)^2$
(c) a^3+b^3 (d) a^2+b^2

35. Find the coefficient of the independent term in the expansion of $(x^{1/2}+7x^{-1/3})^{10}$.
(a) ${}^{10}C_4 7^4$ (b) ${}^{10}C_6 7^6$
(c) ${}^{10}C_6 7^5$ (d) ${}^{10}C_4 7^7$

36. Find the term which has the exponent of x as 8 in the expansion of $\left(x^{5/2}-\frac{3}{x^3\sqrt{x}}\right)^{10}$.
(a) T_2 (b) T_3
(c) T_4 (d) Does not exist

37. The greatest number which divides $25^n-24n-1$ for all $n \in N$ is ______.
(a) 24
(b) 578
(c) 27
(d) 576

38. If three consecutive coefficients in the expansion of $(1+x)^n$, where n is a natural number are 36, 84 and 126 respectively, then n is ______.
(a) 8
(b) 9
(c) 10
(d) 36

39. Find the value of k for which the term independent of x in $\left(x^2+\frac{k}{x}\right)^{12}$ is 7920.
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$
(c) $\sqrt{2}$ (d) 2

40. Find the coefficient of x^7 in the expansion of $\left(7x+\frac{2}{x^2}\right)^{13}$.
(a) $78\times 8^8\times 4$
(b) $78\times 7^6\times 4^2$
(c) $78\times 7^{11}\times 4$
(d) $78\times 7^{11}\times 4^2$

PRACTICE QUESTIONS

41. The value of $(\sqrt{5}+2)^6+(\sqrt{5}-2)^6$ is
(a) a positive integer.
(b) a negative integer.
(c) an irrational number.
(d) a rational number but not an integer.

42. The ratio of the coefficients of x^4 to that of the term independent of x in the expansion of $\left(x^2+\dfrac{9}{x^2}\right)^{18}$ is ______.
(a) 1 : 6 (b) 3 : 8
(c) 1 : 10 (d) 1 : 8

43. $\sum_{r=2}^{16} {}^{16}C_r =$ ______.
(a) $2^{15}-15$ (b) $2^{16}-16$
(c) $2^{16}-17$ (d) $2^{17}-17$

44. Number of non-zero terms in the expansion of $(5\sqrt{5}x+\sqrt{7})^6+(5\sqrt{5}x-\sqrt{7})^6$ is ______.
(a) 4 (b) 10
(c) 12 (d) 14

45. Find the value of $(98)^4$ by using the binomial theorem.
(a) 92236846 (b) 92236816
(c) 92236886 (d) 92236806

46. If the number of terms in the expansion $(2x+y)^n-(2x-y)^n$ is 8, then the value of n is ______. (where n is odd)
(a) 17 (b) 19
(c) 15 (d) 13

47. If the expansion $\left(2x^5+\dfrac{1}{3x^4}\right)^n$ contains a term independent of x, then the value of n can be ______.
(a) 6 (b) 18
(c) 3 (d) 12

48. Find the sum of the coefficients in the expansion of $\left(5x^6-\dfrac{4}{x^9}\right)^{10}$.
(a) 5^{10} (b) 1
(c) 4^{10} (d) 0

49. $49^n+16n-1$ is divisible by ______ $(n \in N)$.
(a) 64 (b) 28
(c) 48 (d) 54

50. Find the coefficient of x^{11} in the expansion of $(1+2x+x^2)^6$.
(a) 1 (b) 2
(c) 6 (d) 12

Level 3

51. The number of irrational terms in the expansion of $(x^{2/3}+y^{1/4})^{81}$ is ________.
(a) 70
(b) 12
(c) 75
(d) 13

52. Find the independent term in the expansion of $\left(x^4+\dfrac{3}{8x^3\sqrt{x}}\right)^{15}$.
(a) ${}^{15}C_4\left(\dfrac{3}{8}\right)^{16}$ (b) ${}^{15}C_{12}\left(\dfrac{3}{8}\right)^{4}$
(c) ${}^{15}C_8\left(\dfrac{3}{8}\right)^{8}$ (d) ${}^{15}C_7\left(\dfrac{3}{8}\right)^{15}$

53. $\sum_{r=1}^{30} r\dfrac{{}^{30}C_r}{{}^{30}C_{r-1}} =$ ______.
(a) 930 (b) 465
(c) 310 (d) 630

54. For all $n \in N$, $41^n-40n-1$ is divisible by ______.
(a) 41 (b) 40
(c) 300 (d) 500

55. If m and n are the coefficients of x^{a^2} and x^{b^2} respectively in $(1+x)^{a^2+b^2}$, then ______.
(a) $n=2m$
(b) $m+n=0$
(c) $2n=m$
(d) $m=n$

PRACTICE QUESTIONS

56. For each $n \in N$, $5^{3n} - 1$ is divisible by _____.

(a) 115 (b) 124

(c) 5 (d) 6

57. In the expansion $(6 + 9x)^5$ the coefficient of x^3 is _____.

(a) $2^2 \times 3^8$

(b) $2^4 \times 3^7$

(c) $2^3 \times 3^8 \times 5$

(d) $2^4 \times 3^7 \times 5$

58. In the expansion of $(x + y)^n$, the coefficients of the 17th and the 13th terms are equal. Find the number of terms in the expansion.

(a) 18 (b) 22

(c) 28 (d) 29

59. Find the coefficient of x^{-2} in the expansion of $\left(x^2 + \dfrac{4}{x^5}\right)^6$.

(a) 240 (b) 150

(c) 100 (d) 180

60. In the 10th term of $(x + y)^n$, the exponent of x is 3, then the exponent of x in the 7th term is _____.

(a) 3 (b) 6

(c) 5 (d) 7

61. $\sqrt{20}\{(\sqrt{20}+1)^{100} - (\sqrt{20}-1)^{100}\}$ is a/an _____.

(a) irrational number (b) whole number

(c) negative number (d) rational number

62. If $x = -\,{}^nC_1 + {}^nC_2\,(2) - {}^nC_3\,(2)^2 + \ldots$ (where n is odd), then $x =$ _____.

(a) 1 (b) −1

(c) 0 (d) 12

63. Find the independent term in the expansion of $\left(5x^2 - \dfrac{1}{x^4}\right)^6$.

(a) 8250 (b) 8560

(c) 9250 (d) 9375

64. The sum of the first three coefficients in the expansion $\left(x + \dfrac{1}{y}\right)^n$ is 22. Find the value of n.

(a) 8 (b) 7

(c) 6 (d) 5

65. If ${}^{12}C_0$ ${}^{12}C_1$, …, ${}^{12}C_{12}$ are the binomial coefficients of the expansion $(1 + x)^{12}$, then ${}^{12}C_0 - {}^{12}C_1 + {}^{12}C_2 - {}^{12}C_3 + \cdots + {}^{12}C_{12} =$ _____.

(a) 4096 (b) 1024

(c) 0 (d) −1024

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $p(n)$ is true for all natural numbers
2. When $x(n)$ is true for $n = 1$ and also true for $n + 1$.
3. $3! = 6$
4. Not a prime number
5. $(q - 1)!$
6. binomial
7. 1
8. $n + 1$
9. 10
10. 15
11. $(x + y)^{48}$
12. 123
13. $1 \cdot 2 \cdot 3 \ldots (n - r - 1) \cdot (n - r)$
14. $\dfrac{(n+1)!}{(n-r+1)!\,r!}$
15. 0 or 6
16. ${}^nC_r\, x^{n-r} y^r$
17. 21
18. 1
19. 810
20. 1
21. $x^3 + 3x^2y + 3xy^2 + y^3$
22. t_{13}
23. 4
24. $3x^4$; $3x^2$
25. 3^{16}
26. $n(n + 1)$
27. n^2
28. 1, 5, 10, 10, 5, 1
29. 190
30. all integers

Short Answer Type Questions

40. $(3x^2)^6 + 6(3x^2)^5 \dfrac{5}{y^2} + 15(3x^2)^4 \left(\dfrac{5}{y^2}\right)^2 + 20(3x^2)^3 \left(\dfrac{5}{y^2}\right)^3 + 15(3x^2)^2 \left(\dfrac{5}{y^2}\right)^4 + 6(3x^2)\left(\dfrac{5}{y^2}\right)^5 + \left(\dfrac{5}{y^2}\right)^6.$
41. $(5x)^8 + 8(5x)^7(3y) + 28(5x)^6(3y)^2 + 56(5x)^5(3y)^3 + 70(5x)^4(3y)^4 + 56(5x)^3(3y)^5 + 28(5x)^2(3y)^6 + 8(5x)(3y)^7 + (3y)^7$
42. ${}^9C_4\, 5^4x^5y^4$ and ${}^9C_5\, 5^5x^4y^5$
43. ${}^6C_3$
44. ${}^9C_6(5x)^3\left(\dfrac{1}{7y}\right)^6$

Essay Type Questions

46. 152
47. $\dfrac{-16}{5^7}$
48. ${}^{10}C_4\, \dfrac{6^6}{7^4}$
49. $\dfrac{4}{3}$
50. ${}^{21}C_{15}\, \dfrac{2^6}{3^{15}}$

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (d)
11. (a)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (b)	**30.** (d)

Level 2

31. (a)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (d)	**37.** (d)	**38.** (b)	**39.** (c)	**40.** (c)
41. (a)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (d)

Level 3

51. (c)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (d)	**56.** (b)	**57.** (c)	**58.** (d)	**59.** (a)	**60.** (b)
61. (b)	**62.** (b)	**63.** (d)	**64.** (c)	**65.** (c)					

CONCEPT APPLICATION

Level 1

1. Substitute different natural numbers for 'n' in the given expression.
2. Use the formula, $r = \frac{np}{p+q}$ to find the independent term in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$.
3. For $n = 2$, find the sum of first two terms and check for which option this sum is obtained.
4. For $n = 1, 2, 3$ and 4, find the value of $2^{3n} - 1$ and obtain a general conclusion.
5. Evaluate the given expression for $n = 1, 2, 3, 4$ and 5.
6. Among the options, identify the value that satisfies the given inequality.
7. The rth term from the end in the expansion $(x + y)^n$ is $(n - r + 2)$th term from the beginning.
8. The exponent of x of the terms in the expansion of $(x + y)^n$ decreases as we go from left to right.
9. The sum of the elements in the nth row of Pascal triangle is 2^n.
10. $(x + y)^n - (x - y)^n$ has $\frac{n}{2}$ terms when n is even and $\frac{n+1}{2}$ terms when n is odd.
11. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
13. $(x + y + z)^n$ has ${}^{n+2}C_2$ terms.
14. To find the coefficient of x^k in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ use the formula $r = \frac{np-k}{p+q}$.
15. For each choice, check if the given expression is divisible by it for the given values of n.
16. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression.
17. In the given expression substitute different values of n and then identify the factor.
18. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression.
19. As we go from left to right, the exponent of y in the expansion of $(x + y)^n$ increases.
20. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
21. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
22. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
23. To find the independent term in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ use the formula, $r = \frac{np}{p+q}$.
24. To find the independent term in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ use the formula, $r = \frac{np}{p+q}$.
25. Put $x = 1$ and $y = 1$ in the given expression.
26. Use the binomial expansion of $(x + y)^n + (x - y)^n$.
27. Use the formula $r = \frac{np-k}{p+q}$ in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ to find the coefficient of x^k.
28. Use $T_{r+1} = n_{c_r} x^{n-r} y^r$.
29. The number of terms in the expansion of $(x + y)^n$ is $n + 1$.
30. Put $y = 1$ and proceed.

Level 2

31. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$. Find the number of values of r for which both r and as $(n - r)$ are integers.
32. (i) $(8 + 1)^{49} + (8 - 1)^{49}$.
 (ii) Use the binomial expansion $(x + y)^n + (x - y)^n$ and simplify.
33. The product of n consecutive integers is always divisible by $n!$
34. Substitute $n = 1, 2, 3, 4, \ldots$ and verify from the options.

HINTS AND EXPLANATION

35. To find the independent term, in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$. Use the formula $r = \frac{np}{p+q}$.

36. To find the coefficient of x^k in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$. Use the formula $r = \frac{np-k}{p+q}$.

37. Substitute $n = 1, 2, 3, 4 \ldots$ in the given expression.

38. Let the three consecutive coefficients in the expansion $(1 + x)^n$ be ${}^nC_r$, ${}^nC_{r+1}$ and ${}^nC_{r+2}$.

39. To find the independent term in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$, use the formula $r = \frac{np}{p+q}$.

40. To find the coefficient of x^k in the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, use $r = \frac{np-k}{p+q}$.

41. $(x + a)^n + (x - a)^n = 2({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{x-4}\, y^4 + \ldots)$.

42. In the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ to find

 (i) the independent term, use the formula $r = \frac{np}{p+q}$ and

 (ii) the coefficient of x^k, use the formula $r = \frac{np-k}{p+q}$.

43. Use ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

44. The number of terms in the expansion of $(x + y)^n + (x - y)^n$ is $\frac{n}{2} + 1$ when n is even and $\frac{n+1}{2}$ when n is odd.

45. $(98)^4 = (100 - 2)^4$. Expand using binomial theorem.

46. The number of terms in the expansion $(x + y)^n - (x - y)^n$ is $\frac{n+1}{2}$, when n is odd.

 The given number of terms = 8

 $\therefore \frac{n+1}{2} = 8$

 $n + 1 = 16 \Rightarrow n = 15$.

47. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the independent term is T_{r+1} where $r = \frac{np}{p+q}$.

 $r = \frac{5n}{5+4} = \frac{5n}{9}$. Since r is are integer, 'n' must be a multiple of 9.

 From the options, the value of n can be 18.

48. The given expression is $\left(5x^6 - \frac{4}{x^9}\right)^{10}$,

 Put $x = 1$, so we get the sum of the coefficients, i.e., $(5 - 4)^{10} = 1^{10} = 1$.

49. Put $n = 1$, $49 + 16 - 1 = 64$.

 $\therefore$ It is divisible by 64.

50. $(1 + 2x + x^2)^6 = [(1 + x)^2]^6 = (1 + x)^{12}$.

 Coefficient of x^{11} is ${}^{12}C_{11} = 12$.

Level 3

54. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression and verify from the options.

55. (i) Use ${}^nC_r = {}^nC_{n-r}$.

 (ii) $m = (a^2 + b^2)c_{a^2}x^{a^2}$ and

 $n = (a^2 + b^2)c_{b^2}x^{b^2}$

 (iii) Find.

56. $5^{3n} - 1 = (5^3)^n - 1 = (125)^n - 1$

 $= (125 - 1)\ ((5^3)^{n-1} + (5^3)^{n-2} + \cdots + 1)$

 $= 124m$ (where 'm' is some positive integer).

 $\therefore$ $5^{3n} - 1$ is always divisible by 124.

57. Given expression is $(6 + 9x)^5$

 $T_{3+1} = {}^5C_3(6)^2(9x)^3 = 2 \times 5 \times (2 \times 3)^2 \times 3^6 x^3$.

 $\therefore$ The required coefficient $= 2^3 \times 3^8 \times 5$.

58. The coefficient of the 17th term is ${}^nC_{16}$.

 The coefficient of the 13th term is ${}^nC_{12}$.

 Given that the coefficients are equal.

 ${}^nC_{16} = {}^nC_{12}$

 $n = 16 + 12 = 28$.

 $\therefore$ The number of terms in the expansion is $28 + 1 = 29$.

59. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the term containing x^k is T_{r+1} where $r = \frac{np-k}{p+q}$.

Here, $r = \frac{12+2}{5+2} = \frac{14}{4} = 2.$

$T_{2+1} = T_3 = {}^6C_2(x^2)^4\left(\frac{4}{x^5}\right)^2 = {}^6C_2(16)x^{-2}$

$= (15)\ (16)\ x^{-2}$

$\therefore$ The required coefficient = 240.

60. The given expansion is $(x + y)^n$

$T_{10} = T_{9+1} = {}^nC_9(x)^{n-9}(y)^9$

Given $n - 9 = 3$

$n = 12$

$T_7 = T_{6+1} = {}^{12}C_6\ (x)^6\ (y)^6$

$\therefore$ The exponent of x is 6.

61. Let $x = \sqrt{20}$ and $n = 100$

$= x((x + 1)^{100} - (x - 1)^{100})$

$= x[{}^nC_1x^{n-1} + {}^nC_3x^{n-3} + \cdots]$

$= 2[{}^nC_1x^n + {}^nC_3x^{n-2} + {}^nC_5x^{n-4} + \cdots]$

Now $n = 100$, $n - 2$, $n - 4$, … are all even number.

But $x = \sqrt{20}$

$\therefore$ $x^n \cdot x^{n-2}$, x^{n-4}, are all integers and ${}^nC_1$, ${}^nC_3$, …, ${}^nC_5$ are all integers.

Hence, the given expression is a whole number.

62. Given $x = -{}^nC_1 + {}^nC_2\ (2)\ \ldots$

$2x = -{}^nC_1\ (2) + {}^nC_2\ (2)^2 - {}^nC_3\ (2)^3 + \cdots$

$1 + 2x = 1 - {}^nC_1\ (2) + {}^nC_2\ (2)^2 - {}^nC_3\ (2)^3 + \cdots$

$1 + 2x = (1 - 2)^n$

$1 + 2x = (-1)^n$

$1 + 2x = -1$ (given n is odd)

$2x = -2$

$x = -1.$

63. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the independent term is T_{r+1}, where $r = \frac{np}{p+q}$.

Here, $r = \frac{6\times 2}{6} = 2.$

The independent term is T_3.

$T_3 = T_{2+1} = {}^6C_2(5x^2)^4\left(\frac{-1}{x^4}\right)^2$

$= 5^4 \times 15 = 9375.$

64. Given that ${}^nC_0 + {}^nC_1 + {}^nC_2 = 22$

$\Rightarrow\quad 1 + n + \frac{n(n-1)}{2} = 22$

$\Rightarrow 2 + 2n + n^2 - n = 44$

$\Rightarrow n^2 + n - 42 = 0$

$\Rightarrow (n + 7)(n - 6) = 0$

$\Rightarrow n = -7$ or $n = 6$.

65. We know that

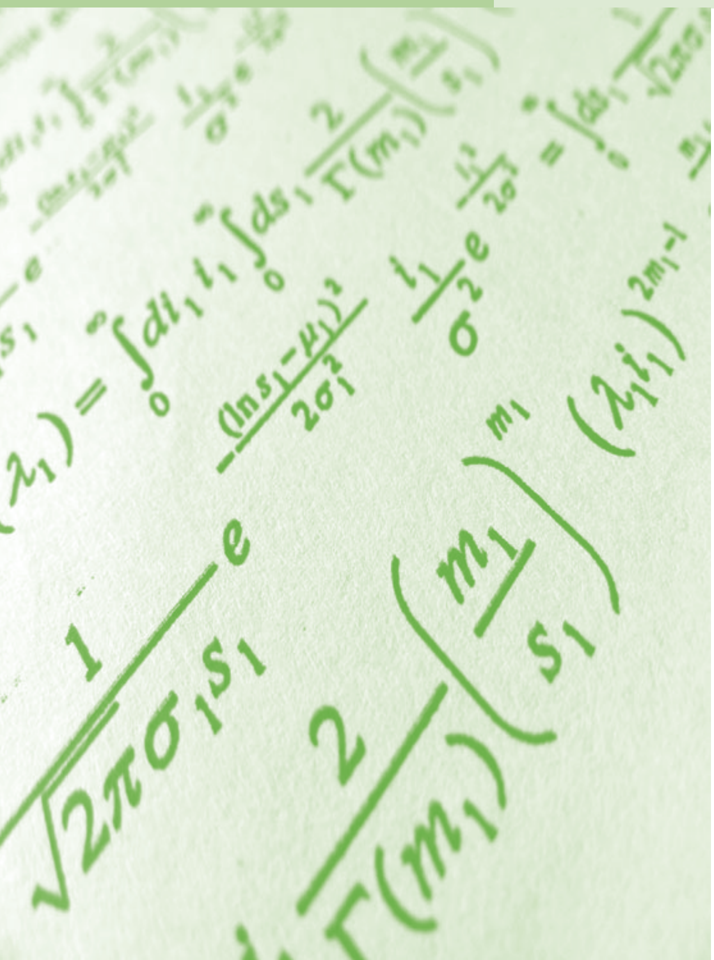
$(1 + x)^{12} = {}^{12}C_0 + {}^{12}C_1\ x + {}^{12}C_2\ x^2 + {}^{12}C_3\ x^3 + \cdots + {}^{12}C_{12}\ x^{12}$

Put $x = -1$

$0 = {}^{12}C_0 - {}^{12}C_1 + {}^{12}C_2 - {}^{12}C_3 + \cdots + {}^{12}C_{12}.$

Chapter

17 Modular Arithmetic



REMEMBER

Before beginning this chapter, you should be able to:

- Have basic knowledge of numbers and their types
- Apply operations on numbers

KEY IDEAS

After completing this chapter, you would be able to:

- Find congruency of numbers and set of residues
- Understand modular addition and multiplication to solve numerical problems
- Represent modular arithmetic system by Caley's table
- Study linear congruence and its solutions

INTRODUCTION

Suppose, you have set out for a drive from your place. Let your wrist watch show time as 5 O'clock. After 6 hours of driving, the time would be 11 O'clock. We can deduce that the time is 11 O'clock by adding 6 hours to 5 O'clock. Now, if the journey continues, after another 7 hours, the time would be 6 O'clock.

It is not going to be, 11 O'clock + 7 = 18 O'clock.

Thus, 18 have to be replaced with 6.

How did we get this 6 from 18?

How is 6 related to 18?

The number 6 is the remainder obtained, when 18 is divided by 12. We say that 18 is congruent to 6 modulo 12.

Similarly, 21 is congruent to 9 modulo 12.

CONGRUENCE

Let a and b be two integers, and m be a positive integer. We say that a is congruent to b modulo m if m is a factor of $(a - b)$.

It is denoted by $a \equiv b \pmod{m}$ or $(a - b) \equiv 0 \pmod{m}$.

Notes

1. If m is not a factor of $a - b$, then $a \not\equiv b \pmod{m}$.
2. If m is a factor of a, then $a \equiv 0 \pmod{m}$.
   Example: 5 is a factor of 15. So, $15 \equiv 0 \pmod{5}$.
3. If r is the remainder on dividing a by m, then $a \equiv r \pmod{m}$.
   Example: $125 \equiv 5 \pmod{6}$ as 125 leaves a remainder of 5 on being divided by 6.
4. If $a \equiv b \pmod{m}$ and c is an integer, then $a + c \equiv b + c \pmod{m}$.
5. If $a \equiv b \pmod{m}$ and c is an integer, then $a \cdot c \equiv b \cdot c \pmod{m}$.
6. If $a \equiv b \pmod{m}$ and c is a positive integer, then $a^c \equiv b^c \pmod{m}$.
7. If p is a prime number, then $x^p \equiv x \pmod{p}$.

Set of Residues

When a positive integer is divided by 2, then the remainder will be either 0 or 1. Hence, 0 and 1 are called residues of 'modulo 2'.

$\therefore$ $\{0, 1\}$ is the set of residues 'modulo 2'. It is denoted by Z_2.

So,

1. $Z_4 = \{0, 1, 2, 3\}$
2. $Z_m = \{0, 1, 2, \ldots, (m - 1)\}$

Modular Addition

Let a and b be two integers, and 'm' be a fixed positive integer. Then, an 'addition modulo m' b is denoted by $a \oplus_m b$ is defined as the remainder when $a + b$ is divided by m.

Examples:

1. $5 \oplus_6 4 = 3$
2. $3 \oplus_4 5 = 0$

Notes

1. If $a \oplus_m b = r$, then $a + b \equiv r \pmod{m}$.
2. $a \oplus_m b = b \oplus_m a$.

Modular Multiplication

Let a and b be two integers, and 'm' be a fixed positive integer.

Then, a 'multiplication modulo m' b is denoted by $a \otimes_m b$ is defined as the remainder when $a \cdot b$ is divided by m.

Examples:

1. $4 \otimes_5 4 = 1$
2. $8 \otimes_3 4 = 2$

Notes

1. If $a \otimes_m b = r$, then $ab \equiv r \pmod{m}$.
2. $a \otimes_m b = b \otimes_m a$.

Construction of Caley's Table

Consider, $Z_2 = \{0, 1\}$

1. All the possible results under addition modulo 2 are:

   $0 \oplus_2 0 = 0$, $1 \oplus_2 0 = 1$,

   $1 \oplus_2 0 = 0$ and $1 \oplus_2 1 = 0$

   These results can be tabulated as follows:

$\oplus_2$	0	1
0	0	1
1	1	0

2. All the possible results under multiplication modulo 2 are:

   $0 \otimes_2 0 = 0$, $1 \otimes_2 0 = 0$,

   $0 \otimes_2 1 = 0$ and $1 \otimes_2 1 = 1$

   These results can be tabulated as follows:

$\otimes_2$	0	1
0	0	0
1	0	1

Note Caley's table is the representation of modular arithmetic system.

EXAMPLE 17.1

(a) Construct Caley's table for $A = \{1, 2, 3\}$ under addition modulo 5.

SOLUTION

$\oplus_5$	1	2	3
1	2	3	4
2	3	4	0
3	4	0	1

(b) Construct Caley's table for the set $\{0, 1, 2, 3, 4\}$ under multiplication modulo 6.

$\otimes_6$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	0	2
3	0	3	0	3	0
4	0	4	2	0	4

Linear Congruence

A polynomial congruence of degree 1 is called a *linear congruence*.

A linear congruence is of the form $ax \equiv b \pmod{m}$, where $a \neq 0 \pmod{m}$.

Examples:

1. $5x \equiv 8 \pmod{4}$
2. $6x \equiv 3 \pmod{5}$

Solution of Linear Congruence

An integer x_1 is said to be a solution of linear congruence $ax \equiv b \pmod{m}$ if $ax_1 \equiv b \pmod{m}$. That is, $ax_1 - b$ is divisible by m.

1. Consider the linear congruence $5x \equiv 8 \pmod{4}$.

   Clearly, $x = 0$ is a solution as $5(0) - 8 = -8$ is divisible by 4.

2. Consider the linear congruence $6x \equiv 3 \pmod{5}$.

   Clearly, $x = 3$ is a solution as $6(3) - 3 = 15$ is divisible by 5.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $15 \equiv -3 \pmod 9$. (True/False)
2. $5 \equiv 2 \pmod 4$. (True/False)
3. $4 \otimes_3 9 =$ ______.
4. The 19th hour of the day is equivalent to ______ hour.
5. $6 \oplus_4 7 =$ ______.
6. In a certain month, the first Sunday falls on the fifth day of the month. In the same month, the fourth Sunday falls on the ______ day.
7. In a certain non-leap year, 1st February is Wednesday. Then the last day of the month is also Wednesday. (True/False)
8. If $63 \equiv 2 \pmod a$ and $a > 1$, then a is ______.
9. If x belongs to the set of residues modulo 4 and $2 + x \equiv 5 \pmod 4$, then $x =$ ______.
10. If $x \equiv y \pmod m$, then $6x - 5 \equiv 6y - 5 \pmod m$. (True/False)
11. In the set of integers modulo 5, $16 \oplus_5 7 =$ ______.
12. In the set of integers modulo 6, $35 \otimes_6 5 =$ ______.
13. If $a + 2 \equiv 3 \pmod 6$, then a is ______.
14. If $6x \equiv 5 \pmod 7$, then find x.
15. If $x - 4 \equiv 8 \pmod 5$, then x is ______.

Short Answer Type Questions

16. If x belongs to the set of residues modulo 6 and $5 + x \equiv 3 \pmod 6$, then find x.
17. If x belongs to the set of residues modulo 4 and $6x - 3 \equiv -1 \pmod 4$, then find x.
18. If $46 \equiv 11 \pmod a$, and a is a prime number, then find the greatest possible value of a.
19. If 1st July 2006 was a Saturday, then what day of the week will be 18th July, 2007?
20. If you were born on March 8, 1990 and the day of the week was a Thursday, then on what day of the week did your birthday fall in 1991?
21. Find the remainder when $(26)^{31}$ is divided by 31.
22. Find the remainder when 8^{15} is divided by 5.
23. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then list out all the pairs of distinct numbers from set A which are congruent to each other under modulo 5.
24. Construct Caley's table for Z_7 under multiplication modulo 7.
25. Construct Caley's table for the set $A = \{2, 4, 6, 8, 10\}$ under addition modulo 12.

Essay Type Questions

26. Construct Caley's table for the set $B = \{1, 3, 5, 7, 9\}$ under multiplication modulo 10.
27. Construct Caley's table for Z_8 under addition modulo 8.
28. Find the remainder when 3^{31} is divided by 31.
29. If $a \otimes_m b = 1$, then b is called there ciprocal of a under modulo m. Find the reciprocal of 8 under modulo 17.
30. How many two digit numbers satisfy the equation, $3x \equiv 5 \pmod 7$?

CONCEPT APPLICATION

Level 1

1. In the set of integers modulo 8, $28 \otimes_8 2 =$ ______.
   (a) 0 (b) 1
   (c) 2 (d) 3

2. If $25 \equiv 4 \pmod{p}$, where p is a prime number, then p is ______.
   (a) 3 (b) 5
   (c) 7 (d) Either (a) or (c)

3. Solve for x, if $5x \equiv 0 \pmod{4}$.
   (a) 0 (b) 3
   (c) 2 (d) 4

4. In the set of integers modulo 12, $38 \oplus_{12} 28 =$ ______.
   (a) 6 (b) 5
   (c) 4 (d) 3

5. The largest single-digit number that satisfies $14x \equiv 4 \pmod{3}$ is ______.
   (a) 5 (b) 7
   (c) 8 (d) 9

6. If 8-8-2009 is a Saturday, then 15-8-2010 falls on ________.
   (a) Saturday (b) Sunday
   (c) Wednesday (d) Thursday

7. If $23 \equiv 7 \pmod{x}$, then which of the following cannot be the value of x?
   (a) 4 (b) 6
   (c) 8 (d) 16

8. Find the remainder when 13^{15} is divided by 5.
   (a) 4 (b) 3
   (c) 2 (d) 1

9. Now the time is 1: 30 pm. If I woke up 8 hours ago, then I woke up at ______.
   (a) 4:30 am (b) 5:30 am
   (c) 3:30 am (d) 6:30 am

10. If $37 \equiv 18 \pmod{p}$, where p is a prime number, then find p.
    (a) 3 (b) 7
    (c) 19 (d) 18

11. If $15 \equiv 3 \pmod{x}$, then which of the following cannot be the value of x?
    (a) 3 (b) 4
    (c) 6 (d) 8

12. Find the remainder when 5^{18} is divided by 19.
    (a) 1 (b) 4
    (c) 11 (d) 17

13. The largest two-digit number that satisfies $5x \equiv 6 \pmod{4}$ is ______.
    (a) 96 (b) 97
    (c) 98 (d) 99

14. If you were born on 15-4-1993 which was a Tuesday, then on which day of the week did your birthday fall in 1994?
    (a) Tuesday (b) Wednesday
    (c) Thursday (d) Monday

15. Find the remainder when 11^{12} is divided by 7.
    (a) 0 (b) 1
    (c) 3 (d) 5

Level 2

16. If $a \equiv b \pmod{m}$ and the remainder obtained when 'a' is divided by m is 2, then find the remainder when 'b' is divided by m.
    (a) 2 (b) 1
    (c) 0 (d) 3

17. If $x \equiv y \pmod{2}$, then which of the following is/are correct?
    (A) x is even and y is odd.
    (B) Both x and y are odd.
    (C) Both x and y are even.

PRACTICE QUESTIONS

(a) Only (C)

(b) Only (A)

(c) Both (B) and (C)

(d) Both (A) and (B)

18. If 1-1-2010 is a Friday, then the fifth Sunday of January, 2011 will fall on ______.

(a) 26th day (b) 27th day

(c) 29th day (d) 30th day

19. Anand started a work on Sunday at 9:30 am. He finished the work after 87 hours. Then he finished the work on ______.

(a) Wednesday at 11:30 pm

(b) Thursday at 0:30 am

(c) Wednesday at 0:30 am

(d) Thursday at 11:30 pm

20. Which of the following are the common solutions of $3x \equiv 0 \pmod 6$ and $2x \equiv 0 \pmod 4$?

(A) 0 (B) 2 (C) 4

(a) Both (A) and (B)

(b) Both (A) and (C)

(c) Both (B) and (C)

(d) All of (A), (B) and (C)

21. If $15x \equiv 2 \pmod 3$, then which of the following is a possible value of x?

(a) 3 (b) 315

(c) 0 (d) None of these

22. Which of the following is a common solution for $6x \equiv 0 \pmod 8$ and $8x \equiv 0 \pmod{10}$?

(a) 0 (b) 4

(c) 6 (d) 8

23. Find the remainder when 2^{24} is divided by 35.

(a) 2 (b) 31

(c) 1 (d) 29

24. Which of the following is correct?

(a) $5 \oplus_3 2 \equiv 3 \otimes_3 6 \pmod 4$

(b) $4 \oplus_3 2 \equiv 3 \otimes_4 5 \pmod 6$

(c) $5 \oplus_5 3 \equiv 6 \otimes_8 9 \pmod 3$

(d) $5 \oplus_5 3 \equiv 6 \otimes_8 9 \pmod 4$

25. Which of the following is/are correct?

(a) $6 \oplus_4 3 \equiv 7 \otimes_9 8 \pmod 5$

(b) $10 \oplus_5 4 \equiv 9 \otimes_{11} 9 \pmod{11}$

(c) $14 \oplus_8 8 \equiv 15 \otimes_{16} 12 \pmod 4$

(d) $10 \oplus_5 4 \equiv 9 \otimes_{11} 9 \pmod{10}$

26. If x belongs to the set of residues modulo 10, then the common solution of $5 + x \equiv 0 \pmod 3$ and $6 + x \equiv 0 \pmod 5$ is ________.

(a) 1 (b) 2

(c) 4 (d) 5

27. By which of the following numbers should 3^5 be divided to obtain a remainder 3?

(a) 7 (b) 11

(c) 5 (d) 3

28. Find the remainder when $6^{11} - 6$ is divided by 11.

(a) 5 (b) 1

(c) 0 (d) None of these

29. Find x, if $9x \equiv 2 \pmod 7$.

(a) 1 (b) 2

(c) 3 (d) 4

30. Find the remainder when 3^{19} is divided by 19.

(a) 3 (b) 15

(c) 16 (d) 19

31. In the set of integers modulo 9, $15 \otimes_9 10 =$ ______.

(a) 3 (b) 6

(c) 0 (d) 1

32. If $7x \equiv 1 \pmod 5$, then which of the following is a possible value of x?

(a) 10 (b) 11

(c) 12 (d) None of these

33. In the set of integers modulo 17, $19 \oplus_{17} 15 =$ ______.

(a) 0 (b) 1

(c) 2 (d) 3

34. In order to enter her name in the Guinness Book of world records, Sangeeta started singing on

PRACTICE QUESTIONS

Monday at 10.30 am. If she sings continuously for 36 hours, then she will finish her singing on

(a) Tuesday at 10.30 am

(b) Wednesday at 10.30 am

(c) Tuesday at 10.30 pm

(d) Wednesday at 10.30 pm

35. Which of the following is a common solution of $3x \equiv 2 \pmod 5$ and $4x \equiv 0 \pmod 6$?

(a) 9

(b) 4

(c) 6

(d) 3

Level 3

36. Find the remainder when 3^{215} is divided by 43.

(a) 35 (b) 28

(c) 33 (d) 30

37. Kishore reached his school on Monday at 8:30 am, and then immediately started on a tour to GOA. After $106\frac{1}{2}$ hours, he reached his house. Then, Kishore reached his home on

(a) Saturday at 7 pm.

(b) Friday at 6 pm.

(c) Saturday at 6 pm.

(d) Friday at 7 pm.

38. If 1-8-2012 is Wednesday, then find the day on which we shall celebrate our Independence Day in the year 2015.

(a) Saturday (b) Sunday

(c) Friday (d) Thursday

39. Find the remainder when 5^{97} is divided by 97.

(a) 5 (b) 97

(c) 92 (d) 100

40. If $a \equiv b \pmod m$, then which of the following is not always true?

(a) $a^2 \equiv b^2 \pmod m$

(b) $a + m \equiv b + m \pmod{2m}$

(c) $am \equiv bm \pmod{m^2}$

(d) $a + m \equiv b - m \pmod{2m}$

41. If $x^3 \equiv x \pmod 3$, then x can be ________.

(a) 2 (b) 5

(c) 4 (d) All of these

42. A part of Caley's table for $\otimes_6$ is given below. Find the values of x, y and z.

$\otimes_6$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	0	2
3	0	3	0	y	0
4	0	4	z	0	x

(a) $x = 4$, $y = 3$, $z = 2$

(b) $x = 2$, $y = 4$, $z = 3$

(c) $x = 3$, $y = 4$, $z = 2$

(d) $x = 4$, $y = 2$, $z = 3$

43. A part of Caley's table for $\oplus_7$ is given below. Find the value of p, q, x and y.

$\oplus_7$	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	p	x
4	5	Q	y	1

(a) $x = y = 0$, $p = q = 5$

(b) $x = y = 1$, $p = q = 6$

(c) $x = y = 3$, $p = q = 4$

(d) $x = y = 0$, $p = q = 6$

44. The Independence Day of India in 2007 was celebrated on a Wednesday, then Children's day in 2008 was celebrated on a ________.

(a) Friday (b) Saturday

(c) Sunday (d) Monday

PRACTICE QUESTIONS

45. If $13 \equiv 3 \pmod{p}$, then p can be ________.

(a) 2 (b) 5

(c) 10 (d) All of these

46. If the 1st January of a certain year, which was not a leap year, was a Thursday, then what day of the week was the 31st December of that year?

(a) Monday (b) Thursday

(c) Sunday (d) Saturday

47. If $x + 10 \equiv 1 \pmod{8}$, then x can be ________.

(a) 1 (b) 0

(c) 6 (d) 7

48. If x belongs to the set of residues modulo 6 and $3 + x \equiv 2 \pmod{6}$, then $x =$ ________.

(a) 1 (b) 3

(c) 4 (d) 5

49. The 2-1-2009 is a Friday. The fourth Sunday of January 2010 falls on the ______.

(a) 23rd day

(b) 24th day

(c) 25th day

(d) 26th day

50. Which of the following is/are correct?

(a) $5 \oplus_2 4 \equiv 17 \otimes_5 3 \pmod{7}$

(b) $6 \oplus_4 7 \equiv 19 \otimes_9 3 \pmod{3}$

(c) $9 \oplus_7 3 \equiv 8 \otimes_7 9 \pmod{9}$

(d) $5 \oplus_2 4 \equiv 17 \otimes_5 3 \pmod{5}$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. True
2. False
3. 0
4. 7th
5. 1
6. 26th
7. False
8. 61
9. 3
10. True
11. 3
12. 1
13. 1
14. 2
15. 2

Short Answer Type Questions

16. $x = 4$
17. 1 or 3
18. 7
19. Wednesday
20. Friday
21. 31
22. 2
23. [0, 5], [0, 10], [1, 6], [2, 7] [3, 8], [4, 9], [5, 10]

Essay Type Questions

28. 2
29. 3
30. 4

CONCEPT APPLICATION

Level 1

1. (a)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (c)
11. (d)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (b)					

Level 2

16. (a)	**17.** (c)	**18.** (d)	**19.** (b)	**20.** (d)	**21.** (d)	**22.** (a)	**23.** (c)	**24.** (c)	**25.** (b)
26. (c)	**27.** (c)	**28.** (c)	**29.** (a)	**30.** (a)	**31.** (b)	**32.** (d)	**33.** (a)	**34.** (c)	**35.** (a)

Level 3

36. (b)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (b)	**41.** (d)	**42.** (a)	**43.** (d)	**44.** (a)	**45.** (d)
46. (b)	**47.** (d)	**48.** (d)	**49.** (b)	**50.** (a)					

CONCEPT APPLICATION

Level 1

1. Recall the concept of modular multiplication.
2. Check from the options.
3. Check from the options.
4. Recall the concept of modular addition.
5. Check from the options.
6. Use the concept, '365 under modulo 7'.
7. Check from the options.
8. Use the concept of congruence modulo.
9. Check from the options.
10. Check from the options.
11. Check from the options.
12. Start with the step $5^2 \equiv 6 \pmod{19}$ and proceed.
13. Check from the options.
14. Use the concept '365 under modulo 7'.
15. Use the concept of congruence modulo.
16. If $a \equiv b \pmod{m}$, then the remainder obtained when a is divided by m is equal to the remainder obtained when b is divided by m.
17. If $a - b$ is divisible by 2, then both a and b are either even numbers or odd numbers.
18. 1-1-2010 is a Friday. 1-1-2011 is a Saturday. First Sunday in 2011 is 2nd January.
19. 87 hours = 3 days + 15 hours.
20. Verify whether the given options are common solutions are not.
21. $15x$ is always divisible by 3.
22. Verify whether each option is a solution of both the equations or not.
23. Use, if p is prime then $a^p \equiv a \pmod{p}$.
24. $a \oplus_m b$ means 'the remainder when $a + b$ is divided by m' and $a \otimes_m b$ means.

    'The remainder when ab is divided by m'.
25. (i) Use the definitions of addition modulo m and multiplication modulo m.

    (ii) Substitute the values in the options in the given inequations.

    (iii) The point which satisfies the given inequations is the required point.
26. $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $x \in Z_{10}$

    $5 + x \equiv 0 \pmod{3} \Rightarrow x = 1, 4$ and 7

    $6 + x \equiv 0 \pmod{5} \Rightarrow x = 4$ and 9

    $\therefore$ Common solution is $x = 4$.
27. $3^5 \equiv 3 \pmod{x} \Rightarrow 243 = 3 \pmod{x}$

    $\Rightarrow 243 - 3$ is divisible by x

    $\Rightarrow 240$ is divisible by x.

    $\therefore$ From the options, $x = 5$.
28. We have $a^p \equiv a \pmod{p}$ where p is a prime number.

    11 is prime number, $6^{11} \equiv 6 \pmod{11}$

    That is, $6^{11} - 6$ is divisible by 11. The remainder when $6^{11} - 6$ is divided by 11 is zero.
29. Given, $9x \equiv 2 \pmod{7} \Rightarrow 9x - 2$ is divisible by 7.

    From the options, if $x = 1$, then $9 - 2$ is divisible by 7.
30. We have $a^p \equiv a \pmod{p}$, where p is a prime number.

    When 3^{19} is divided by 19, the remainder is 3.
31. The remainder when 15×10, i.e., 150 is divided by 9 is 6.
32. Given, $7x \equiv 1 \pmod{5}$.

    $7x - 1$is divisible by 5.

    $\therefore x = 3$ or 8 or 13 or … satisfies the above relation.
33. The remainder when $19 + 15$, i.e., 34 is divided by 17 is 0.
34. Monday, 10.30 am to Tuesday, 10.30 am is 24 hours.

    Tuesday, 10.30 am to Tuesday, 10.30 pm = 12 hours.

    $\therefore$ She will finish her singing on Tuesday night at 10.30 pm.
35. Given $3x \equiv 2 \pmod{5}$

    $\Rightarrow 3x - 2$ is divisible by 5.

    $x = 4$ or 9 or 14, or … satisfies the above relation.

    $4x \equiv 0 \pmod{6}$

    $\Rightarrow 4x$ is divisible by 6

$x = 3$ or 9 or 15 … satisfies the above relation.

$\therefore$ The common solution of $3x \equiv 2 \pmod 5$ and $4x \equiv 0 \pmod 6$ is 9. (From the options)

37. $106\frac{1}{2}$ hours = 4 days + $10\frac{1}{2}$ hours.

38. The number of days in between August 15, 2015 and August 1, 2012 is 1109.

39. We know that, when p is a prime number $a^p \equiv a \pmod p$.

40. (i) Use the condition $a \equiv b \pmod m \Rightarrow a - b$ is divisible by m.

(ii) Verify whether each option satisfies the above condition or not.

(iii) Refer the properties of congruence modulo.

41. Given $x^3 \equiv x \pmod 3$

$x^3 - x$ is divisible by 3.

$x(x^2 - 1)$ is divisible by 3.

$(x - 1)\, x\, (x + 1)$ is divisible by 3.

$x - 1, x, x + 1$ are three consecutive numbers.

We know that the product of three consecutive numbers is always divisible by 3.

$\therefore$ For any integer of x (where $x \geq 2$), the given relation is true.

42. x is the remainder when 4×4 is divided by 6.

$\therefore x = 4$

y is the remainder when 3×3 is divided by 6.

$\therefore y = 3$

z is the remainder when 4×2 is divided by 6.

$\therefore z = 2$.

43. p is the remainder when $3 + 3$ is divided by $7 = 6$

q is the remainder when $4 + 2$ is divided by $7 = 6$

x is the remainder when $3 + 4$ is divided by $7 = 0$

y is the remainder when $4 + 3$ is divided by $7 = 0$

$\therefore x = y = 0$ and $p = q = 6$.

44. The number of days from 15-10-2007 to 14-11-2008 is 458 days.

Remainder when 458 is divided by 7 is 3.

Since the first day is Wednesday, the third day is Friday.

$\therefore$ Children's day in 2008 was celebrated on Friday.

45. Given $13 \equiv 3 \pmod p$

$\Rightarrow 13 - 3$ is divisible by p

$\Rightarrow 10$ is divisible by p

i.e., p is a factor of 10.

$\therefore p = 1, 2, 5$ or 10

$\therefore$ All the options are true.

46. The given year is a non-leap year and contains 365 days.

(The number of days from 1st January to 31st December is 365)

$365 \equiv 1 \pmod 7$

The 365th day is equivalent to the first day of the year.

$\therefore$ But the first day of the week is Thursday.

$\therefore$ Hence, 31st December is Thursday.

47. Given $x + 10 = 1 \pmod 8$

$\Rightarrow x + 10 - 1$ is divisible by 8

$\Rightarrow x + 9$ is divisible by 8

From the options, if $x = 7$ then $7 + 19$, i.e., 16 is divisible by 8.

$\therefore x = 7$.

48. $Z_6 = \{0, 1, 2, 3, 4, 5\}$ and $x \in Z_6$.

Given, $3 + x \equiv 2 \pmod 6$

$\Rightarrow 3 + x - 2$ is divisible by 6

$\Rightarrow x + 1$ is divisible by 6

$x = 5$ satisfies the above conditions.

49. The number of days from 2-1-2009 to 1-1-2010 is 365 days

$\Rightarrow 365 \equiv 1 \pmod 7$

$\therefore$ 1-1-2010 is Friday ($\therefore$ 2-1-2009 is Friday)

$\therefore$ 1st Sunday falls on 3-1-2010

$\therefore$ 2nd Sunday falls on 10-1-2010

$\therefore$ 3rd Sunday falls on 17-1-2010

$\therefore$ 4th Sunday falls on 24-1-2010.

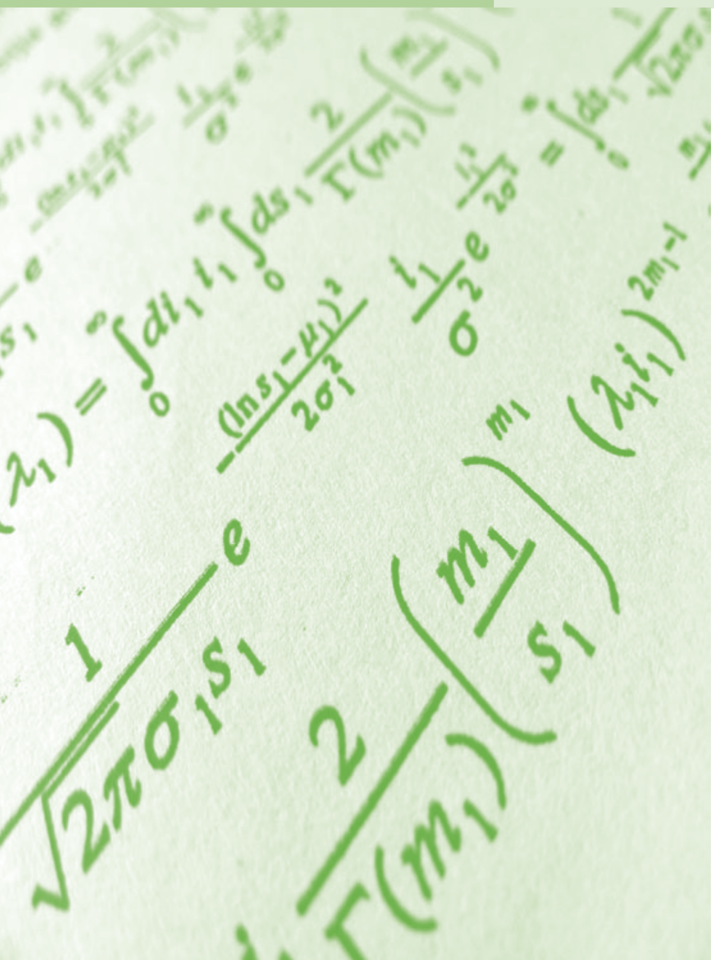
50. $5 \oplus_2 4 \equiv 17 \otimes_5 3 \pmod 7$

$1 \equiv 1 \pmod 7$

$\therefore 1 - 1$ is divisible by 7 (true).

Chapter 18

Linear Programming



REMEMBER

Before beginning this chapter, you should be able to:

- Have basic knowledge of linear equations and inequations
- Represent equations and inequations by graphs

KEY IDEAS

After completing this chapter, you would be able to:

- Study about convex sets and objective functions
- Understand open and closed convex polygon
- State and prove the fundamental theorem
- Study graphical method of solving linear programming problem

INTRODUCTION

In business and industry, certain problems arise, the solutions of which depend on the way in which a change in one variable may affect the other. Hence, we need to study the interdependence between variables, such as, the cost of labour, cost of transportation, cost of material, availability of labour and profit. In order to study the interdependence, we need to represent these variables algebraically. The conditions, which these variables have to satisfy, are represented as a set of linear inequations. We try to find the best or the optimum condition by solving these inequations. This process is called 'linear programming'. In a situation where only two variables exist, the graphical method can be used to arrive at the optimal solution. To start with, let us define some important terms related to linear equations and inequations.

Convex Set

A subset X of a plane is said to be convex, if the line segment joining any two points P and Q in X, is contained in X.

Example: The triangular region ABC (shown below) is convex, as for any two points P and Q in the region, the line segment joining P and Q, i.e., $\overline{PQ}$ is contained in the triangular region ABC.

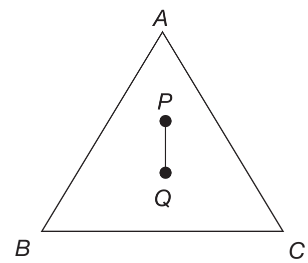


Figure 18.1

Example: The polygonal region $ABCDE$ (shown below) is not convex, as there exist two points P and Q, such that these points belong to the region $ABCDE$, but the line segment $\overline{PQ}$ is not wholly contained in the region.

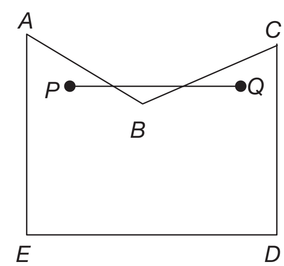


Figure 18.2

Objective Function

Every linear programming problem involving two variables, consists of a function of the form $f = ax + by$, which is to be either maximized or minimized subject to certain constraints. Such a function is called the objective function or the profit function.

Closed-Convex Polygon

Closed convex polygon is the set of all points within, or on a polygon with a finite number of vertices.

Example: If l_1 and l_2 are two lines, which meet the coordinate axes at A, E and D, C respectively, and B is the point of intersection of the lines, then $OABC$ is a closed convex polygon.

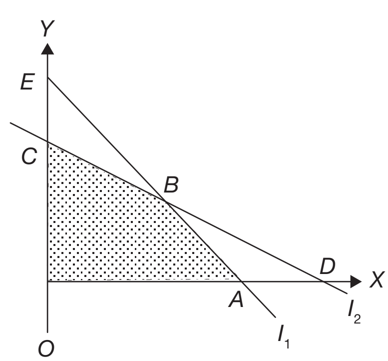


Figure 18.3

Open-Convex Polygon

Consider an infinite region bounded by two non-intersecting rays and a number of line segments, such that the endpoint of each segment is also the endpoint of another segment or one of the rays. If the angle between any two intersecting segments, or between a segment and the intersecting ray, measured through the region, is less than 180°, the region is an open-convex polygon.

For example, in the figure above, the region bounded by the rays $\overrightarrow{AP}, \overrightarrow{DQ}$ and the segments $\overline{AB}, \overline{BC}, \overline{CD}$ is an open convex polygon, because each of the angle $\angle A$, $\angle B$, $\angle C$ and $\angle D$ is less than 180°.

The Fundamental Theorem

When the values of the expression, $f = ax + by$, are considered over the set of points forming a non-empty closed convex polygon, the maximum or minimum value of f occurs on at least one of the vertices of the polygon. To solve any linear programming problem, we use the fundamental theorem.

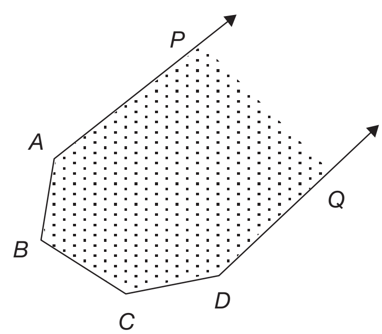


Figure 18.4

Feasible Region

The common region determined by all the constraints of a linear programming problem is called the feasible region.

Feasible Solution

Every point in the feasible region is called a feasible solution.

Optimum Solution

A feasible solution which maximizes or minimizes the objective function is called an optimum solution.

Graphical Method of Solving a Linear Programming Problem

When the solution set of the constraints is a closed convex polygon, the following method can be adopted:

1. Draw the graphs of the given system of constraints (system of inequations).
2. Identify the common region. If it is a closed convex polygon, find its vertices.
3. Find the value of the objective function at each of these vertices.
4. The vertex at which the objective function has the maximum or minimum value, gives the required solution.

The following examples explain the method in detail.

EXAMPLE 18.1

Maximize the function $f = 4x + 5y$ subject to the constraints $3x + 2y \leq 18$, $x + y \leq 7$ and $x \geq 0$, $y \geq 0$.

SOLUTION

Given, $f = 4x + 5y$

$3x + 2y \leq 18 \rightarrow$ (A)

$x + y \leq 7 \rightarrow$ (B)

and $x \geq 0$, $y \geq 0$.

As $x \geq 0$ and $y \geq 0$, the feasible region lies in first quadrant. The graph of $3x + 2y \leq 18$ is shown with horizontal lines and the graph of $x + y \leq 7$ is shown with vertical lines.

$\therefore$ The feasible region is the part of the first quadrant in which there are both horizontal and vertical lines.

The feasible region is a closed-convex polygon $OAED$ in the above graph. The vertices of the closed polygon $OAED$ are $O(0, 0)$, $A(6, 0)$, $E(4, 3)$ and $D(0, 7)$.

Now, $f = 4x + 5y$

At $O(0, 0)$, $f = 0$

At $A(6, 0)$, $f = 4(6) + 5(0) = 24$

At $E(4, 3)$, $f = 4(4) + 5(3) = 31$

At $D(0, 7)$, $f = 4(0) + 5(7) = 35$

$\therefore$ f is maximum at $D(0, 7)$ and the maximum value is 35.

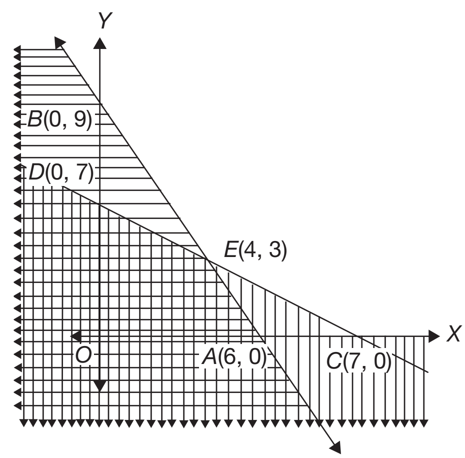


Figure 18.5

EXAMPLE 18.2

A manufacturer makes two models A and B of a product. Each model is processed by two machines. To complete one unit of model A, machines I and II must work 1 hour and 3 hours respectively. To complete one unit of model B, machine I and II must work 2 hours and 1 hour respectively. Machine I may not operate for more than 8 hours per day, and machine II for not more than 9 hours per day. If profits on model A and B per unit are ₹300 and ₹350, then how many units of each model should be produced, per day, to maximize the profit?

SOLUTION

Let the manufacturer produce x units of model A and y units of model B $\Rightarrow x \geq 0$, $y \geq 0$.

$\therefore$ Profit function $f = 300x + 350y$.

To make x and y units of models A and B, machine I should be used only 8 hours per day.

$\therefore$ $x + 2y \leq 8$ and machine II should be used for at the most 9 hours per day.

$3x + y \leq 9$ and $x \geq 0$, $y \geq 0$

Hence, we maximize $f = 300x + 350y$, subject to the constraints.

$$x + 2y \leq 8$$
$$3x + y \leq 9$$
$$x \geq 0,\ y \geq 0$$

As $x \geq 0$ and $y \geq 0$, the feasible region lies in the first quadrant.

The graph of $x + 2y \leq 8$ is shown with horizontal lines and the graph of $3x + y \leq 9$ is shown with vertical lines.

$\therefore$ The feasible region is the part of the first quadrant in which there are both horizontal and vertical lines.

The shaded region is the closed polygon having vertices $O(0, 0)$, $A(3, 0)$, $B(2, 3)$ and $C(0, 4)$.

Profit function $f = 300x + 350y$

At vertex $O(0, 0)$, $f = 300(0) + 350(0) = 0$

At vertex $A(3, 0)$, $f = 300(3) + 350(0) = 900$

At vertex $B(2, 3)$, $f = 300(2) + 350(3) = 600 + 1050 = 1650$

At vertex $C(0, 4)$, $f = 300(0) + 350(4) = 1400$

$\therefore f$ is maximum at the vertex $B(2, 3)$.

Hence, in order to get the maximum profit, the manufacturer has to produce 2 units of model A and 3 units of model B per day.

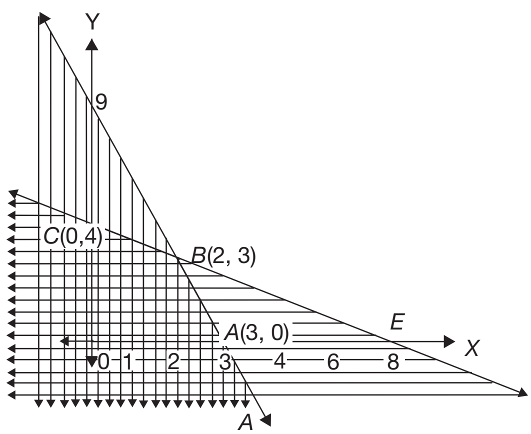


Figure 18.6

EXAMPLE 18.3

A transport company has two main depots P and Q, from where buses are sent to three sub-depots A, B and C in different parts of a region. The number of buses available at P and Q are 12 and 18. The requirements of A, B and C are 9, 13 and 8 buses respectively. The distance between the two main depots and the three sub-depots are given in the following table (in km).

To → / From ↓	A	B	C
P	15	40	50
Q	20	30	70

How should the buses be sent from P and Q to A, B and C, so that the total distance covered by the buses is the minimum?

SOLUTION

Let x buses be sent from P to A, and y buses be sent from P to B.

Since, P has only 12 buses, so $12 - (x + y)$ buses are sent from P to C.

Since, 9 buses are required for A, and x buses are sent from P, so $(9 - x)$ buses should be sent from Q to A. Similarly, the number of buses to be sent from Q to B is $13 - y$, and the number of buses to be sent from Q to C is $18 - (9 - x + 13 - y)$ or $x + y - 4$.

The above can be represented in the following table:

The Number of Buses to be Sent			
To→ From↓	A (9)	B (13)	C (8)
P (12)	x	y	$12 - (x + y)$
Q (18)	$9 - x$	$13 - y$	$x + y - 4$

All the variables are non-negative.

The given conditions are:

$x \geq 0$ and $x \leq 9$

$y \geq 0$ and $y \leq 13$

The region which satisfies the above set of inequalities is the rectangle *OGBC*.

The graph of $x + y \geq 4$ is shown with horizontal lines and the graph of $x + y \leq 12$ is shown with vertical lines.

∴ The feasible region is the area with both horizontal and vertical lines within the rectangular region *OGBC*, i.e., the polygonal region *FGHDE*.

The distance covered by the buses

$d = 15x + 40y + 50(12 - (x + y)) + 20(9 - x) + 30(13 - y) + 70(x + y - 4)$

$\quad = 15x + 40y + 600 - 50x - 50y + 180 - 20x + 390 - 30y + 70x + 70y - 280$

$\quad = 15x + 30y + 890$

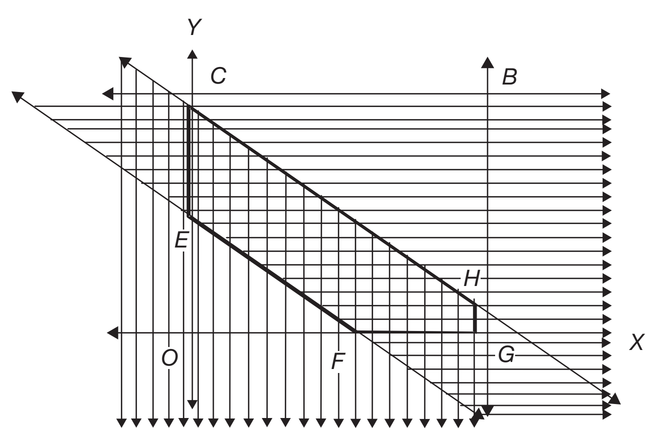


Figure 18.7

∴ We have to minimize the distance d.

In the graph, *DEFGH* is a closed-convex polygon with vertices $F(4, 0)$, $G(9, 0)$, $H(9, 3)$, $D(0, 12)$ and $E(0, 4)$ objective function $d = 15x + 30y + 890$.

At vertex $F(4, 0)$, $d = 15(4) + 30(0) + 890 = 60 + 890 = 950$

At vertex $G(9, 0)$, $d = 15(9) + 30(0) + 890 = 1025$

At vertex $H(9, 3)$, $d = 15(9) + 30(3) + 890 = 1115$

At vertex $D(0, 12)$, $d = 15(0) + 30(12) + 890 = 1250$

At vertex $E(0, 4)$, $d = 15(0) + 30(4) + 890 = 1010$

$\therefore$ d is minimum at $F(4, 0)$.

$\therefore$ $x = 4$; $y = 0$

$\therefore$ 4 buses have to be sent from P to A, and 8 buses have to be sent from P to C. Also, 5 buses have to be sent from Q to A, and 13 buses have to be sent from Q to B to minimize the distance travelled.

The Number of Buses to be Sent

To → / From ↓	A	B	C
P	4	0	8
Q	5	13	0

General Graphical Method for Solving Linear Programming Problems

If the polygon is not a closed-convex polygon, the above method is not applicable. So, we apply the following method:

1. In this method, first we draw graphs of all systems of inequations representing the constraints.
2. Let the objective function be $ax + by$. Now, for different values of the function, we draw the corresponding lines $ax + by = c$. These lines are called the isoprofit lines.
3. Take different values of c, so that the line $ax + by = c$ moves away from the origin till we reach a position where the line has at least one point in common with the feasible region. At this point, the objective function has the optimum value.

This is explained with the following example.

EXAMPLE 18.4

Minimize $3x + 2y$ subject to the constraints $x + y \geq 5$ and $x + 2y \geq 6$, $x \geq 0$, $y \geq 0$.

SOLUTION

As $x \geq 0$ and $y \geq 0$, the feasible region lies in the I quadrant.

The graph of $x + y \geq 5$ is shown with horizontal lines, and the graph of $x + 2y \geq 6$ is shown with vertical lines.

$\therefore$ The feasible region is that part of the I quadrant in which there are both horizontal and vertical lines.

At $B(4, 1)$, $f(x, y) = 3(4) + 2(1) = 14$

At $A(0, 5)$, $f(x, y) = 3(0) + 2(5) = 10$

At $C(6, 0)$, $f(x, y) = 3(6) + 2(0) = 18$.

Clearly, $f(x, y)$ is the minimum at A.

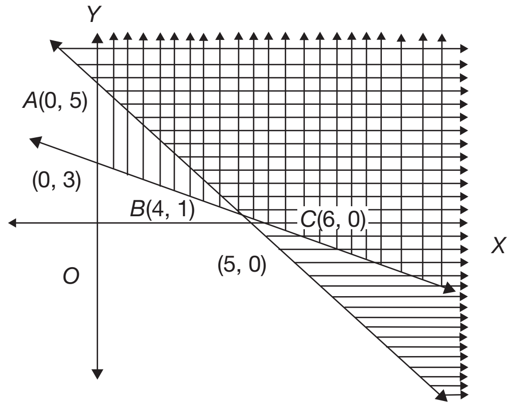


Figure 18.8

EXAMPLE 18.5

In an examination, the marks obtained by Rohan in two subjects are x and y. The total marks in the two subjects is less than or equal to 150. The maximum of marks of each subject is 100. Find the sum of the minimum and the maximum values of $3x + 4y$. (No negative marks in the examination.)

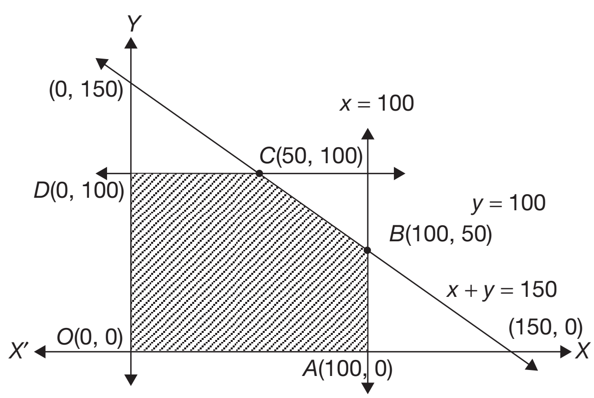
(a) 400 **(b)** 550 **(c)** 500 **(d)** 300

SOLUTION

$x + y \leq 150$

$x \geq 0,\ y \geq 0$ and $x \leq 100$

$y \leq 100$



The shaded region is a closed convex polygon whose vertices are:

$O(0, 0)$, $A(100, 0)$, $B(100, 50)$, $C(50, 100)$ and $D(0, 100)$.

$f = 3x + 4y$

The value off at $O(0, 0)$ is $= 3 \times 0 + 4 \times 0 = 0$.

The value off at $A(100, 0)$ is $= 300$.

The value off at $B(100, 50)$ is $300 + 200 = 500$.

The value off at $C(50, 100)$ is

$= 3 \times 50 + 4 \times 100 = 550$

The value of f at $D(0,100)$ is $= 0 + 400 = 400$.

The sum of minimum value and maximum value $= 0 + 550 = 550$.

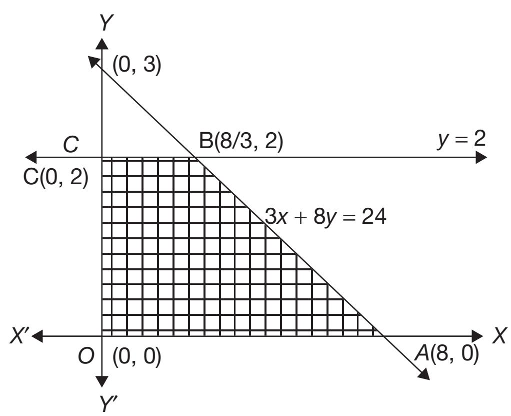
EXAMPLE 18.6

Find the maximum value of $4x + 7y$ with the conditions $3x + 8y \leq 24$, $y \leq 2$, $x \geq 0$ and $y \geq 0$.

(a) 14 **(b)** $\frac{74}{3}$ **(c)** 21 **(d)** 32

SOLUTION

Given condition $3x + 8y \leq 24$
$y \leq 2;\ x \geq 0,\ y \geq 0.$



The vertices of the feasible polygon region are (0, 0), (8, 0), (8/3, 2) and (0, 2).

The maximum value of the objective function attains at (8, 0).

$f = 4x + 7y$

The maximum value $= 4 \times 8 + 7 \times 0 = 32$.

EXAMPLE 18.7

Which of the following is a point in the feasible region determined by the linear inequations $3x + 2y \geq 6$ and $8x + 7y \leq 56$?

(a) (3, 1) **(b)** (–3, 1) **(c)** (1, –3) **(d)** (–3, –1)

HINT

(3, 1) satisfies both the given inequations.

EXAMPLE 18.8

The solution of the system of inequalities $x \geq 0$, $y \geq 0$, $5x + 2y \geq 10$, $6x + 5y \leq 30$ is a polygonal region with the vertices ______.

(a) (0, 0), (2, 0), (0, 5), (0, 6)

(b) (0, 0), (5, 0), (6, 0), (0, 2)

(c) (2, 0), (5, 0), (0, 6), (0, 5)

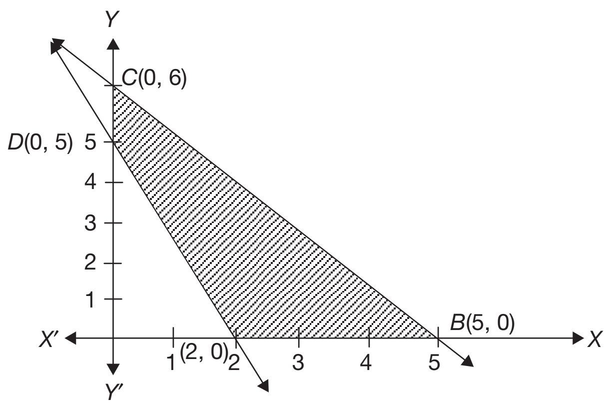
(d) (0, 0), (0, 5), (6, 0), (2, 0)

SOLUTION

Given constraints are:

$5x + 2y \geq 10$; $6x + 5y \leq 30$ and $x \geq 0$, $y \geq 0$

The given lines form a closed-convex polygon with the vertices (2, 0), (5, 0), (0, 6) and (0, 5)



EXAMPLE 18.9

The length and breadth of a rectangle (in cm) are x and y respectively $x \leq 30$, $y \leq 20$, $x \geq 0$ and $y \geq 0$. If a rectangle has a maximum perimeter, then its area is ______.

(a) 400 cm^2 **(b)** 600 cm^2 **(c)** 900 cm^2 **(d)** None of these

SOLUTION

The perimeter function is $p = 2(x + y)$

p is maximum, when $x = 30$ and $y = 20$

$\Rightarrow p = 2(50) = 100$

$\therefore$ The area of the rectangle $= 30 \times 20 = 600$ cm^2.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Does the point (1, 3) lie in the region specified by $x - y + 2 > 0$?

2. The region specified by the inequality $4x + 6y \leq 12$ contains the origin. (True/False)

3. Does the point (0, 0) lie in the region specified by $x + y > 6$?

4. In a rectangular coordinate system, the region specified by the inequality $y \geq 1$ lies below the X-axis. (True/False)

5. The feasible solution that maximizes an objective function is called ______.

6. If the line segment joining any two points A and B, belonging to a subset Y of a plane, is contained in the subset Y, then Y is called ______.

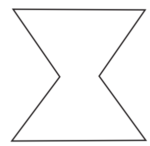
7. Does the line $y - x + 3 = 0$ pass through the point (3, 0)?

8. If $x > 0$ and $y < 0$, then the point (x, y) lies in the ______ quadrant of a rectangular co-ordinate system.

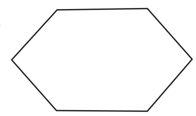
9. State whether the point (−3, 4) lies on the line, $3x + 2y + 1 = 0$ or not?

10. State which of the following figures are convex?

(i)



(ii)



(iii)

11. Define the feasible region.

12. Define the feasible solution.

13. The distances of a point P from the positive X axis and positive Y axis are 3 units and 5 units, respectively. Find the coordinates of P.

14. State which of the following points belong to the region specified by the corresponding inequation that is, $3x + 4y < 4$.
(a) (0, 2) (b) (3, 2) (c) (1, −2) (d) (4, 5)

15. State which of the following points belong to the region specified by the corresponding inequation that is, $5x - 6y + 30 > 0$.
(a) (0, 2) (b) (−4, 8) (c) (2, 3) (d) (4, 5)

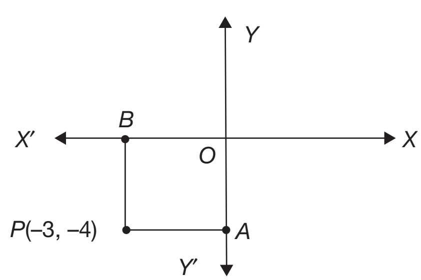
Short Answer Type Questions

16. If (0, 0), (0, 4), (2, 4) and (3, 2) are the vertices of a polygonal region subject to certain constraints, then the maximum value of the objective function $f = 3x + 2y$ is ______.

17. If (3, 2), (2, 3), (4, 2) and (2, 4) are the vertices of a polygonal region subject to certain constraints, then the minimum value of the objective function $f = 9x + 5y$ is ______.

18. A profit of ₹300 is made on class I ticket, and ₹800 is made on class II ticket. If x and y are the number of tickets of class I and class II sold, then the profit function is ______.

19. In the following figure, find AP and BP.



20. Draw the graphs of the following inequations.

$x - 4y + 8 \geq 0$

21. Draw the graphs of the following inequations.

$4x - 5y - 20 \leq 0$

22. Draw the polygonal region represented by the given systems of inequations.

$x \geq 1$, $y \geq 1$, $x \leq 4$, $y \leq 4$

23. Minimize $x + y$, subject to the constraints:

$2x + y \geq 6$

$x + 2y \geq 8$

$x \geq 0$ and $y \geq 0$

24. Define the following:

Convex set

25. Define the following:

Feasible solution

Essay Type Questions

26. A dietician wishes to mix two types of items in such a way that the mixture contains at least 9 units of vitamin A and at least 15 units of vitamin C. Item (1) contains 1 unit/kg of vitamin A and 3 units/kg of vitamin C while item (2) contains 3 units/kg of vitamin A and 5 units/kg of vitamin C. Item (1) costs ₹6.00/kg and item (2) costs ₹9.00/kg. Formulate the above information as a linear programming problem.

27. A manufacturer produces pens and pencils. It takes 1 hour of work on machine A and 2 hours on machine B to produce a package of pens while it takes 2 hours on machine A and 1 hour on machine B to produce a package of pencils. He earns a profit of ₹4.00 per package on pens and ₹3.00 per package on pencils. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at most 12 hours a day? Formulate the above information mathematically and then solve.

28. Santosh wants to invest a maximum of ₹150,000 in saving certificates and national saving bonds, which are in denominations of ₹4000 and ₹5000, respectively. The rate of interest on saving certificate is 10% per annum, and the rate of interest on national saving bond is 12% per annum. Formulate the above information as a linear programming problem.

29. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 8 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 10 minutes each for cutting and 15 minutes each for assembling. There are 4 hours available for cutting and 5 hours available for assembling. The profit is 60 paise on each item of type A and 75 paise on each item of type B. Formulate the above information as a linear programming problem.

30. Find the ratio of the maximum and minimum values of the objective function $f = 3x + 5y$ subject to the constraints: $x \geq 0$, $y \geq 0$, $2x + 3y \geq 6$ and $9x + 10y \leq 90$.

CONCEPT APPLICATION

Level 1

1. If an isoprofit line coincides with the edge of the polygon, then the problem has

(a) no solution (b) one solution

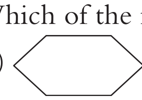
(c) infinite solutions (d) None of these

2. Which of the following is a convex set?

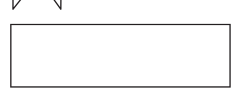
(a) A triangle (b) A square

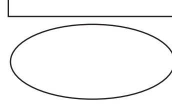
(c) A circle (d) All of these

3. Which of the following is not a convex set?

(a) 

(b) 

(c) 

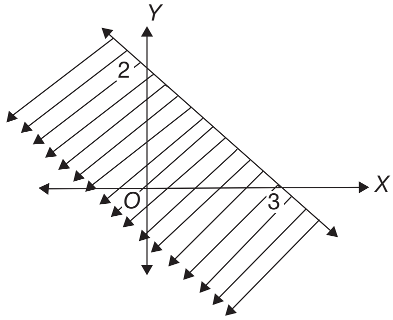
(d) 

4. Which of the following points belongs to the region indicated by the inequation $2x + 3y < -6$?

(a) (0, 2) (b) (−3, 8)

(c) (3, −2) (d) (−2, −2)

5. The inequation represented by the following graph is



(a) $2x + 3y + 6 \le 0$

(b) $2x + 3y - 6 \ge 0$

(c) $2x + 3y \le 6$

(d) $2x + 3y + 6 \ge 0$

6. The minimum value of $2x + 3y$ subjected to the conditions $x + 4y \ge 8$, $4x + y \ge 12$, $x \ge 0$ and $y \ge 0$ is

(a) $\frac{28}{3}$ (b) 16

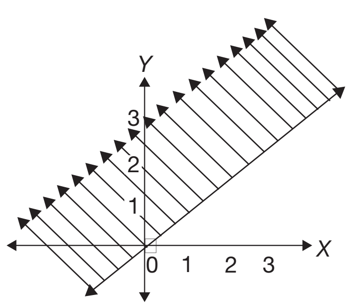
(c) $\frac{25}{3}$ (d) 10

7. Find the maximum value of $x + y$ subject to the conditions $4x + 3y \le 12$, $2x + 5y \le 10$, $x \ge 0$, $y \ge 0$.

(a) 3 (b) $\frac{20}{7}$

(c) 4 (d) $\frac{23}{7}$

8. The inequation represented by the graph given below is:



(a) $x \ge y$ (b) $x \le y$

(c) $x + y \ge 0$ (d) $x + y \le 0$

9. The solution of the system of inequalities $x \ge 0$, $x - 5 \le 0$ and $x \ge y$ is a polygonal region with the vertices as

(a) (0, 0), (5, 0), (5, 5)

(b) (0, 0), (0, 5), (5, 5)

(c) (5, 5), (0, 5), (5, 0)

(d) (0, 0), (0, 5), (5, 0)

10. If the isoprofit line moves away from the origin, then the value of the objective function ______.

(a) increases

(b) decreases

(c) does not change

(d) becomes zero

11. The solutions of the inequations $x \ge 0$, $y \ge 0$, $y = 2$ and $x = 2$ form the polygonal region with the vertices (0, 0) (0, 2) (2, 0) and (2, 2) and the polygon so formed by joining the vertices is a ______.

(a) parallelogram

(b) rectangle

(c) square

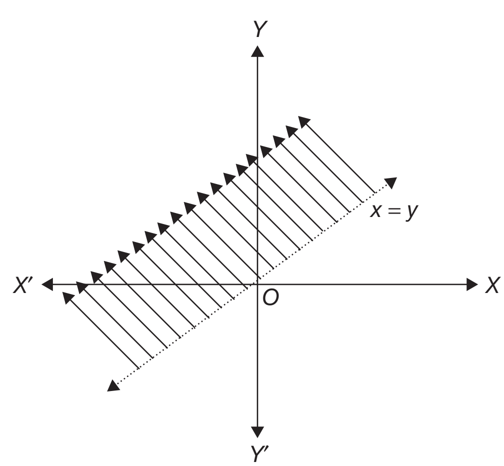
(d) rhombus

12. Maximize $5x + 7y$, subject to the constraints $2x + 3y \le 12$, $x + y \le 5$, $x \ge 0$ and $y \ge 0$.

(a) 29 (b) 30

(c) 28 (d) 31

13. The inequation that best describes the graph given below is ________.

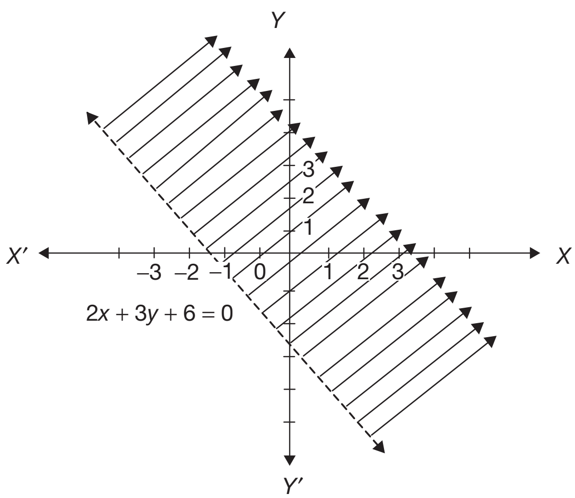


PRACTICE QUESTIONS

(a) $x > y$ (b) $x < y$

(c) $x \leq y$ (d) $x \geq y$

14. The inequation that best describes the following graph is ________.



(a) $2x + 3y + 6 \leq 0$

(b) $2x + 3y + 6 \geq 0$

(c) $2x + 3y + 6 > 0$

(d) $2x + 3y + 6 < 0$

15. The vertices of a closed-convex polygon determined by the inequations $7x + 9y \leq 63$ and $5x + 7y \geq 35$, $x \geq 0$, $y \geq 0$ are:

(a) (7, 0) (5, 0) (9, 0) (0, 9)

(b) (7, 0) (9, 0) (0, 7) (0, 5)

(c) (9, 0) (6, 0) (5, 0) (0, 8)

(d) (0, 9) (0, 5) (7, 0) (3, 0)

Level 2

16. The vertices of a closed convex polygon representing the feasible region of the objective function are (6, 2), (4, 6), (5, 4) and (3, 6). Find the maximum value of the function $f = 7x + 11y$.

(a) 64 (b) 79

(c) 94 (d) 87

17. If the vertices of a closed-convex polygon are $A(8, 0)$, $O(0, 0)$, $B(20, 10)$, $C(24, 5)$ and $D(16, 20)$, then find the maximum value of the objective function $f = \frac{1}{4}x + \frac{1}{5}y$.

(a) $7\frac{1}{2}$ (b) 8

(c) 6 (d) 7

18. Find the profit function p, if it yields the values 11 and 7 at (3, 7) and (1, 3), respectively.

(a) $p = -8x + 5y$ (b) $p = 8x - 5y$

(c) $p = 8x + 5y$ (d) $p = -(8x + 5y)$

19. A shopkeeper can sell upto 20 units of both books and stationery. If he makes a profit of ₹2 on each book and ₹3 on each unit of stationery, then the profit function is ______, if x and y denote the number of units of books and stationery sold.

(a) $p = 2x - 3y$ (b) $p = 2x + 3y$

(c) $p = 3x - 2y$ (d) $p = 3x + 2y$

20. The vertices of a closed-convex polygon representing the feasible region of the objective function f are (4, 0), (2, 4), (3, 2) and (1, 4). Find the maximum value of the objective function $f = 7x + 8y$.

(a) 39 (b) 46

(c) 49 (d) 38

21. The cost of each table and each chair cannot exceed ₹7. If the cost of 3 tables and 4 chairs cannot exceed ₹30, form the inequations for the above data.

(a) $x > 0$, $y > 0$, $x \leq 7$, $y \geq 7$, $3x + 4y \leq 30$

(b) $x < 0$, $y < 0$, $x \leq 7$, $y \leq 7$, $3x + 4y \leq 30$

(c) $0 < x < 7$, $0 < y \leq 7$, $3x + 4y \leq 30$

(d) $x > 0$, $\leq y > 0$, $x \geq 7$, $y \geq 7$ and $3x + 4y \leq 30$

22. The vertices of the closed-convex polygon determined by the inequations $3x + 2y \geq 6$, $4x + 3y \leq 12$, $x \geq 0$ and $y \geq 0$ are

(a) (1, 0), (2, 0), (0, 2), (0, 1)

(b) (2, 0), (3, 0), (0, 4) and (0, 3)

(c) (1, 0), (2, 0), (0, 2) and (2, 2)

(d) (1, 0), (0, 2), (2, 2) and (1, 1)

PRACTICE QUESTIONS

23. Which of the following is a point in the feasible region determined by the linear inequations $2x + 3y \leq 6$ and $3x - 2y \leq 16$?

(a) $(4, -3)$ (b) $(-2, 4)$

(c) $(3, -2)$ (d) $(3, -4)$

24. The maximum value of the function $f = 5x + 3y$ subjected to the constraints $x \geq 3$ and $y \geq 3$ is ______.

(a) 15 (b) 9

(c) 24 (d) Does not exist

25. A telecom company manufactures mobile phones and landline phones. They require 9 hours to make a mobile phone and 1 hour to make a landline phone. The company can work not more than 1000 hours per day. The packing department can pack at most 600 telephones per day. If x and y are the sets of mobile phones and landline phones, respectively, then the inequalities are:

(a) $x + y \geq 600$, $9x + y \leq 1000$, $x \geq 0$, $y \geq 0$

(b) $x + y \leq 600$, $9x + y \geq 1000$, $x \geq 0$, $y \geq 0$

(c) $x + y \leq 600$, $9x + y \leq 1000$, $x \leq 0$, $y \leq 0$

(d) $9x + y \leq 1000$, $x + y \leq 600$, $x \geq 0$, $y \geq 0$

26. If the isoprofit line moves towards the origin, then the value of the objective function ______.

(a) increases

(b) does not change

(c) becomes zero

(d) decreases

27. The minimum cost of each tablet is ₹10 and each capsule is ₹10. If the cost of 8 tablets and 5 capsules is not less than ₹150, frame the inequations for the given data.

(a) $x \geq 10$, $y \geq 10$, $8x + 5y \geq 150$

(b) $x \geq 10$, $y \geq 10$, $8x + 5y \leq 150$

(c) $x \leq 10$, $y \leq 10$, $8x + 5y \geq 150$

(d) $x \leq 10$, $y \leq 10$, $8x + 5y \leq 150$

28. Find the profit function p in two variables x and y, if it yields the values 23 and 7 at (3, 2) and (2, 3) respectively.

(a) $p = 11x + 5y$

(b) $p = 5x + 11y$

(c) $p = 11x - 5y$

(d) $p = 5x - 11y$

29. The maximum value of the function, $f = 3x + 5y$, subject to the constraints $x \geq 5$ and $y \geq 5$, is ______.

(a) 40 (b) 24

(c) 8 (d) Does not exist.

30. The vertices of a closed-convex polygon representing the feasible region of the objective function $f = 5x + 3y$, are (0, 0), (3, 0), (3, 1), (1, 3) and (0, 2). Find the maximum value of the objective function.

(a) 6 (b) 18

(c) 14 (d) 15

Level 3

31. The cost of each table or each chair cannot exceed ₹9. If the cost of 4 tables and 5 chairs cannot exceed ₹120, then the inequations which best represents the above information are:

(a) $x < 9$, $y < 9$, $5x + 4x \geq 120$

(b) $x > 9$, $y > 9$, $4x + 5y \geq 120$

(c) $0 < x \leq 9$, $0 < y \leq 9$, $4x + 5y \leq 120$

(d) $0 < x \leq 9$, $0 < y \leq 9$, $5x + 4y \geq 120$

32. The vertices of a closed-convex polygon determined by the inequations $5x + 4y \leq 20$, $3x + 7y \leq 21$, $x \geq 0$ and $y \geq 0$ are

(a) $(0, 0)(7, 0)(0, 3)\left(\frac{148}{69}, \frac{45}{23}\right)$

(b) $(4, 0)(0, 3)(0, 5)\left(\frac{148}{69}, \frac{45}{23}\right)$

(c) $(0, 0), (4, 0)(0, 3)\left(\frac{56}{23}, \frac{45}{23}\right)$

(d) $(0, 0)(7, 0)(4, 0)(0, 3)$

33. The profit function p which yields the values 61 and 57 at (4, 7) and (5, 6), respectively, is ________.

(a) $2x + 5y$ (b) $7x + 3y$

(c) $5x + 2y$ (d) $3x + 7y$

34. The vertices of a closed-convex polygon representing the feasible region of the objective function f are (5, 1) (3, 5) (4, 3) and (2, 5). Find the maximum value of the function $f = 8x + 9y$.

(a) 61 (b) 69

(c) 59 (d) 49

35. The cost of each table or each chair cannot exceed ₹13. If the cost of 5 tables and 7 chairs cannot exceed ₹250, then the inequations which best represents the above information are

(a) $x > 13$, $y > 13$, $5x + 7y > 250$

(b) $x > 0$, $y > 0$, $5x + 7y < 250$

(c) $x < 13$, $y < 13$, $5x + 7y \leq 25$

(d) $0 < x \leq 13$, $0 < y \leq 13$, $5x + 7y \leq 250$

36. The minimum value of $f = x + 4y$ subject to the constraints $x + y \geq 8$, $2x + y \geq 10$, $x \geq 0$, $y \geq 0$ is ______.

(a) 4 (b) 26

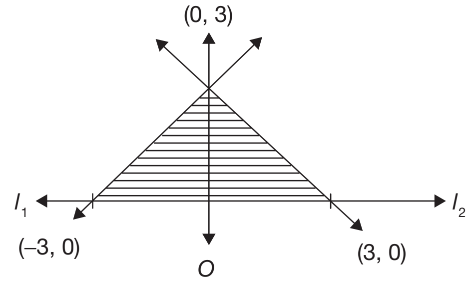
(c) 5 (d) 8

37. A tailor stitches trousers and shirts and each piece is completed by two machines I and II. To complete each trousers, machines I and II must work 3/2 hours and 2 hours respectively, and to complete each shirt, machines I and II must work 2 hours and 1 hour respectively. Machine I may not operate for more than 12 hours per day and machine II not more than 11 hours per day. If the profit on each trouser and each shirt is ₹150 and ₹100 respectively, then the maximum profit is ______.

(a) ₹900 (b) ₹500

(c) ₹375 (d) ₹600

38. Which of the following inequations represent the shaded region in the given figure



(a) $y \geq 0$, $x + y \leq 3$, $x - y \geq -3$

(b) $x \geq 0$, $x \pm y \leq 3$

(c) $y \geq 0$, $x \pm y \geq -3$, $-3 \leq x$

(d) $x \geq 0$, $x \pm y \leq -3$

39. A telecom company offers calls for day and night hours. Calls can be availed 8 hours during the day and 4 hours at night and at most 10 hours a day. The profit on the day calls is ₹60 per hour, and on night calls ₹50 per hour. How many hours during the day and at night a customer must use to fetch a maximum profit to the company?

(a) 6 hours during day and 4 hours at night

(b) 5 hours during day and 5 hours at night

(c) 8 hours during day and 2 hours at night

(d) 6 hours during day and 2 hours at night

40. On a rainy day, a shopkeeper sells two colours (black and red) of umbrellas. He sells not more than 20 umbrellas of each colour. At least twice as many black ones are sold as the red ones. If the profit on each of the black umbrellas is ₹30 and that of the red ones is ₹40, then how many of each kind must be sold to get a maximum profit?

(a) 20, 10 (b) 30, 15

(c) 40, 20 (d) 10, 5

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. No
2. True
3. No
4. False
5. optimum solution
6. a convex set
7. Yes
8. fourth (Q_4)
9. The point lies on the line
10. (ii) Convex set
13. $P = (5, 3)$
14. $(1, -2)$
15. $(-4, 8)$

Short Answer Type Questions

16. 14
17. 33
18. $f = 300x + 800y$
19. $AP = 3$ units, $BP = 4$ units
23. $\frac{14}{3}$

Essay Type Questions

26. $x + 3y \geq 9$,
    $3x + 5y \geq 15$ and
    $x \geq 0, y \geq 0$
27. The manufacturer can manufacture 4 packages of each pens and pencils daily to obtain the maximum profit of ₹28.
28. $4x + 5y \leq 150$ and $x \geq 0$, $y \geq 0$.
29. $8x + 10y \leq 240$
    $10x + 15y \leq 300$
    $x \geq 0, y \geq 0$
30. 5 : 1

CONCEPT APPLICATION

Level 1

1. (c)	2. (d)	3. (b)	4. (d)	5. (c)	6. (a)	7. (d)	8. (b)	9. (a)	10. (a)
11. (c)	12. (a)	13. (b)	14. (c)	15. (b)					

Level 2

16. (c)	17. (b)	18. (a)	19. (b)	20. (b)	21. (c)	22. (b)	23. (c)	24. (d)	25. (d)
26. (d)	27. (a)	28. (c)	29. (d)	30. (b)					

Level 3

31. (c)	32. (c)	33. (d)	34. (b)	35. (d)	36. (d)	37. (a)	38. (a)	39. (c)	40. (a)

CONCEPT APPLICATION

Level 1

2. Recall the definition of convex set.

3. Recall the definition of convex set.

4. Check the point which satisfies the given inequation.

5. The corresponding equation of the line is $\frac{x}{3}+\frac{y}{1}=1$.

6. Find the open-convex polygon formed by the given inequations and proceed.

7. Find the closed-convex polygon formed by the given inequations and proceed.

8. The corresponding equation of the line is $x = y$.

9. Check the points which belongs the given inequations.

10. Increases (standard result).

11. Check which quadrilateral is formed by the points (0, 0), (0, 2), (2, 0) and (2, 2).

12. Represent the regions of given inequations and identify the vertices of the convex polygons.

17. (i) Substitute the given points in f.

 (ii) The maximum/minimum value of f occurs at one of the vertices of the closed-convex polygon.

 (iii) Substitute the given points in f.

 (iv) Identify the maximum value of f.

18. (i) Profit function $p = ax + by$.

 (ii) Take the profit function as $p = ax + by$.

 (iii) Substitute the given points and obtain the equations in a and b.

 (iv) Solve the above equations to get a and b.

19. (i) $p = ax + by$ – find a and b.

 (ii) Profit on books is ₹$2x$ and profit on stationary is ₹$3y$.

 (iii) Profit function = Profit on books + Profit on stationary.

20. (i) Substitute the given points.

 (ii) The maximum/minimum value of f occurs at one of the vertices of the closed-convex polygon.

 (iii) Substitute the given points in f.

 (iv) Identify the maximum value of f.

21. (i) Let the cost of each table and chair be x and y.

 (ii) Consider the cost of each table and each chair as ₹x and ₹y respectively.

 (iii) Frame the inequations according to the given conditions.

22. (i) Check the points which belong to the given inequation.

 (ii) Represent the given inequations on the graph.

 (iii) Detect the closed-convex polygon and its vertices.

23. (i) Check the point which satisfies the given inequation.

 (ii) Substitute the values in the options in the given inequations.

 (iii) The point which satisfies the given inequations is the required point.

24. (i) Find the convex polygon.

 (ii) Represent the given inequations on the graph.

 (iii) Find the vertices of the closed convex polygon.

 (iv) Substitute the vertices of the polygon in f and check for the maximum value of f.

25. (i) Time taken to manufacture x mobiles and y landlines is $9x$ and y hours respectively.

 (ii) $x \geq 0$ and $y \geq 0$. As the number of mobiles cannot be negative.

 (iii) Use the above information and frame the inequations.

26. If the isoprofit line moves towards the origin, then the value of the objective function decreases.

27. Let the cost of each tablet be x and the cost of each capsule be y. Minimum cost of each tablet and each capsule is ₹10.

HINTS AND EXPLANATION

$\therefore$ $x \geq 10$ and $y \geq 10$.

The cost of 8 tablets and 5 capsules should be greater than or equal to 150, i.e., $8x + 5y \geq 150$.

28. Let the profit function be $p = ax + by$

Given at (3, 2), p attains the value 23.

$3a + 2b = 23$ (1)

At (2, 3), p attains the value 7.

$2a + 3b = 7$ (2)

Solving Eqs. (1) and (2), we get $a = 11$, $b = -5$

$\therefore$ Profit function $(p) = 11x - 5y$.

29. The given constraints $x \geq 5$ and $y \geq 5$ form an open-convex polygon.

$\therefore$ The maximum value does not exist.

30. Objective function $f = 5x + 3y$

The vertices of the convex polygon are (0, 0), (3, 0), (3, 1), (1, 3) and (0, 2).

The maximum value attains at (3, 1).

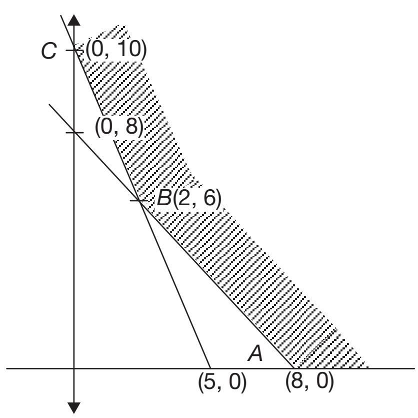
$\therefore$ The maximum value $= 5 \times 3 + 3 \times 1$

$= 15 + 3 = 18$.

36. $x + y \geq 8$; $2x + y \geq 10$, $x \geq 0$; $y \geq 0$

Solving $x + y = 8$; $2x + y = 10$, we get $(x, y) = (2, 6)$

ABC is an open-convex polygon.



The vertices are $A(8, 0)$, $B(2, 6)$, $C(0, 10)$

$F = x + 4y$

$A(8, 0)$, $f = 8 + 4(0) = 8$

$C(0, 10)$, $f = 0 + 4(10) = 40$

$B(2, 6)$, $f = 2 + 4(6) = 26$

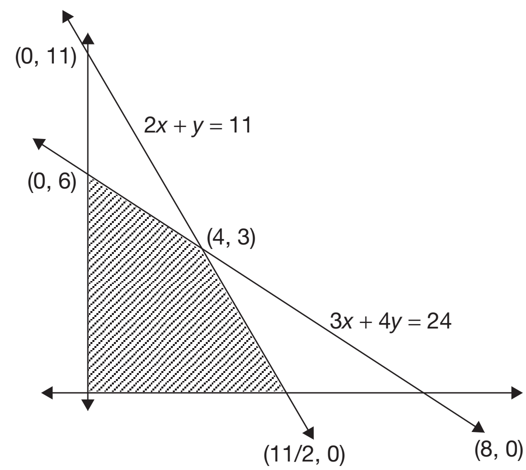
$\therefore$ The minimum value of f is 8.

It attains at $A(8, 0)$.

37. Let the number of trousers $= x$

Let the number of shirts $= y$

	Machine I (≤ 12 hours)	Machine II (≤ 11 hours)
Trousers (x)	$\frac{3}{2}$ hours	2 hours
Shirts (y)	2 hours	1 hour



According to the given condition:

$\Rightarrow \frac{3x}{2} + 2y \leq 12$ and $2x + y \leq 11$

$\Rightarrow 3x + 4y \leq 24$ and $2x + y \leq 11$

$x \geq 0$, $y \geq 0$

The profit function, $p = 15x + 10y$

The shaded region is a closed-convex polygon with vertices $\left(\frac{11}{2}, 0\right)$, (4, 3), (0, 6) (0, 0).

$\therefore$ $ABCD$ is the feasible reason.

$\therefore$ The maximum profit is at $C(4, 3)$, i.e.,

$P = 150 \times 4 + 100 \times 3 = 900$.

38. In the shaded region, y is non-negative.

$\therefore$ $y \geq 0$

Intercepts of l_1 are 3, 3.

And the region contains origin $\Rightarrow x + y \leq 3$.

Intercepts of l_2 are -3, 3, and the region contains the origin.

$x - y \geq -3$.

39. Let the number of hours used during the day = x

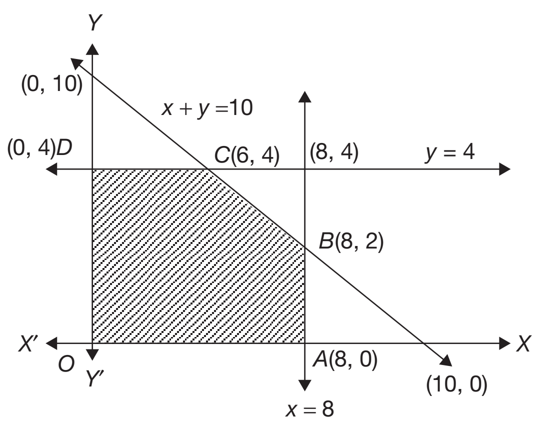
Let the number of hours used at night = y

Given, $x + y \leq 10$, $x \leq 8$ and $y \leq 4$.

Profit on day calls = ₹60 per hour

Profit on night calls = ₹50 per hour

Profit function $p = 60x + 50y$



The maximum profit is at $B(8, 2)$

$\therefore\ P = 60x + 50y$

$= 60 \times 8 + 50 \times 2 = 480 + 100$

$= 580$

$\therefore$ 8 hours during day and 2 hours at night.

40. Let the number of black umbrellas sold be x and that of red umbrellas be y.

$x \geq 0$ and $x \leq 20$

$y \geq 0$ and $y \leq 20$

$x \geq 2y$.

Profit function $p = 30x + 40y$

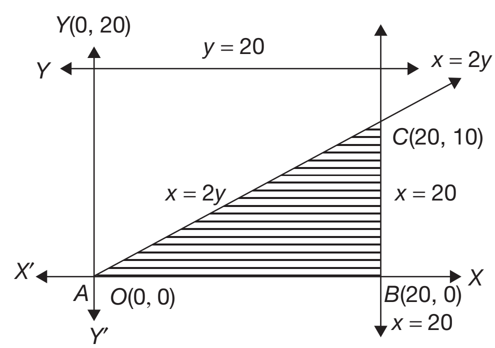
The maximum profit attains at $C(20, 10)$.

The maximum profit = $30 \times 20 + 40 \times 10$

$= 600 + 400 =$ ₹1000.

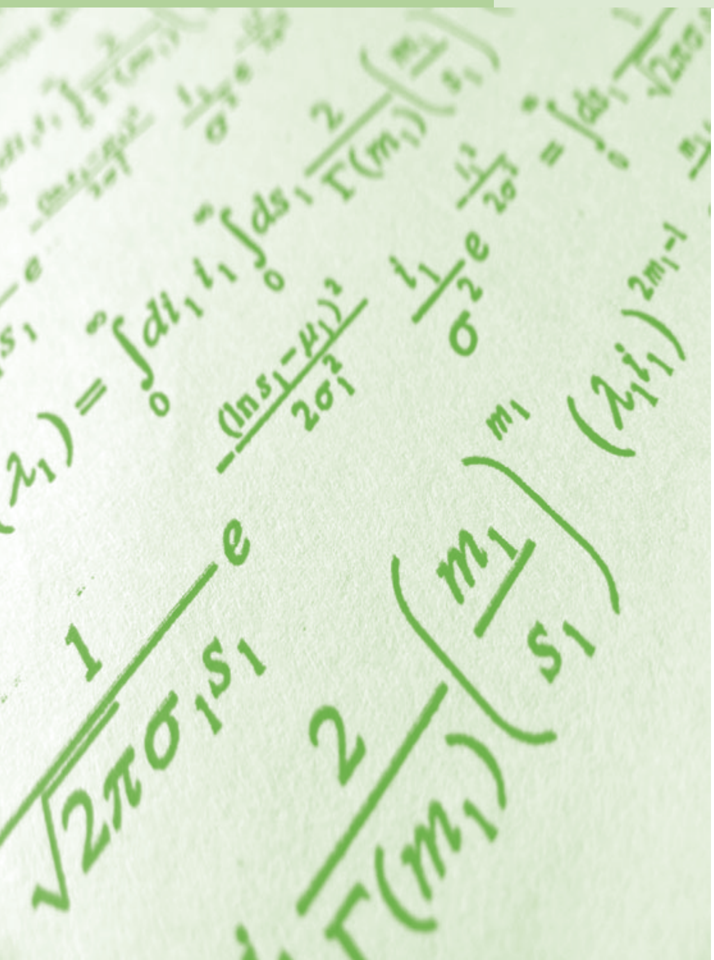
Number of black umbrellas = 20

Number of red umbrellas = 10.



Chapter 19

Computing



REMEMBER

Before beginning this chapter, you should be able to:

- Use terms related to a computer
- Operate on a computer

KEY IDEAS

After completing this chapter, you would be able to:

- Understand characteristics and architecture of a computer
- Apply operators to perform various operations in a computer
- Study rules for declaration of variables
- Explain conditional statements

INTRODUCTION

We are living in the age of information technology. Computers play a key role in our everyday life. These are extensively used in various fields, such as, the banking, insurance, transportation, science and technology, entertainment. Complex tasks can be easily solved with the help of computers.

A computer is a multipurpose electronic device which is used for storing information, processing large amount of information, and to accomplish tasks with high speed and absolute accuracy.

A computer can be defined as an electronic device which accepts input data, processes it following the set of instructions called 'programs', and gives the 'output information'.

The idea of a computer was first developed by **Charles Babbage** in the 18th century. Since the time computer was invented, its architecture has undergone numerous changes. At the initial structure of a computer vacuum tubes were used. Then, computers used to be large in size. Some computers were as large as that of a room. Thereafter, the vacuum tubes were replaced with transistors. That was the beginning of the second generation of computers. On the later years, small-scale integrated circuits (IC) were used in the third generation computers. With rapid advancements in the field of science and technology, VLSI (Very Large Scale Integrated) circuits were fabricated. The present days computers use VLSI circuits to achieve high speed, small size, vast memory, and higher accuracy. Today, various types of mini-computers, such as laptops, notebooks, and PDAs (Personal Digital Assistants) are available in the market.

Historical Development of Computing

The method of computing began in the early 4000–1200 BC. By that time, people used to record transactions on clay tablets. Around 3000 BC, the Babylonians invented the Abacus from where computing started. No significant development took place until the 17th century. **Blaire Pascal** invented a machine which was named as **Pascaline**. It was the first mechanical adding machine in the history of computing. Then punch cards were used in the early 18th century. In 1822, Charles Babbage developed the first mechanical computer. For this reason, he is known as the father of the computer. In 1854, George Boole, developed the Boolean logic. It is the basis of computer design. Before 20th century, all machines were mechanical. The first electronic computer built in 1966, was named ENIAC (Electronic Numerical Integrator and Computer).

In 1962, instead of individual transistors integrated circuits were used in the computers. The most famous machine, at that time, was the IBM 360 and DEC DPBQ. Later, microprocessor was invented. It enabled reduction of size and performance enhancement of the computers. In 1981, IBM launched PC. Since then, there has been a significant development in the field of micro-electronics and micro-processors.

Today, we use computers with 2 to 3 GHz processors, with .5 to 1 GB of memory and 80 to 120 GB of storage space (while only 500 KB storage space was considered significant even till a few years ago.)

In the evolution of computers, major changes have occurred in both the structure and the functioning, which have a tremendous impact on the way these machines appear and the extent to which people use these. We refer to these changes as introducing a new

generation of computers. Till date, the following five generations of computers are readily identifiable:

First generation (1945–1956): The computers of this generation were mechanical or electro mechanical in which vacuum tubes were used. These computers were huge in size, inflexible and slow in comparison to the later generations of computers.

Second generation (1957–1963): In 1948, transistor was invented. It made a major impact on the development of computers. Transistors replaced the large vacuum tubes. This led to a significant reduction in the size of the computers. In the early 1960s, several commercially successful computers were used in business establishments and universities. Several high-level programming languages, such as COBOL, FORTRAN were introduced.

Third generation (1964–1971): By this time, integrated circuits (ICs) were manufactured. In an IC, hundreds of transistors were assembled in a tiny silicon chip. This led to the development of the third generation computers.

Fourth generation (1971 to the present): In this generation, LSI (Large Scale Integrated), VLSI (Very Large Scale Integrated) and ULSI (Ultra Large Scale Integrated) chips were introduced. These significantly increased the efficiency and reliability of the computers.

In 1981, IBM introduced personal computers (PCs) for use at home or office. These PCs could be linked together or networked to share both software and the hardware.

Fifth generation of computers (present days and beyond): With developments in artificial intelligence (AI), computers are enabled to hold conversation with humans. They can use visual inputs and learn from their own experience. A robot is one fine example of the application of artificial intelligence.

Characteristics of a Computer

Some of the characteristics of computers are:

1. Speed
2. Reliability
3. Storage Capacity
4. Productivity

Architecture of a Computer

A computer consists of three essential components. Those are:

1. Input device (e.g., key board)
2. Central Processing Unit (CPU)
3. Output device (e.g., Monitor)

   The CPU is an important component in a computer. It consists of:

   (i) Memory unit

   (ii) Control unit

   (iii) Arithmetic and logic unit (ALU)

Block diagram of a computer is shown below:

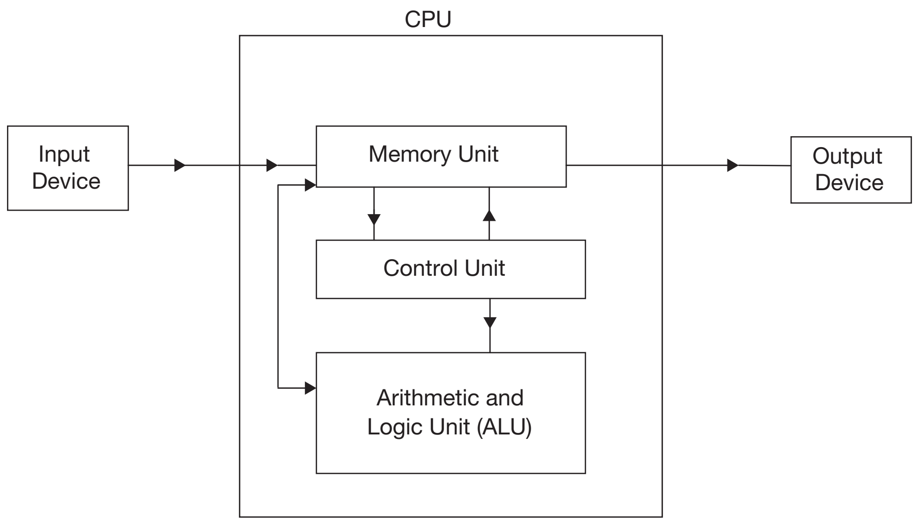


Figure 19.1

The instructions or processed data is received by computers the input devices. This information is stored in the memory unit. If arithmetical operations are to be performed, then with the help of the control unit, the arithmetic and logic unit (ALU) performs the operations and stores the result in the memory unit. Finally the results can be viewed through output devices.

The three physical components, i.e., input devices, CPU, and output devices are together referred as **'hardware'** of a computer.

Software

To accomplish a particular task by using a computer, one requires writing a set of instructions in a language that can be understood by the computer. Any of our usual languages cannot be read or understood by a computer. So, we need to feed information into a computer using a language called 'programming language'. BASIC, PASCAL, C, C^{++}, Java, etc., are some of the popular programming languages.

A set of instructions that are written in a language which can be understood by a computer is called a 'program'. A set of programs is called **'software'**.

A program is to be written to accomplish a particular task. This program is fed into the memory of a computer by using an input device (i.e., key board). The control unit reads the instructions (given in the program) from the memory and processes the data according to these instructions. The result can be displayed with a device, such as a monitor or a printer. These devices are called 'output devices'.

Note that the arithmetic and logic unit (ALU) performs all the arithmetic and logical operations, such as addition, subtraction, multiplication, division, comparison under the supervision of the control unit (CU). The CU decodes these instructions to execute and the output unit receives the results from the memory unit and converts these results into a suitable form which the user can understand.

Algorithm

A comprehensive and detailed step-by-step plan or a design that is followed to solve a problem is called an algorithm. Thus, an algorithm is a set of systematic and sequential steps in arriving at a solution to a problem.

For example, if you want to buy some articles from a grocery store, then the following steps are to be followed:

1. Make a list of articles which you intend to purchase.
2. Go to the grocery store.
3. Give the storekeeper the list of articles.
4. List the prices of the articles on paper. Add them to get the total amount of money you need to pay.
5. Pay the money and collect the articles.
6. Verify if you have received all the articles.
7. Return home with the groceries.

Steps (1) to (7) form the algorithm for the task of buying grocery items. Even though it is a simple task, we follow several steps in a systematic way to achieve the task. Similarly, to solve a task using a computer, at first, we need to make a blue print, i.e., algorithm of the steps that are to be followed. Once an algorithm is ready, we can represent it on a flow chart.

Flowchart

A flowchart is a pictorial representation of an algorithm. Flowchart clearly depicts the points of input, decision-making, loops and output. Thus, with the help of a flow chart, we can plan more clearly and logically,to solve a given task.

To draw a flowchart, we use certain symbols or boxes to represent the information appropriately. Following are the notations used in a flowchart.

1. **The operation box:** This box is used to represent the operations, such as addition, subtraction.
2. **The data box:** This box is used to represent the data that is needed to solve a problem, also to represent information regarding the output of solution.

   Therefore this box is used for input and output.

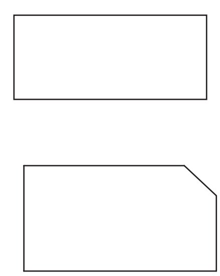


Figure 19.2

3. **Decision box:** A diamond-shaped (or rhombus) box is used whenever a decision is to be taken. The points of decision can be represented by using this box. Usually, the answer to the decision is **'yes'** or **'no'**.
4. **Terminal box:** This box indicates the start or termination of a program.
5. **Flow lines '→':** The arrows which are used in flowcharts are known as flow lines. These arrows are very important in flow charts.
6. **Connectors:** The circle is a connector in a flowchart. The connector is used in the flowcharts only if it needs to be continued on the next page. Connectors are always used in pairs. The flowchart will have an outwards connector on a page which can be continued with an inwards connector in the next page.

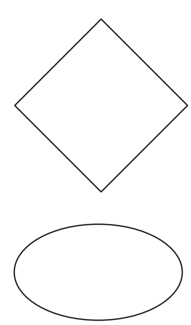


Figure 19.3

Both 'in' and 'out' connectors should contain the same alphabet.

Once a flowchart is ready, we can translate it into a programming language, and feed it into a computer's memory.

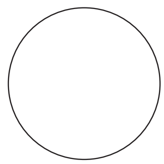


Figure 19.4

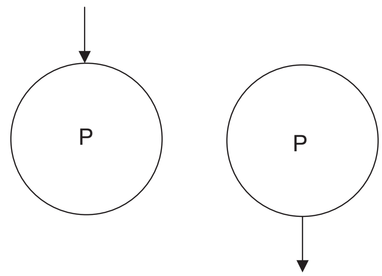


Figure 19.5

To accomplish a task on a computer, the following steps are to be followed:

1. Identify and analyze the problem.
2. Design a systematic solution to the problem, and write an algorithm.
3. Represent the algorithm in a flowchart.
4. Translate a flowchart into a program.
5. Execute the program and get the output.

Examples:

1. You are given the principle and the rate of simple interest (SI) per month. Write an algorithm to calculate the cumulative simple interest at the end of each year up to 10 years, and also draw a flowchart.

Algorithm:

Step 1: Read the values of principle (P), rate of interest (R).

Step 2: Take $T = 1$.

Step 3: $\text{SI} = \dfrac{12 \times P \times T \times R}{100}$

Step 4: Print the SI.

Step 5: Calculate $T = T + 1$.

Step 6: If $T \leq 10$, then repeat Steps 3, 4 and 5.

Step 7: Otherwise, stop the program.

2. Write an algorithm and draw a flowchart to find the sum of first 50 natural numbers.

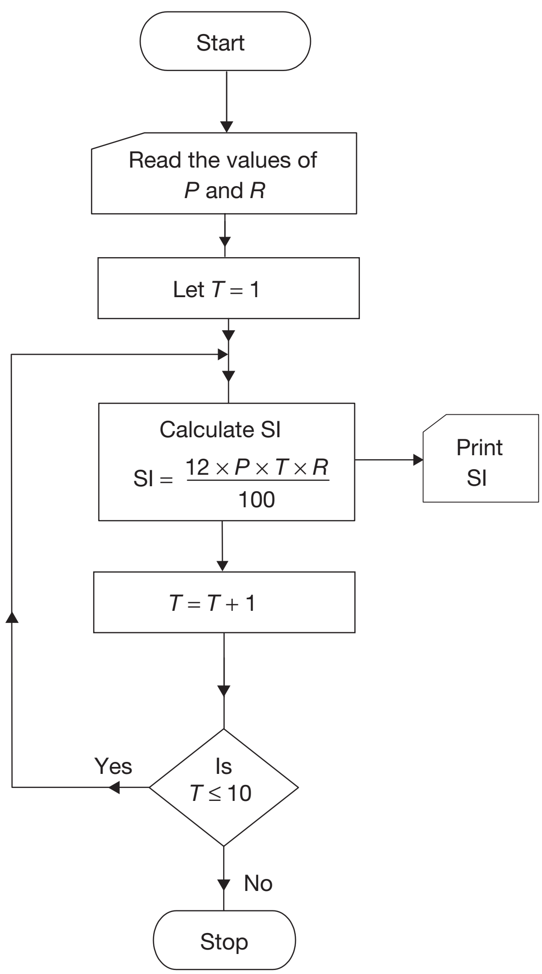


Figure 19.6

Algorithm:

Step 1: Set count = 1, Sum = 0.

Step 2: Add count to sum.

Step 3: Increase count by one, i.e., count = count + 1.

Step 5: Check whether count is 51.

Step 6: In Step (5), if count is 51.

Display the sum and stop the program else Go to Step (2).

From this flowchart, we can observe that there is a loop among Boxes 3, 4 and 5.

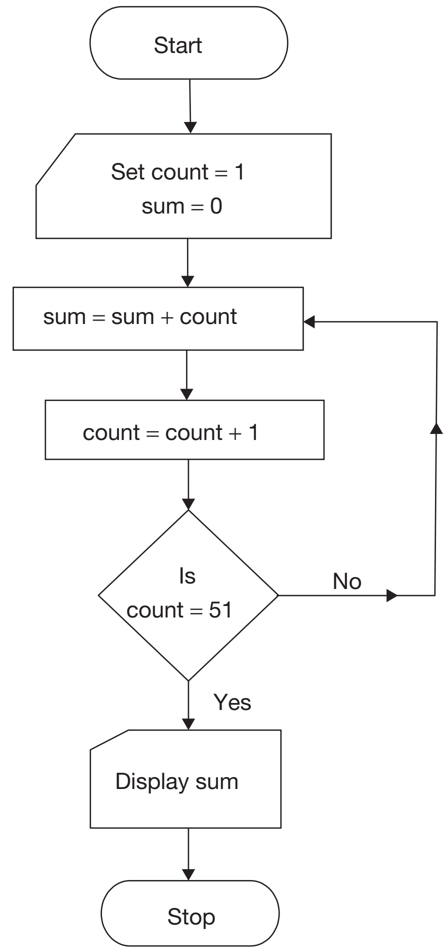


Figure 19.7

Operators

Operators are used to perform various types of operations.

Example: Addition can be done by '+' operator.
There are different types of operators:

1. Shift operators
2. Logical operators
3. Relational operators
4. Arithmetic Operators

Computer performance is measured in three ways:

1. Storage Capacity
2. Processing Speed
3. Data transfer Speed

Storage Capacity is measured in Bits, Bytes, Kilobytes, Megabytes or Giga Bytes.

1 nibble = 4 bits

1 Byte = 8 bits

1 Kilo Byte (KB) = 1024 Bytes = 2^{10} Bytes

1 Mega Byte (MB) = 1024 KB = 2^{20} Bytes

1 Giga Byte (GB) = 1024 MB. = 2^{30} Bytes

1 Tera Byte (TB) = 1024 GB. = 2^{40} Bytes

Processing Speed is measured in Hertze, i.e., cycles/second. It explains about the processor speed.

Example: 800 MHz, 1.5 GHz.
Data transfer speed is measured in Bytes per second.

Example: 256 KB/second, 128 KB/second.

Examples:

1. Write an algorithm to calculate the sum of the squares of the first five natural numbers, and also draw the flowchart.

Algorithm:

Step 1: Take $N = 0$, Sum = 0

Step 2: Calculate $N = N + 1$

Step 3: Calculate Temp = $N * N$

Step 4: Calculate Sum = Sum + Temp.

Step 5: If $N < 5$ repeat Steps 2, 3 and 4. Otherwise, print the sum and stop the program.

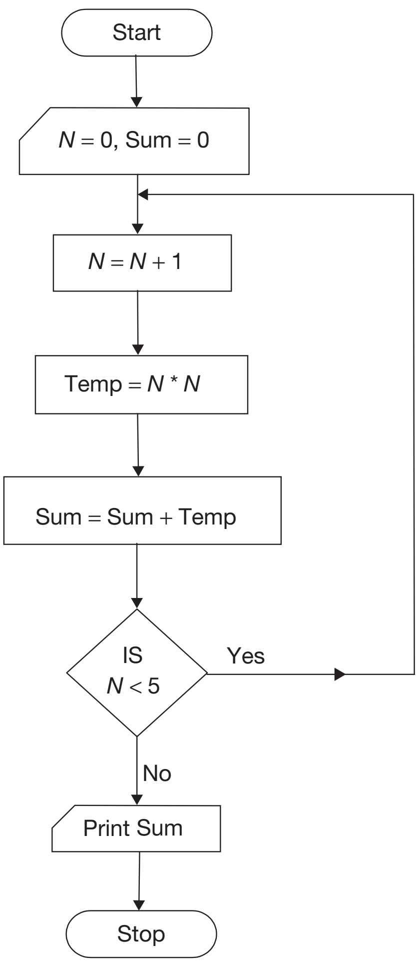


Figure 19.8

2. Write an algorithm to generate the Fibonocci Series up to n terms, and also draw a flowchart.

Algorithm:

Step 1: Let $F_1 = 0$, $F_2 = 1$ and $K = 2$

Step 2: Read N for number of terms

Step 3: Print F_1 and F_2

Step 4: Calculate $F_3 = F_1 + F_2$

Step 5: Print F_3

Step 6: Calculate $K = K + 1$

Step 7: $F_1 = F_2$ and $F_2 = F_3$

Step 8: If $K < N$, then go to Step 4, else stop the program.

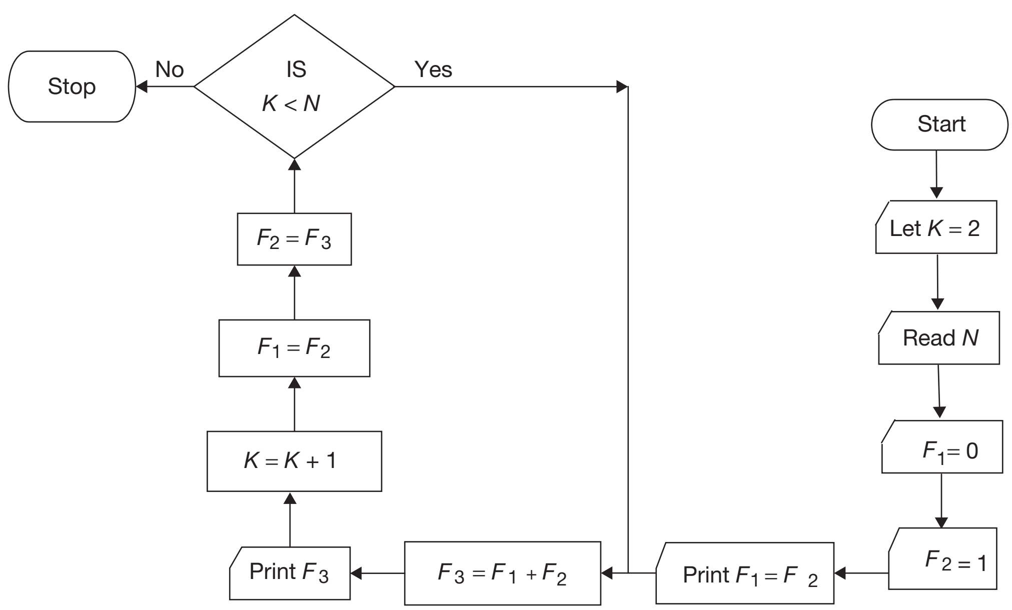


Figure 19.9

Basic

Basic (Beginners All Purpose Symbolic Instruction Code) is a high-level and user-friendly language. The instructions can be given in simple English language along with some keywords and specific syntax. This language is useful in the field of Business, Engineering, Mathematics and other applications.

Constants

Constants are those which do not change their values in the program. They can be classified as:

1. Numeric Constants
2. Alphanumeric Constants

Numeric

All the whole numbers ranging between –32767 to +32767 are numeric constants.

Note Commas are not allowed in a constant. For an example, 23,456 are not valid.

Alphanumeric

Set of alphabets or numeric's or alphanumeric's which are enclosed within double quotation marks are treated as alphanumeric or string constants.

Example: 'A', '576', 'AP007'

Variables

A variable is a name that represents a number, a character or a string. Variables are of two types:

Numeric variable

It must start with a letter.
Example: A, BASIS.

Alphanumeric variables or string variables

It must begin with a letter. It can be followed by other letters or numbers, but must end with a dollar sign ($).
Example: A$, NAME 2$.

Rules for Declaration of Variables

1. A variable must start with an alphabet.
2. Keywords should not be used as a variable name.
3. In a variable name declaration, commas and blank spaces are not allowed.
4. The variable length should not exceed 40 characters.
5. The special characters, like %, #, and $ are not allowed in variables.

Examples:

1. 9ALSAB Invalid, it should start with an alphabet.
2. 201 Valid
3. SAP GIM No space is allowed in variable name, hence it is invalid.
4. K21A $ Invalid, $ cannot be used here.
5. R, PT20 Invalid, commas are not allowed.

Basic Operators

Operators are used to relate variables and constants to form expressions.
Operators are of three types:

1. Arithmetic operators
2. Logical operators
3. Relational operators

Arithmetic Operators

These are used for mathematical operations.
Addition '+'
Subtraction '–'
Multiplication '*'
Division '/'
Exponential '^'

Logical Operators

There are three logical operators:

1. AND: AND expression will be executed when both of the conditions are true.
2. OR: OR expression will be executed when any one of the conditions is true.
3. NOT: NOT expression will be executed when a condition has to be negated.

Relational Operators

These operators are used to form a relational expression. These are:
'=' equality, '<' less than, '>' greater than, '< =' less than or equal to, '> =' greater than or equal to, '<>' Not equal to.

Example: $x < = y$, $A > B$

Order in which Arithmetic Operators are Evaluated

1. Parenthesis ( )
2. Exponentiation ^, ↑
3. Multiplication or division *, /
4. Addition or subtraction

Operations of equal priority are performed from left to right.

Basic Statements

BASIC statements are primarily of two types: (1) executable, and (2) non-executable. Executable statements are those which are executed by the computer, while non-executable statements are those which are ignored by the computer and used for the user to understand the nature of the program. The following statements are generally used in BASIC Programming.

1. REM
2. LET
3. INPUT
4. READ … DATA
5. END
6. GOTO
7. PRINT
8. BRANCHING
9. STOP

REM

To declare non-executable statements, REM statement is used.
Syntax: Ln REM comment

Example: 10 REM ** BASIC LANGUAGE**

LET

To assign numeric or string values to a variable LET statement is used.
Syntax: Ln LET variable = constant/expression

Example: 20 Let X = 10
30 Let SI = $P*T*R$/100

INPUT

To enter data into the computer during the process of execution is called INPUT. The entered value will be stored in memory variable.
Syntax: Ln INPUT variables

Example: 10 INPUT X

> **Note** A single INPUT statement can have many variables either same data type or different data type.

The user has to enter the values in the same order in which the variable appears.

Input statement also allows the user to enter relevant data at the time of its execution.

10 INPUT 'enter marks of student'; A.

READ … DATA

It is used to assign the values to the variables. In READ statement variables are declared and in DATA statement, the respective values are provided for the declared variables.

Syntax: Ln READ list of variables

Ln DATA list of values

Notes

1. The number values given through DATA should be more or equal to the number of variables of same data type or different data type declared in READ statement. Otherwise, an error message will be displayed in the program and it will be terminated.
2. The constants in DATA statement must match the variable type.
3. The string constant in the DATA statement need not be enclosed within quotes.
4. There can be many READ and DATA statements in a program.

Example: 10 READ P, Q, R
20 DATA 10, 20, 30.

PRINT

It is used to display the output of the program.
Syntax: Ln PRINT variables

Examples:

1. 100 PRINT x, 4$
2. 100 PRINT $A\$$, $B\$$ when $A\$$ = NEW and $B\$$ = YORK
   Result NEWYORK
3. 100 PRINT A; B; C when $A = 15$, $B = -4$ and $C = 25$
   Result: $b15b - 4b - 25b$
   Where b denotes blank space.
   Print statement can also be written with message enclosed within double quotes.
   150 PRINT 'THIS IS AN ANIMAL'; B$.

END

It is used to terminate the execution of the program.
Syntax: Ln END

Example: 100 END

STOP

It terminates the execution of the program temporarily; it can be re-executed by typing CONT or by pressing the F5 key.
Syntax: Ln STOP

Example: 100 STOP

CONDITIONAL STATEMENTS

The statements which are dependent on certain conditions are known as conditional statements. Only if the test of expression is true, the statements which are dependent on conditions will be executed. Otherwise, they will be skipped.

Conditional statements are of two types:

1. Branching statements
2. Looping statements

Branching Statements

1. IF–THEN statement
2. IF–THEN–ELSE statement

IF–THEN Statement

This is a conditional branching statement. A condition will be specified here, and if it is true the action is carried out.
Syntax: Ln IF conditional THEN actions.

EXAMPLE 19.1

5 REM OPERATION ON TWO NUMBERS.

10 Let $A = 8$

20 Let $B = 20$

30 IF $B > A$ THEN $C = B - A$

40 IF $A = B$ THEN $C = B + A$

50 IF $A > B$ THEN $C = A - B$

60 Print C

70 END.

SOLUTION

Here, $A = 8$, $B = 20$,
i.e., $A < B$.
$C = B - A = 20 - 8 = 12$.
$\therefore$ The output of program is 12.

IF–THEN–ELSE

It is a conditional statement. If a condition is satisfied, then a particular action is executed. Otherwise, another action is executed.
Syntax: Ln IF conditional
THEN action1
ELSE action 2

EXAMPLE 19.2

5 REM Arranging the two numbers in ascending order.

10 Let $A = 10$

20 Let $B = 20$

30 IF $A > B$ THEN

40 PRINT B; A;

50 ELSE

60 PRINT A; B;

SOLUTION

Here,

$A = 10,\ B = 20,$

i.e., $A < B$

Control is transferred to ELSE block.

∴ The output of the program 10, 20.

Looping Statements

Here, a condition is specified with a set of statements. The statements will be executed until the condition gets violated. Generally, an incrementing or a decrementing statement will keep track of the loop.

Syntax: In variable $[u = e_1]$ TO $[e_2]$ STEP $[e_3]$

EXAMPLE 19.3

5 REM Summing of squares of odd numbers

10 LET $S = O$

20 FOR $N = 1$ to 10 STEP 2

30 LET $S = S + N * N$

40 NEXT N

50 PRINT S

60 END

SOLUTION

Here, in the first time, the value of S is $1 \times 1 = 1$. STEP 2 implies that the value of N increases by 2 and becomes 3. This process keeps on going and, finally, prints the value of S as 165.

Unconditional Statements

GOTO

It transforms the control to another part of the program which will be executed.
It is unconditional branching statement.
Syntax: Ln GOTO line number

Example: 50 GOTO 200

EXAMPLE 19.4

Find the output of the following program.

10 Let $A = 1$

20 READ P, Q

30 DATA 2, 8

40 Let $R = P$

50 Let $R = R * P$

60 Let $A = A + 1$

70 If $A = Q$, then GOTO 80 ELSE GOTO 50

80 PRINT R

90 END

(a) 64 **(b)** 256 **(c)** 512 **(d)** 128

SOLUTION

Initial $A = 1$

Initial $R = 2$

Next $R = (2)(2) = 4$

Next $A = 2$

$A \neq Q$ $\therefore$ GOTO 50 takes place

Next $R = (4)(2) = 8 = 2^3$

Next $A = 3$

$A \neq Q$

$\therefore$ GOTO 50 takes place

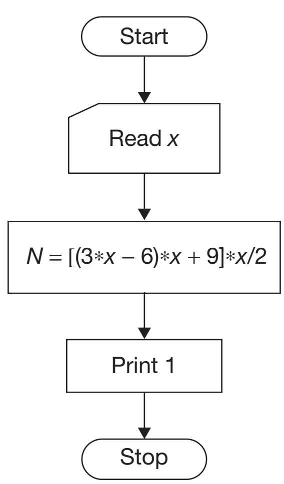
Next $R = (8)(2) = 16 = 2^4$

Similarly when $A = 7$ $A \neq Q$, $R = 2^7$

Next $A = 8$

$\therefore$ GOTO 80, OUTPUT is $2^8 = 256$.

EXAMPLE 19.5



If $x = 12$, then find the output of the flowchart given above.

(a) 2126 **(b)** 2348 **(c)** 2214 **(d)** 2434

SOLUTION

$$\text{Output} = [((3)(12) - 6)12 + 9]\left(\frac{12}{2}\right)$$

$= [(36 - 6)12 + 9] \times 6$

$= (369)(6) = 2214.$

EXAMPLE 19.6

Find the output of the following program.

10 Let $a = 3$

20 Let $b = 2$

30 Let sum $= a + b$

40 Let $b = a$

50 Let $a =$ sum

60 If sum < 90 THEN GOTO 30 ELSE GOTO 70

70 PRINT 'The sum is'; sum

80 END

(a) The sum is 134 **(b)** The sum is 124 **(c)** The sum is 144 **(d)** The sum is 154

SOLUTION

Initial $a = 3$

Initial $b = 2$

Initial sum $= 3 + 2 = 5$

Next $b = 3$

Next $a = 5$

Next sum $= 5 + 3 = 8$

sum < 90

Next $b = 5$

Next $a = 8$

Next sum $= 5 + 8 = 13$

sum < 90

In this manner, the next sums are

$8 + 13 = 21$

$13 + 21 = 34$, $21 + 34 = 55$, $34 + 55 = 89$,

$55 + 89 = 144$

$144 > 90$

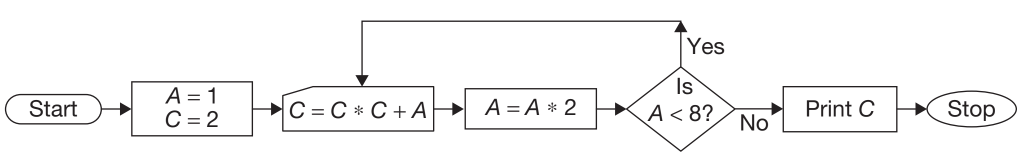
GOTO 70 will take place when

sum $= 144$.

$\therefore$ Output: The sum is 144.

EXAMPLE 19.7

Find the output of the flowchart.



(a) 212 **(b)** 532 **(c)** 244 **(d)** 733

SOLUTION

Initially, $A = 1,\ C = 2$

$C = 2^2 + 1 = 5$

$A = 1 \times 2 = 2$

$A < 8$

$\therefore\ C = 5 \times 5 + 2 = 27$

As $A = 2 \times 2 = 4$

$A < 8$

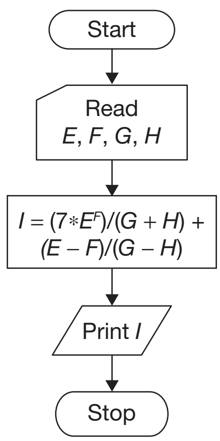
$\therefore\ C = 27^2 + 4 = 733$

$A = 4 \times 2 = 8$

As $A = 8$

$\therefore$ The output is '733'.

EXAMPLE 19.8



If the values of E, F, G and H are 2, 10, 9 and 5 respectively, then find the output of the flowchart.

(a) 510 **(b)** 520 **(c)** 500 **(d)** 490

SOLUTION

$$\text{Output } = \frac{7 * 2^{10}}{9+5} + \frac{2-10}{9-5}$$

$$= \frac{7(1024)}{14} + \left(\frac{-8}{4}\right)$$

$$= \frac{1024}{2} - 2 = 512 - 2 = 510.$$

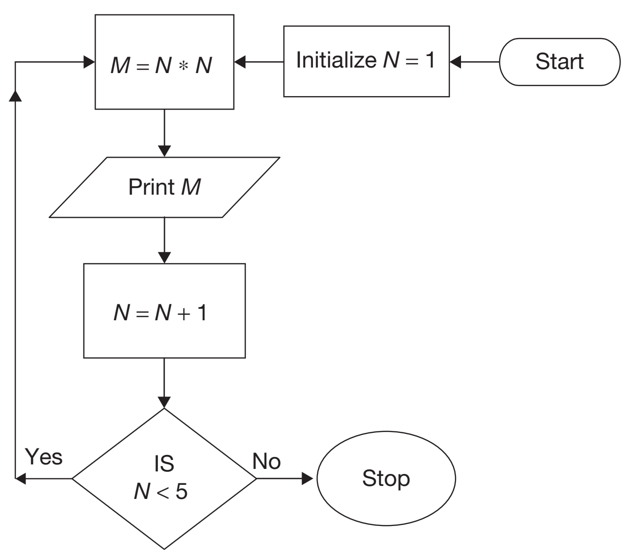
TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Computer works according to instructions. This set of instructions is called a ________.
2. What are the basic units of CPU?
3. The language understood by the computer is called a ________.
4. In first generation of computers, ________ are used.
5. Software acts as a mediator between the user and computer hardware. [True/False]
6. Numeric variables must begin with ________ in BASIC language.
7. In fourth generation of computers, ________ are used.
8. RAM is a secondary memory. [True/False]
9. To display the output of the program, ________ keyword is used.
10. 1 MB = ________ KB.
11. The box indicating the decision in a flowchart is called a ________.
12. Every statement in BASIC language input starts with a ________.
13. In BASIC language, REM keyword is used to write ________.
14. ________ is pictorial representation of algorithm.
15. To assign any value to the variable, ________ key word is used.
16. BASIC is a ________ language.
17. To enter numerical or string data during the time of execution, ________ keyword is used.
18. Go to is used to skip the ________.
19. The maximum length of numeric constant in BASIC language is ________.
20. The usage of numbers that are allowed in BASIC language is ________ to ________.

Short Answer Type Questions

21. Write an algorithm to find the sum of squares of the first ten natural numbers.
22. Write an algorithm to find $S = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5$.
23. Write a program in BASIC to find the centroid of a triangle.
24. Write a program in BASIC to find the sum of the first N natural numbers without using the formula $\frac{N(N+1)}{2}$.
25. What will be the output of the following flowchart?



Essay Type Questions

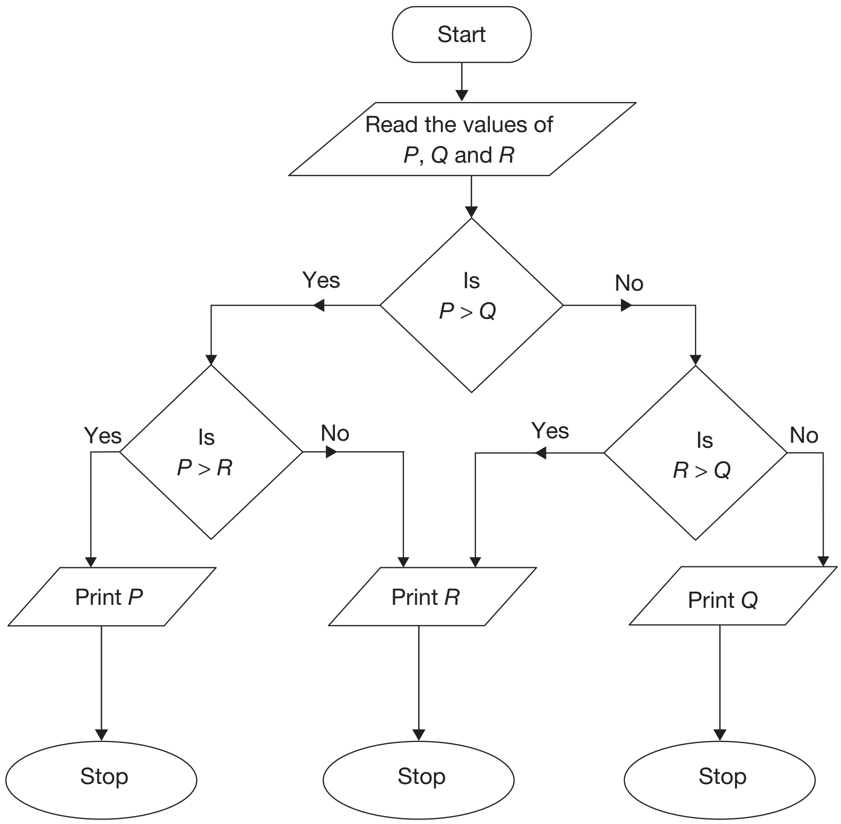
26. Write an algorithm to calculate the value of the expression $S = a * c - d * e + f \div m$, if the values of a, b, c, d, e, f and m are given.
27. Write an algorithm to calculate the value of the expression $R = \sqrt{x^2 + y^2 + z^2}$.

PRACTICE QUESTIONS

28. Write a program in BASIC to find the area of a triangle when the lengths of three sides are given.

29. Write an algorithm to find the largest number among the given ten natural numbers.

30. If 12, 120 and 105 are the values of P, Q and R respectively, then what is the output of the following flow chart?



CONCEPT APPLICATION

Level 1

1. BASIC is

(a) Business Arithmetic System Instruction Code.

(b) Beginners All Purpose Symbolic Instruction Code.

(c) Basic All Purpose System Instruction Code.

(d) Beginners All Purpose System Instruction Code.

2. Which of the following is correct?

(a) LET $A = 20, 420$

(b) LET $B = 21, 445$

(c) LET $A = 40$

(d) None of these

3. Low-level languages or machine languages use strings of

(a) Zeros and twos

(b) Ones and twos

(c) Zeros and Ones

(d) Both (b) and (c)

4. PRINT keyword is useful to assign the values during

(a) compilation

(b) program

(c) the process of execution

(d) storage

5. Evaluate the expression, as done by a computer: $13 - 7 \times 4 \div 2 + 3 - 2 \times 5 - 8$

(a) 0 (b) 57

(c) −16 (d) −3

6. Which of the following statements is true?

(a) Every variable in the INPUT statement must have a corresponding constant or value in the READ statement.

(b) Every variable in the READ statement must have a corresponding constant or value in the DATA statement.

(c) There can be one READ statement and many Data statements.

(d) Every variable in the DATA statement must have many corresponding constants.

7. What is the output of the following program?

10 Read P, R and N

20 Data 1000, 10, 2;

30 Let $A = P * \left(1 + \dfrac{N * R}{100}\right)$

40 Print A

50 End

(a) 1050 (b) 1120

(c) 1230 (d) 1200

8. Which of the following is not an alphanumeric?

(a) ‘S’

(b) ‘SIX’

(c) ‘S92’

(d) ‘123’

9. Which of the following is correct?

(a) 10 READ $x\$$, y
20 DATA 20, ‘TIME’

(b) 10 READ $x\$$, y
20 DATA ‘TIME’, 20

(c) 10 READ $x\$$, $y\$$.
20 DATA 20, 30

(d) 10 READ $x\$$, y
20 DATA ‘TIME’, 20

10. Find the result of the following program.

10 LET $A = 20$, $B = 30$ 20 LET $C = (A + B)/2$

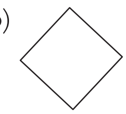
30 PRINT C 40 END

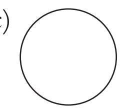
(a) 50 (b) 25

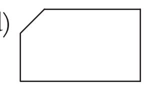
(c) 10 (d) 15

11. In a flowchart representation, to connect a flow diagram from one page to another page of a program, which of the following diagram is used?

(a) 

(b) 

(c) 

(d) 

12. What will be the output of the following program?

10 REM AREA OF TRIANGLE IF THREE SIDES ARE GIVEN.

20 READ a, b, c, s

30 DATA 12,18, 10

40 LET $D = [s*(s - a)*(s - b)*(s - c)] \wedge \dfrac{1}{2}$

50 PRINT ‘THE VALUE OF D = ’; D

60 PRINT D

70 END

(a) 1850 (b) 3200

(c) 6400 (d) Error

13. What will be the output of the following program?

10 REM DISCRIMINANT OF THE QUADRATIC EQUATION

20 READ a, b, c

30 DATA 10, 20, 5

40 LET $D = b \wedge 2 - 4 * a * c$

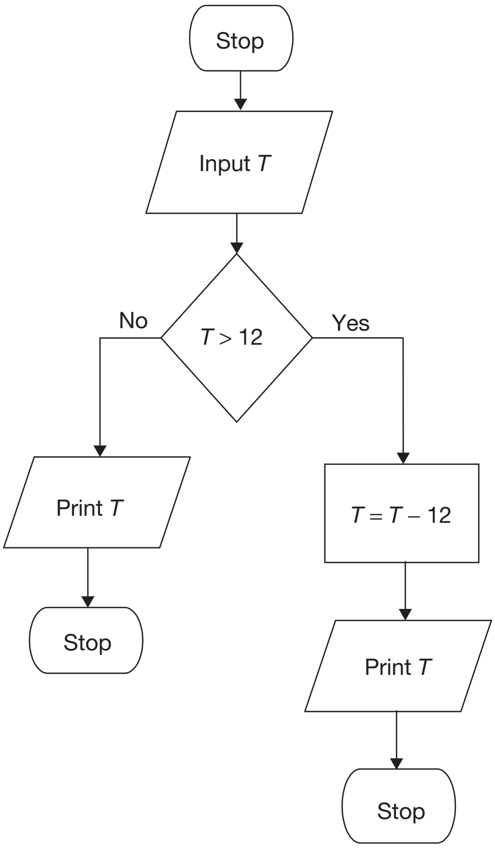
50 PRINT ‘THE VALUE OF D IS = ’; D

60 END

(a) 400 (b) 200

(c) 300 (d) 150

14. What will be the output of the following flowchart, if $T = 15$?



(a) 3 (b) 12

(c) 15 (d) 5

15. Evaluate the following expressions as a computer would do.

I. $(l + m)/n + q \times t \div 5$

II. $80 + 30 - 27 * 4 - 25/(5 * 5)$

(a) $(m + l)/n + \dfrac{qt}{5}$, 1

(b) $(m + l)/n + qt/5$, 2

(c) $(m + l)/n$, 3

(d) $(m + l)/n + qt/5$, 2

16. Which of the following is an algorithm to find the area of a square?

(a) (i) Read an arm of the square.
(ii) Find the area by using $A = 4 * a$.
(iii) Display the area.

(b) (i) Find the area by using $A = 4 * a$.
(ii) Display the area.
(iii) Read an arm of square (a).

(c) (i) Read an arm of square (a).
(ii) Find the area using $A = (a * a)$.
(iii) Display the area.

(d) (i) Read an arm of the square (a).
(ii) Display the area.
(iii) Find the area by using $A = (a * a)$.

17. REM is non-executable statement and is short for ________.

(a) REMAT

(b) REMSTATE

(c) REMARKDATA

(d) REMARK

18. What will be the output of the following program?

10 Read A, B, C, D

20 Data 8, 10, 6, 2

30 Let $S = D - A/(B - C) + A$

40 Print S

50 End

(a) 0.5 (b) 10

(c) 8 (d) 9

19. Find the output of the following program.

10 LET $A = 36$, $B = 4$

20 LET $C = (A/B)$ ^ (1/2)

30 PRINT C

40 END

(a) 9 (2) 36

(c) 3 (d) 6

20. What will be the output of the following program?

10 REM AVERAGE OF FOUR NUMBERS

20 READ N, N_1, N_2, N_3, N_4

30 DATA 4, 16, 40, 42, 90

40 PRINT 'AVERAGE = $(N_1 + N_2 + N_3 + N_4)/N$

50 END

(a) AVERAGE = 43

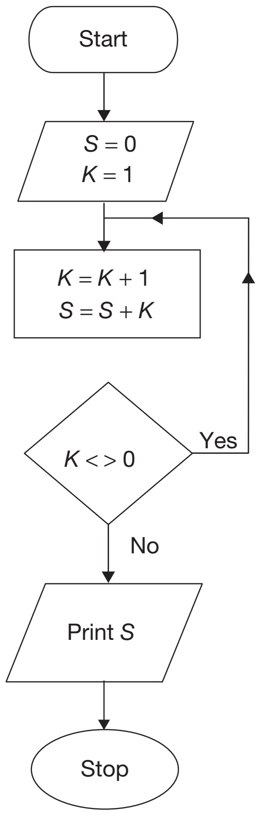
(b) AVERAGE = 47

(c) AVERAGE = 39

(d) Error

Level 2

21. What will be the output of the following flowchart?



(a) 1 (b) Infinite loop
(c) 27 (d) 54

22. Read the following algorithm:

10 Let $P = 1$, Sum = 0

20 If $P \leq N$, then (N is the given Number)

30 Sum = Sum + P

40 $P = P + 1$

50 Repeat this loop

60 End

Above algorithm is used to ________.

(a) find first N natural numbers
(b) find sum of first N natural numbers
(c) find the sum of first N even numbers
(d) find the sum of first N odd numbers

23. 10 REM 'Scholarship Test'

20 INPUT 'Enter marks secured',

S 30

40 If $S > 50$ AND $S < 60$ THEN

Let 50 $S = S + 1000$

60 If $S > 70$ AND < 80 THEN

Let 70 $S = S + 2000$

80 If $S > 75$ THEN $S = S + 2500$

90 If $S > 80$ THEN $S = S + 3000$

100 If $S < 50$ THEN

110 PRINT 'You are not eligible for the Scholarship Test'

120 PRINT 'Your Scholarship amount is:'

S 100 END

If $S = 72$, then what is the scholarship that a student gets?

(a) ₹2572 (b) ₹3072
(c) ₹2072 (d) ₹4572

24. Study the following program:

10 LET $P = 0$

20 LET $a = 0$

30 LET $a = a + 1$

40 Read M

50 If $P > M$ then Go to 70

60 LET $P = M$

70 If $a < 5$ then Go to 30

80 Print P;

90 DATA 3, 5, 4, 2, 6

100 END

What is the output of the above program?

(a) 2 (b) 4 (c) 6 (d) 5

25. What is the output of the following program?

20 Let $I = 1$

30 Read x, y;

40 Data 5, 4;

50 Let $S = X$

60 Print S

70 If $I = Y$, then go to 100

80 Let $S = S * X$

90 Let $I = I + 1$

100 Go to 60

110 End

PRACTICE QUESTIONS

(a) 5 25 625 (b) 625 25 5

(c) 5 25 125 625 (d) 125 25 5

26. Find the output of the program given below.

10 Read P, N, R

20 Data 2000, 2, 20

30 Let $A = P\left[\left(\dfrac{1+R}{100}\right)^{N}\right]$

40 Print A

50 END

(a) 3240 (b) 2880

(c) 3840 (d) 3360

27. BASIC stands for

(a) beginners all purpose system instruction code.

(b) business arithmetic system instruction code.

(c) basi call purpose symbolic instruction code.

(d) beginners all purpose symbolic instruction code.

28. Find the output of the program given below.

10 READ P, Q, R, S

20 DATA 5, 12, 58, 30

20 $T = [(Q + R) / (P + S)] + P$

40 PRINT T

50 End

(a) 7 (b) 9 (c) 6.5 (d) 8.5

29. Find the output of the program given below.

10 Let $P = 7$

20 Let $Q = P$

30 Let $Q = Q - 1$

40 Let $P = P * Q$

50 If $Q > 2$, then GOTO 30 else PRINT P

60 End

(a) 720 (b) 5040

(c) 120 (d) 40320

30. Find the output of the program given below, if $x = 48$ and $y = 60$.

10 READ x, y

20 Let $x = \dfrac{x}{3}$

30 Let $y = x + y + 8$

40 $z = \dfrac{y}{4}$

50 PRINT z

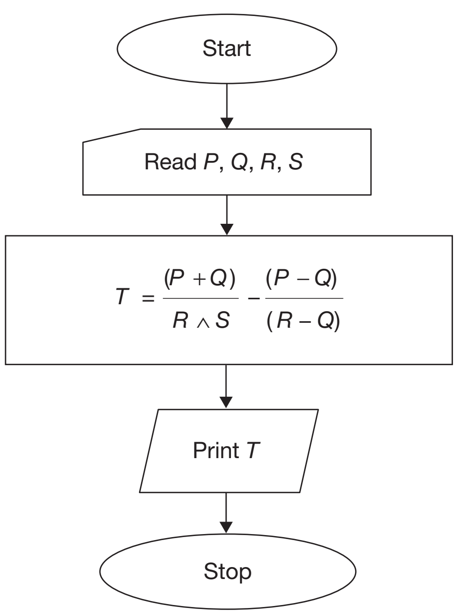
60 End

(a) 21 (b) 22

(c) 23 (d) 24

Level 3

31.



If the values of P, Q, R and S are 8, 6, 7 and 2, then what is the output of above flowchart?

(a) $\dfrac{12}{7}$ (b) $\dfrac{14}{5}$

(c) $\dfrac{-12}{7}$ (d) $\dfrac{-27}{8}$

32. 10 Read A, B, C

20 Read S

30 Read L, M, N, X

40 Read P, Y

50 Data 5, 6, 7

60 Data 9

70 Data 2, $\sqrt{6}$, 2, 4

80 Data $\sqrt{6}$, 2

90 Let $R = [S*(S - A)*(S - B)*(S - C)]$ ^ 0.5

100 Let $T = L* M/(N* P) + X \wedge Y$

110 Let $Z = R/T$

120 Print Z

130 End

(a) $\dfrac{6\sqrt{3}}{17}$ (b) $\dfrac{6\sqrt{6}}{17}$

(c) $\dfrac{6\sqrt{5}}{17}$ (d) $\dfrac{6\sqrt{2}}{17}$

33. 10 LET $x = 0$

20 LET $S = 1$

30 $x = x + 1$

40 $y = x * x$

50 $S = S + y$

60 INPUT 'Enter the value for K'; K

70 If $x < K$ THEN GOTO 30 ELSE

PRINT 'Sum of the squares of the numbers'; S

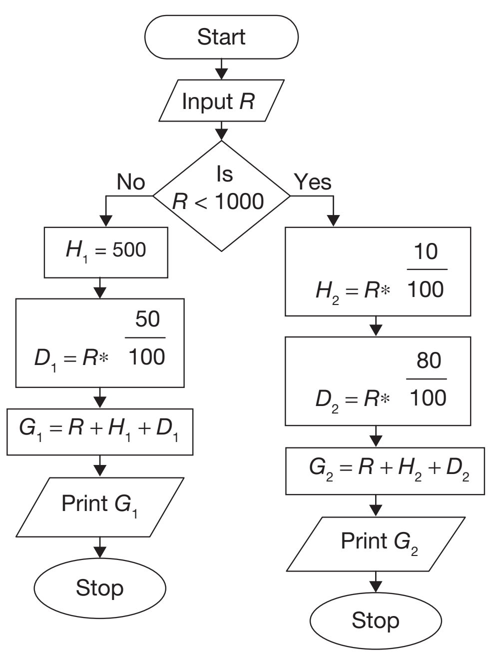
80 END

What is the output of the above program if $K = 8$?

(a) 120 (b) 204

(c) 140 (d) 205

34. Study the following flowchart.



In the above flowchart, what will be the output, if $R = 800$?

(a) 1044 (b) 844

(3) 735 (d) 1520

35. What is the output of the following program?

10 Let Sum = 0

20 Let $N = 8$

30 if $N < 1$ then Go to 70

40 Let Sum = Sum + $N * N$

50 Let $N = N - 2$

60 Go to 30

70 Print Sum

80 end

(a) 120 (b) 125

(c) 145 (d) 170

36. Find the output of the following program.

10 REM Area of a triangle when the sides are given

20 READ x, y, z

30 DATA 8, 15, 17

40 Let $S = (x + y + z)/2$

50 Let $A = [S*(S - x) * (S - y) * (S - Z)]^{1/2}$

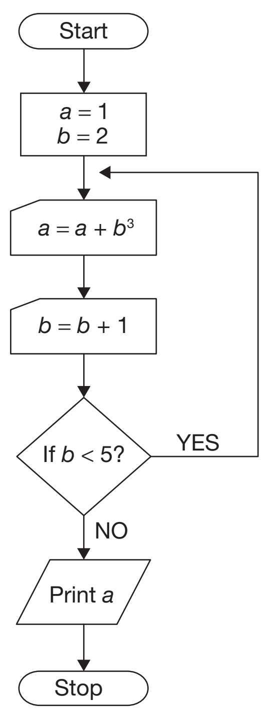
60 PRINT 'The area =', A

70 END

(a) 120 (b) 60

(c) 90 (d) The triangle is not feasible

37.



PRACTICE QUESTIONS

Find the output of the given flowchart.

(a) 100 (b) 73

(c) 120 (d) 144

38. Find the output of the following program.

10 REM Average of ten numbers

20 READ $n_1, n_2, n_3, n_4, \ldots, n_{10}$

30 DATA 2, 4, 14, 8, 22, 32, 44, 58, 74, 92

40 Let Average = $(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10})/10$

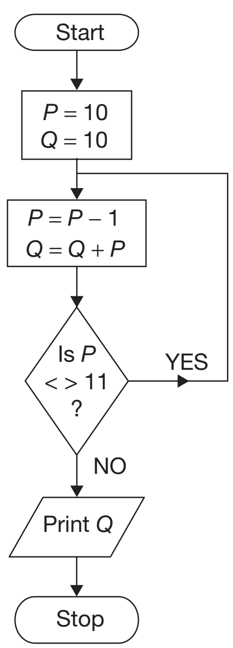
50 PRINT 'Average = '; Average

60 END

(a) Average = 30 (2) Average = 35

(c) Average = 38 (4) Average = 42

39. Find the output of the flowchart below.



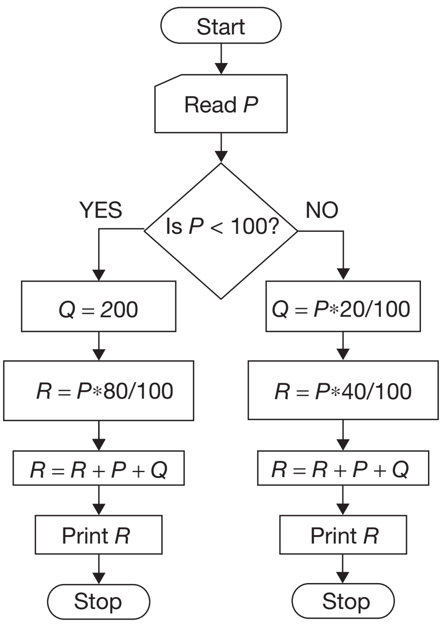
(a) 20

(b) 19

(c) 18

(d) Infinite loop exists

40.



Find the output of the above flowchart, if $P = 500$.

(a) 750

(b) 700

(c) 900

(d) 800

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. program
2. Control unit, ALU and memory unit.
3. machine level language or binary language or low-level language.
4. vacuum tubes
5. True
6. a letter or alphabet
7. VLSI (Very Large Scale Integrated Circuits)
8. False
9. PRINT
10. 1024 KB
11. decision box
12. statement number
13. non-executable statements
14. Flowchart
15. LET
16. high level
17. INPUT
18. Statements
19. 8 digits
20. −32768 to 32767

Short Answer Type Questions

25. 1 4 9 16.

Essay Type Questions

30. 120.

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (b)	**10.** (b)
11. (c)	**12.** (d)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (c)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (b)

Level 2

21. (b)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (c)	**26.** (b)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (a)

Level 3

31. (c)	**32.** (b)	**33.** (d)	**34.** (d)	**35.** (a)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (d)	**40.** (d)

ANSWER KEYS

CONCEPT APPLICATION

Level 1

2. We cannot assign two values to a single variable.

5. Use the order of priority of the arithmetic expressions and evaluate the expression.

6. Identify the syntax of READ and DATA keywords and also their assignments.

7. Use the order of priority of the arithmetic expressions and evaluate the expression.

9. A string constant should be within double quotation marks.

12. (i) Value of s is not given.

(ii) Observe the statement numbered 50. Here, identify the task to be done by operator $\uparrow$ and $*$ and their priorities and precedence.

13. (i) It calculates $D = b^2 - 4ac$.

(ii) Observe the statement numbered 50. Here, identify the task to be done by operator $\uparrow$ and $*$ and their priorities and precedence.

14. The given program prints $T = T - 12$, as $T > 12$.

15. Use BODMAS rule.

16. Steps of algorithms must be in précised order. So, read the statements carefully and complete the task in proper sequence.

17. Recall the keywords of BASIC.

18. Use BODMAS rule.

20. (i) The program prints the average of given numbers.

(ii) Keep track of each statement. Solve clearly the statement numbered 40.

Level 2

21. $K \neq 0$ in every loop.

23. (i) Given $S = 72$, Statement 60, i.e., $S > 70$ and $S < 80$ executes.

(ii) Keep track of the order of the statements that is to be executed and follow with the conditional statements.

24. Keep track of the keyword GOTO which is unconditional branching statement. Be careful when the loop is to be terminated.

25. Find the values in every iterative loop.

26. $P = 2000$, $N = 2$ and $R = 20$

$$A = P\left[\left(1 + \frac{R}{100}\right)^N\right]$$

$$\therefore\ A = 2000\left(1 + \frac{20}{100}\right)^2 = 2000(1.2)^2$$

$$= 2000(1.44) = 2880.$$

27. BASIC stands for 'beginners all purpose symbolic instruction code'.

28. The first statement conveys P, Q, R and S are numeric quantities. The second statement conveys $P = 5$, $Q = 12$, $R = 58$ and $S = 30$.

The third statement conveys $T = \dfrac{12 + 58}{5 + 30} + 5$

$$= \frac{70}{35} + 5 = 7.$$

29. Initial $P = 7$

Initial $Q = 7$

Next $Q = 6$

Next $P = 7 \times 6 = 42$

$Q > 2$

$\therefore$ Next $Q = 5$

Next $P = 42 \times 5 = 210.$

$Q > 2.$

$\therefore$ Next $Q = 4$

Next $P = 210 \times 4 = 840.$

$Q > 2.$

$\therefore$ Next $Q = 3$

Next $P = 840 \times 3 = 2520$

$Q > 2$

$\therefore$ Next $Q = 2$

Next $P = 2520 \times 2 = 5040$.

Q is not greater than 2.

$\therefore$ Output is 5040.

30. Initial $x = 48$

Initial $y = 60$

Next $x = \dfrac{48}{3} = 16$.

Next $y = 16 + 60 + 8 = 84$

$z = \dfrac{84}{4} = 21$.

Level 3

31. Substitute the values of P, Q, R and S in T and follow the operator precedence.

32. (i) Calculate R, T and R/T.

(ii) Use the order of priority of the arithmetic expressions and evaluate the expression.

35. Keep track of the key word GOTO which is unconditional branching statement. Be careful when the loop is to be terminated.

36. $x = 8$, $y = 15$ and $z = 17$

$$S = \frac{8+15+17}{2} = 20$$

$$A = [(20)\cdot(20-8)\cdot(20-15)\cdot(20-17)]^{1/2}$$

$$= [(20)(12)(5)(3)]^{1/2} = \sqrt{3600} = 60$$

Output is = 60.

37. Initial $a = 1$

Initial $b = 2$

Next $a = 1 + 2^3 = 1 + 8 = 9$

Next $b = 3$

$b < 5$

$\therefore$ Next $a = 9 + 3^3 = 9 + 27 = 36$

Next $b = 3 + 1 = 4$

$b < 5$. Next $a = 36 + 4^3 = 100$

Next $b = 5$

$\therefore$ Output $= a = 100$.

38. Average

$$= \frac{2+4+8+14+22+32+44+58+74+92}{10}$$

$$= \frac{350}{10} = 35$$

$\therefore$ Output is average = 35.

39. As P decreases, P never be 11

$\therefore$ The loop represents an infinite loop.

The output would be displayed only if the loop execution stops, i.e., when $P = 11$.

$\therefore$ No output would be displayed.

40. $P = 500$, it is not less than 100.

$\therefore$ The loop on the right of the flowchart will be executed.

$\therefore$ $Q = (500)\left(\dfrac{20}{100}\right) = 100$ and initial.

$R = (500)\left(\dfrac{40}{100}\right) = 200$

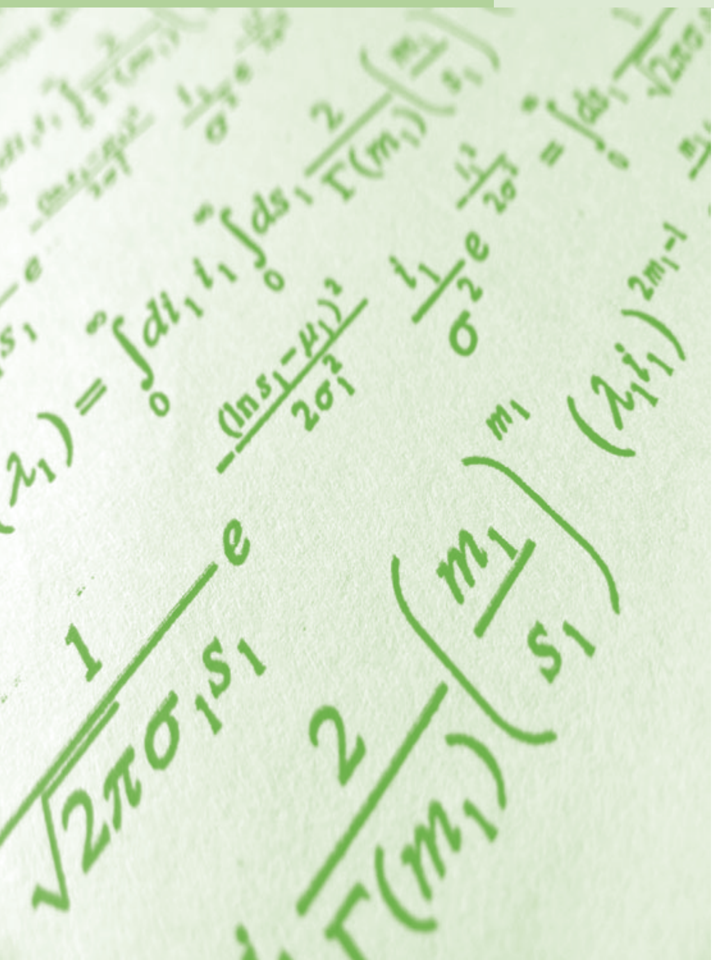
Next $R = 200 + 500 + 100 = 800$.

$\therefore$ Output = 800.

HINTS AND EXPLANATION

Chapter 20

Permutations and Combinations



REMEMBER

Before beginning this chapter, you should be able to:

- Have a knowledge of sets and terms related to it
- Apply operators on sets

KEY IDEAS

After completing this chapter, you would be able to:

- Explain sum rule of disjoint counting and its general form
- Understand multiplication/product rule and its generalization
- Learn about permutation, its factorial notation and general formula
- Study combination and its general formula

INTRODUCTION

This chapter offers some techniques of counting without direct listing of the number of elements in a particular set or the number of outcomes of a particular experiment. We now present the two fundamental rules of counting, namely:

1. The Sum Rule
2. The Multiplication Rule or Product rule.

Sum Rule of Disjoint Counting

If there are two sets say A and B with A having m elements and B having n elements with no element in A appearing in B, then the number of elements in A or B is $(m + n)$.

Symbolically,

$$n(A \cup B) = n(A) + n(B), \text{ when } A \text{ and } B \text{ are disjoint.}$$

The symbol '$\cup$' stands for Union.

EXAMPLE 20.1

$A = \{1, 2, 3, 4\}$ and $B = \{a, e, i, o, u\}$ are two sets. In how many ways can a number from A or a letter from B be chosen?

SOLUTION

As no element of A is in B, we can apply the sum rule of disjoint counting.

$\therefore\ n(A \cup B) = n(A) + n(B) = 4 + 5 = 9.$

General Form of Sum Rule

If A and B are two sets, then the number of elements in A or B (not necessarily disjoint) is given by

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

The symbol '$\cap$' stands for intersection. It means 'common to'.

EXAMPLE 20.2

In how many ways can a prime or an odd number be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

SOLUTION

We form two sets P and O as follows.

$P = \{2, 3, 5, 7\}$ (primes) and $O = \{1, 3, 5, 7, 9\}$ (odd numbers).

On applying the general form of Sum Rule we get,

$n(P \cup O) = n(P) + n(O) - n(P \cap O) = 4 + 5 - 3 = 6.$

We note that the numbers 3, 5 and 7 are counted among primes and also among odd numbers. So, we discount 3 (common numbers) from the sum $n(P) + n(O)$.

Note The usage of the word OR prompts you to add.

Product Rule or Multiplication Rule

If two operations must be performed, and if the first operation can be performed in p_1 ways and the second in p_2 ways, then there are $p_1 \times p_2$ different ways in which the two operations can be performed one after the other.

EXAMPLE 20.3

A caterer's menu is to include 4 different sandwiches and 3 different desserts. In how many ways can one order for a sandwich and a dessert?

SOLUTION

Let S_1, S_2, S_3 and S_4 denote 4 sandwiches and D_1, D_2, D_3 denote 3 desserts.

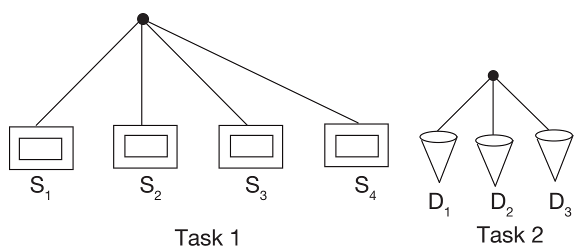


Figure 20.1

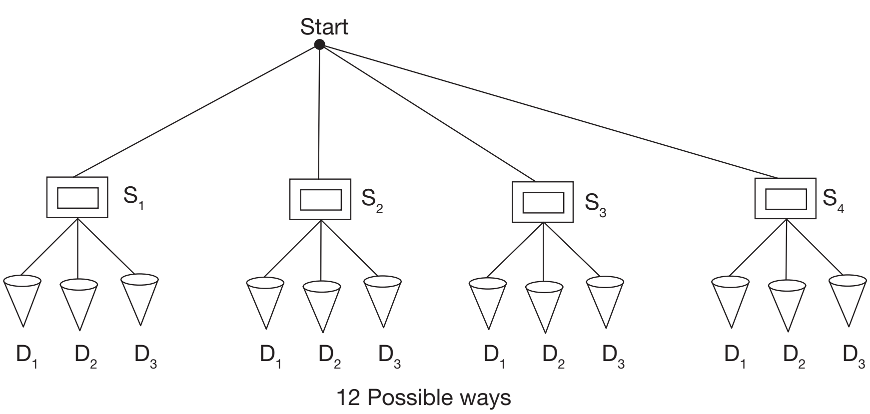


Figure 20.2

The tree diagram clearly suggests that there are 4 ways to choose a sandwich and for each of these 4 ways there are 3 ways to choose a dessert. There are $4 \times 3 = 12$ ways of choosing a sandwich and a dessert.

Generalization of Product Rule

Suppose that tasks $T_1, T_2, T_3, \ldots, T_r$ are to be performed in a sequence. If T_1 can be performed in p_1 ways, and for each of these ways, T_2 can be performed in p_2 ways, and for each of these $p_1 \times p_2$ ways of performing T_1, T_2 in sequence, T_3 can be performed in p_3 ways and so on, then the sequence $T_1, T_2, T_3, \ldots, T_r$ can be performed in $p_1 \times p_2 \times p_3 \times \cdots \times p_r$ ways.

EXAMPLE 20.4

A man has 7 trousers and 10 shirts. How many different outfits can he wear?

SOLUTION

Task 1: He may choose the trouser in 7 ways.

Task 2: He may choose the shirt in 10 ways.

According to the Product Rule, the total number of different outfits is 7×10, i.e., 70.

EXAMPLE 20.5

A class has 20 boys and 15 girls. If one representative from each gender has to be chosen, in how many ways can this be done?

SOLUTION

Task 1: Choosing a representative from boys.
This can be done in 20 ways.

Task 2: Choosing a representative from girls.
This can be done in 15 ways.

By the product rule, the number of ways of performing the two tasks is 20×15, i.e., 300 ways.

EXAMPLE 20.6

How many different outcomes arise from first tossing a coin and then rolling a die?

SOLUTION

There are 2 possibilities (i.e., head or tail) for the first task (tossing a coin) and after each of these outcomes there are 6 possibilities (i.e., any number from 1 to 6) for the second task (rolling a dice). Thus, by the product rule, there are 2×6, i.e., 12 possible outcomes, for the given compound task.

EXAMPLE 20.7

A password of 4 letters is to be formed with vowels alone. How many such passwords are possible if

(a) repetition of letters is allowed,

(b) repetition of letters is not allowed?

SOLUTION

The tasks T_1, T_2, T_3 and T_4 are about filling the 1st, 2nd, 3rd and 4th slots in the password.

(a) The first slot can be filled in 5 ways (a, e, i, o or u).
The second can also be filled in 5 ways (with repetition being allowed).
The third and fourth can also be filled in 5 ways each.
Using the generalization, we get $5 \times 5 \times 5 \times 5 = 625$ passwords.

(b) The first slot can be filled in 5 ways (a, e, i, o or u). The second slot can be filled in 4 ways (as repetition is not allowed). The third and fourth in 3 and 2 ways respectively. Thus the total number of possible passwords are $5 \times 4 \times 3 \times 2 = 120$.

Note The usage of AND prompts you to multiply.

PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a Permutation. Permutation implies 'arrangement', i.e., order of things is important.

EXAMPLE 20.8

Consider 4 elements a, b, c and d. List all permutations taken two at a time.

SOLUTION

We have two cases to deal

(a) with repetition allowed,

(b) with repetition not allowed.

Now list for:

Case 1: aa, ab, ba, ac, ca, ad, da, bb, bc, cb, bd, db, cc, cd, dc, dd, i.e., there are 16 possibilities.

Case 2: ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc, i.e., there are 12 possibilities.

We have seen all the possibilities in both cases.

To count the number of permutations without actual listing of the arrangements, we use the product rule as a technique.

We have two tasks to perform–namely filling up the first slot in the arrangement and filling up the second slot in the arrangement.

Case 1: Repetition allowed.

$$\underset{\text{Task 1}}{\boxed{4}} \times \underset{\text{Task 2}}{\boxed{4}} = 16$$

Case 2: Repetition not allowed.

$$\underset{\text{Task 1}}{\boxed{4}} \times \underset{\text{Task 2}}{\boxed{3}} = 12$$

Note In Case (2), aa, bb, cc, dd are not allowed.

EXAMPLE 20.9

There are 10 railway stations between a station X and another station Y. Find the number of different tickets that must be printed so as to enable a passenger to travel from one station to any other.

SOLUTION

Including X and Y there are 12 stations. From any one station to any other, we need 11 different types of tickets. Since there are 12 stations, the different tickets possible are (12)(11) = 132.

Factorial Notation

If n is a positive integer, then the product of the first n positive integers is denoted by $n!$ or $\lfloor\underline{n}$ (read as n factorial). We define zero factorial as 1.

Accordingly,

$0\,! = 1$

$1\,! = 1$

$2\,! = 1 \times 2 = 2$

$3\,! = 1 \times 2 \times 3 = 6$

$4\,! = 1 \times 2 \times 3 \times 4 = 24$

$5\,! = 120$, $6\,! = 720$, $7\,! = 5040$

Note In some problems the answers will be left as factorials. We need not simplify numbers like 10!, 12!, 25!, etc. However a problem like $\frac{30!}{28!}$ can be simplified as $\frac{30 \times 29 \times 28!}{28!} = 30 \times 29 = 870.$

General Formula for Permutations (Repetitions not Allowed)

The number of permutations of n distinct objects, taken r at a time without repetition is ${}^{n}P_{r} = \frac{n!}{(n-r)!}$, for $r = 0, 1, 2, 3, \ldots, n$. ${}^{n}P_{r}$ is read as nPr.

Explanation

Consider r boxes, each of which can hold one item. When all the r boxes are filled, what we have is an arrangement of r items taken from the given n items. Hence, the number of ways in which we can fill up the r boxes by taking r things from the given n things is equal to the number of permutations of n things taken r at a time.

$$\boxed{n} \quad \boxed{n-1} \quad \boxed{n-2} \cdots \boxed{n-r+1}$$

First box can be filled in n ways (because any one of the n items can be used to fill this box); having filled the first box, to fill the second box we now have only $(n - 1)$ items; any one of these items can be used to fill the second box and hence the second box can be filled in $(n - 1)$ ways; similarly, the third box in $(n - 2)$ ways and so on. Thus the rth box in $\{n - (r - 1)\}$ ways, i.e., $(n - r + 1)$ ways. Hence all the r boxes together can be filled up in $n(n - 1)(n - 2) \cdots (n - r + 1)$ ways.

So, ${}^{n}P_{r} = n(n - 1)(n - 2) \cdots (n - r + 1)$. This can be simplified by multiplying and dividing the right hand side by $(n - r)(n - r - 1) \ldots 3 \cdot 2 \cdot 1$, giving us

$${}^{n}P_{r} = n(n-1)(n-2) \cdots [n-(r-1)] = \frac{n!}{(n-r)!}.$$

The number of permutations of n distinct items taken r at a time is:

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

If we take n things at a time, then we get ${}^{n}P_{n}$. From the discussion we had for filling the boxes, we can find that the number of permutations of n things taken all at a time is $n!$.

$$\boxed{{}^{n}P_{n} = n!}$$

The value of ${}^{n}P_{r}$ without factorials is, ${}^{n}P_{r} = n(n - 1)(n - 2) \ldots (n - r + 1)$, for $r \neq 0$.

EXAMPLE 20.10

In how many ways can 8 athletes finish a race for Gold, Silver and Bronze medals?

SOLUTION

This is the number of permutations of 8 distinct objects taken three at a time without repetitions (here it means same person cannot get both silver and bronze).

Thus, ${}^{8}P_{3} = 8 \times 7 \times 6 = 336$ ways.

General Formula for Permutations with Repetitions Allowed

The number of permutations of n distinct objects taken r at a time with repetition allowed is n^r, for any integer $r \geq 0$.

Explanation We have r boxes with each box ready to accept one or more of the n distinct objects. Using product rule, the total ways of filling up these r boxes is $n \times n \times n \times n \times \cdots$ for r times $= n^r$.

$$\boxed{n} \quad \boxed{n} \quad \boxed{n} \cdots \boxed{n} = n^r$$
$$(r \text{ times})$$

EXAMPLE 20.11

In how many ways can 3 letters be put into 5 letter boxes when each box can take any number of letters?

SOLUTION

As each box can taken any number of letters, we can post each letter in 5 ways.

$$\underset{\text{Letter 1}}{\boxed{5}} \times \underset{\text{Letter 2}}{\boxed{5}} \times \underset{\text{Letter 3}}{\boxed{5}} = 5^3 = 125 \text{ ways.}$$

COMBINATIONS

Each of the groups or selections which can be made by taking some or all of a number of things is called a Combination.

In combinations, the order in which the things are taken is not considered as long as the specific things are included.

EXAMPLE 20.12

Consider a, b, c, d. List all combinations taken 3 at a time.

SOLUTION

The list includes abc, abd, acd, bcd.

Here, abc, bca, cab are regarded the same as order is not important.

The number of combinations of n things taken r at a time is denoted by ${}^nC_r$.

General Formula for Combinations

We first look at the permutations of n items taken r at a time from a different perspective. We look at two tasks T_1 and T_2 as:

T_1: Select r objects.

T_2: Arrange all the r objects that got selected in T_1.

We understand that T_1 can be done in ${}^nC_r$ ways by definition, and its value yet to be determined and T_2 can be done in $r!$ ways. But then to get the permutations, we need to perform T_1 followed by T_2.

Thus, by Fundamental Principle of Counting, both tasks can be done in ${}^nC_r \times r!$ ways.

Thus, ${}^nC_r \times r! = {}^nP_r$, i.e., ${}^nC_r = \dfrac{{}^nP_r}{r!} = \dfrac{n!}{(n-r)!r!}$.

Notes

1. ${}^nC_0 = {}^nC_n = 1$
2. ${}^nC_1 = {}^nC_{n-1} = n$
3. If ${}^nC_r = {}^nC_s$, then $r = s$ or $n = r + s$.

EXAMPLE 20.13

In a library there are 10 research scholars. In how many ways can we select 4 of them?

SOLUTION

Out of 10 scholars, we can select 4 of them in ${}^{10}C_4$ ways.

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210 \text{ ways.}$$

EXAMPLE 20.14

In how many ways can we select two vertices in a hexagon?

SOLUTION

A hexagon has 6 vertices. Any 2 vertices can be selected in ${}^6C_2$ ways $= \dfrac{6 \times 5}{1 \times 2} = 15$ ways.

EXAMPLE 20.15

From 8 gentlemen and 5 ladies, a committee of 4 is to be formed. In how many ways can this be done,

(a) when the committee consists of exactly three gentlemen?

(b) when the committee consists of at most three gentlemen?

SOLUTION

(a) We have to select three out of 8 gentlemen and one out of 5 ladies. Hence, the number of ways in which this can be done $= {}^8C_3 \times {}^5C_1 = 280$.

(b) The committee is to contain at most three gentlemen, i.e., it may contain either 1, 2 or 3 gentlemen.

Hence, the total number of ways $= {}^8C_1 \times {}^5C_3 + {}^8C_2 \times {}^5C_2 + {}^8C_3 \times {}^5C_1 = 80 + 280 + 280 = 640$.

EXAMPLE 20.16

Find ${}^nC_3$, if ${}^nC_7 = {}^nC_4$.

SOLUTION

$${}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11$$

So $${}^nC_3 = {}^{11}C_3 = \frac{11 \times 10 \times 9}{1 \times 2 \times 3} = 165.$$

EXAMPLE 20.17

How many distinct positive integers are possible with the digits 1, 3, 5, 7 without repetition?

SOLUTION

Possible number of,

single-digit numbers = 4

two-digit numbers = $4 \times 3 = 12$

three-digit numbers = $4 \times 3 \times 2 = 24$

four-digit numbers = $4 \times 3 \times 2 \times 1 = 24$

Thus, total number of distinct positive integers without repetition = $4 + 12 + 24 + 24 = 64$.

EXAMPLE 20.18

If ${}^nP_r = 990$ and ${}^nC_r$ 165, then find the value of r.

SOLUTION

${}^nP_r = r!\ {}^nC_r$

$$\Rightarrow \frac{990}{165} = r!$$

$\Rightarrow r! = 6 \Rightarrow r = 3$.

Alternative method:

${}^nP_r = 990 = 11 \times 10 \times 9 = {}^{11}P_3 \Rightarrow n = 11,\ r = 3$ also ${}^{11}C_3 = 165$.

$\therefore\ r = 3$.

EXAMPLE 20.19

In a plane there are 12 points, then answer the following questions:

(a) Find the number of different straight lines that can be formed by joining these points, when no combination of 3 points are collinear.

(b) Find the number of different straight lines that can be formed by joining these points, when 4 of these given points are collinear and no other combination of three points are collinear.

(c) Find the number of different triangles that can be formed by joining these points, when no combination of 3 points are collinear.

(d) Find the number of different triangles that can be formed by joining these points, when 5 of these given points are collinear and no other combination of three points are collinear.

SOLUTION

(a) We know passing through two points in a plane we can draw only one line, i.e., we require to select any two points from the given 12 points which is possible in ${}^{12}C_2$ ways.

$\therefore$ The number of different straight lines that can be formed by joining the given 12 points

$$= {}^{12}C_2\ \frac{12 \times 11}{1 \times 2} = 66.$$

(b) Given, out of the 12 points, 4 points are collinear.

We know that collinear points form only one line.

$\therefore$ These four points when they are not collinear will actually form ${}^4C_2$ lines, which are not forming here.

$\therefore$ The number of the required lines $= {}^{12}C_2 - {}^4C_2 + 1 = 66 - 6 + 1 = 61$.

(c) We know, by joining three non-collinear points a triangle forms.

$\therefore$ Three points can be selected from 12 points in ${}^{12}C_3$ ways.

$\therefore$ The required number of triangles $= {}^{12}C_3 = \dfrac{12\times 11\times 10}{1\times 2\times 3} = 220$.

(d) Given, 5 points are collinear.

$\Rightarrow$ ${}^5C_3$ triangles will not form.

$\therefore$ The required number of triangles $= {}^{12}C_3 - {}^5C_3 = 220 - 10 = 210$.

EXAMPLE 20.20

Find the number of diagonals of a polygon of 10 sides.

SOLUTION

Assume that there are 10 points in a plane where no 3 of them are collinear which are the vertices of the given polygon.

The number of different lines that can be formed by joining these 10 points is ${}^{10}C_2$.

We know in any polygon the lines joining non-adjacent vertices are called diagonals.

Hence, The required number of diagonals = Number of lines formed – Number of sides of the polygon $= {}^{10}C_2 - 10 = 35$.

Using the formula, the number of diagonals in the above problem $= \dfrac{10(10-3)}{2} = 35$.

Note The number of diagonals of a polygon of n sides is given by

$${}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Find the value of 6!.
2. What is the value 0!?
3. Factorial is defined for _____ numbers.
4. The number of arrangements that can be made by taking r objects at a time from a group of n dissimilar objects, is denoted as _____.
5. What is the formula for ${}^nP_r$?
6. In ${}^6C_r$, what are the possible values of r?
7. What is the value of ${}^nC_r$?
8. What is the relation between ${}^nP_r$ and ${}^nC_r$?
9. The number of straight lines that can be formed by n points in a plane, where no three points are collinear is _____ and in case p of these points are collinear is _____.
10. The number of triangles that can be formed by n points in a plane where no three points are collinear is _____ and when p of the given points are collinear is _____.
11. Find the number of 3-digit numbers, formed with the digits {2, 5, 4, 6} when repetition of the digits is allowed.
12. If ${}^nP_{100} = {}^nP_{99}$, then find the value of n.
13. If ${}^{100}C_3 = 161700$, then ${}^{100}C_{97}$ is equal to _____.
14. If ${}^nP_3 = 720$, then find the value of ${}^{11}P_n$.
15. Find the number of four-digit numbers that can be formed using the digits 1, 2, 5, 7, 4 and 6, if every digit can occur at most once in any number.
16. Find the number of integers greater than 4000 that can be formed by using the digits 3, 4, 5 and 2, if every digit can occur at most once in any number.
17. How many 6-letter words with distinct letters in each can be formed using the letters of the word EDUCATION? How many of these begin with I?
18. How many words with distinct letters can be formed by using all the letters of the word PLAYER, which begin with P and end with R?
19. In a class, there are 45 students. On a new year eve, every student sends one greeting card to each of the other students. How many greeting cards were exchanged in all?
20. In how many ways can 6 prizes be distributed among 4 students, if each student can receive more than one prize?
21. If ${}^nP_r = 360$ and ${}^nC_r = 15$, then find the value of r.
22. A bag contains 3 yellow balls and 4 pink balls. In how many ways can 2 pink balls and 1 yellow ball be drawn from the bag?
23. In how many ways can 11 players be chosen from a group of 15 players?
24. A committee of 5 members is to be formed from 8 men and 6 women. Find the number of ways of forming the committee, if it has to contain 3 men and 2 women.
25. In how many ways can 3 diamond cards be drawn simultaneously from a pack of cards?
26. In a party there are 20 persons. If every person shook hand with every other person in the party exactly once, find the total number of handshakes exchanged in the party.
27. A regular polygon has 20 sides. Find the number of diagonals of the polygon.
28. How many different straight lines can be formed from 30 points in a plane? (no three points are collinear)
29. If the number of diagonals of a regular polygon is three times the number of its sides, find the number of sides of the polygon.
30. There are 20 points in a plane. How many different triangles can be formed with these points? (no three points are collinear)

Short Answer Type Questions

31. If ${}^nP_r = 1716$ and $r = 3$, then ${}^nC_r =$ _____.
32. A boy has 9 trousers and 12 shirts. In how many different ways can he select a trouser and a shirt?
33. How many three letter words are formed using the letters of the word FAILURE?
34. The number of selections that can be made to select 5 members from a group of 15 members is _____.

35. There are 8 points in a plane, how many different triangles can be formed using these points (no three points are collinear)?

36. A bag contains 9 yellow balls, 3 white balls and 4 red balls. In how many ways can two balls be drawn from the bag?

37. A question paper contains 20 questions. In how many ways can 4 questions be attempted?

38. If a polygon has 8 sides, then the number of diagonals of the polygon is _____.

39. In a class there are 15 boys and 10 girls. How many ways can a pair of one boy and one girl be selected from the class?

40. How many five-digit numbers can be formed using the digits {5, 6, 3, 9, 2}? (no digit can occur more than once in any number)?

41. In how many ways can 3 consonants be selected from the letters of the word EDUCATION?

42. Using all the letters of the word NOKIA, how many words can be formed, which begin with N and end with A?

43. Given 1 and 2 are two parallel lines. How many triangles can be formed with 12 points taking on 1 and 6 points on 2?

44. A question paper contain 15 questions. In how many ways can 7 questions be attempted?

45. A bag contains 5 white balls and 2 yellow balls. The number of ways of drawing 3 white balls is _____.

Essay Type Questions

46. A four-digit number is formed using the digits {0, 6, 7, 8, 9}. How many of these numbers are divisible by 3? (Each digit is occurred at most once in every number).

47. There are 25 points in a plane. Six of these are collinear and no other combination of 3 points are collinear. How many different straight lines can be formed by joining these points?

48. There are 20 points in a plane, of which 5 points are collinear and no other combination of 3 points are collinear. How many different triangles can be formed by joining these points?

49. Using the letters of the word TABLE. How many words can be formed so that the middle place is always occupied by a vowel?

50. Find the value of ${}^6C_2 + {}^6C_3 + {}^7C_4 + {}^0C_5 + {}^9C_6$.

CONCEPT APPLICATION

Level 1

1. ${}^nC_n =$ _______.
   (a) $n!$ (b) 1
   (c) nn (d) n

2. If a polygon has 6 sides, then the number of diagonals of the polygon is _______.
   (a) 18 (b) 12
   (c) 9 (d) 15

3. How many two digit numbers can be formed using the digits {1, 2, 3, 4, 5}, if no digit occurs more than once in each number?
   (a) 10 (b) 20
   (c) 9 (d) 16

4. If ${}^nC_4 = 35$, then ${}^nP_4 =$ ________.
   (a) 120 (b) 140
   (c) 840 (d) 420

5. Using all the letters of the word QUESTION, how many different words can be formed?
   (a) 8! (b) 7!
   (c) 7 × 7! (d) 9!

6. If ${}^nP_r = 24\ {}^nC_r$, then $r =$ ________.
   (a) 24
   (b) 6
   (c) 4
   (d) 2

7. In how many ways can 5 prizes be distributed to 3 students, if each student is eligible for any number of prizes?
   (a) 3^5 (b) 5^3
   (c) ${}^5P_3$ (d) ${}^5C_3$

PRACTICE QUESTIONS

8. Using the letters of the word PUBLIC, how many four letter words can be formed which begin with B and end with P? (Repetition of letters is not allowed)

(a) 360 (b) 12
(c) 24 (d) 30

9. In a class there are 20 boys and 15 girls. In how many ways can 2 boys and 2 girls be selected?

(a) ${}^{35}C_4$ (b) ${}^{35}C_2$
(c) ${}^{20}C_2 \times {}^{15}C_2$ (d) 20×15

10. Using all the letters of the word OBJECTS, how many words can be formed which begin with B but do not end with S?

(a) 120 (b) 480
(c) 600 (d) 720

11. The number of diagonals of a regular polygon is 14. Find the number of the sides of the polygon.

(a) 7 (b) 8
(c) 6 (d) 9

12. In how many ways can 5 letters be posted into 7 letter boxes?

(a) ${}^{7}C_5$ (b) 5^7
(c) 7^5 (d) ${}^{7}P_5$

13. Sunil has 6 friends. In how many ways can he invite two or more of his friends for dinner?

(a) 58 (b) 57
(c) 63 (d) 49

14. Find the value of ${}^{7}C_4 - {}^{6}C_4 - {}^{5}C_3 - {}^{4}C_2$.

(a) 3 (b) 8
(c) 4 (d) 15

15. How many different words can be formed using all the letters of the word SPECIAL, so that the consonants always in the odd positions?

(a) 112 (b) 72
(c) 24 (d) 144

16. In how many ways can 3 consonants be selected from the English alphabet?

(a) ${}^{21}C_3$ (b) ${}^{26}C_3$
(c) ${}^{21}C_5$ (d) ${}^{26}C_5$

17. From 8 boys and 5 girls, a delegation of 5 students is to be formed. Find the number of ways this can be done such that delegation must contain exactly 3 girls.

(a) 140 (b) 820
(c) 280 (d) 410

18. There are 18 stations between Hyderabad and Bangalore. How many second class tickets have to be printed, so that a passenger can travel from one station to any other station?

(a) 380 (b) 190
(c) 95 (d) 100

19. How many numbers greater than 3000 can be formed using the digits 0, 1, 2, 3, 4 and 5, so that each digit occurs at most once in each number?

(a) 1000 (b) 300
(c) 1200 (d) 1380

20. Using all the letters of the word EDUCATION, how many words can be formed which begin with DU? (Repetition is not allowed).

(a) 8! (b) 7!
(c) 6! (d) 9!

21. Anil has 8 friends. In how many ways can he invite one or more of his friends to a dinner?

(a) 127 (b) 128
(c) 256 (d) 255

22. In how many ways can 4 letters be posted in 3 letter boxes?

(a) 4^3 (b) 3^4
(c) 6! (d) 4

23. Using the letters of the word PRIVATE, how many 6-letter words can be formed which begin with P and end with E?

(a) 3! (b) 4!
(c) 7! (d) 5!

24. Find the number of 4 digit odd numbers that can be formed using the digits 4, 6, 7, 9, 3, so that each digit occurs at most once in each number.

(a) 120 (b) 24
(c) 48 (d) 72

25. How many 5-digit numbers that are divisible by 5 can be formed using the digits {0, 1, 3, 5, 7, 6}? (each digit can be repeated any number of times)

(a) 1080 (b) 2160
(c) 6480 (d) 3175

26. How many four-digit even numbers can be formed using the digits {3, 5, 7, 9, 1, 0}? (Repetition of digits is not allowed)

(a) 120 (b) 60
(c) 360 (d) 100

27. There is a three-digit password and it is known that each digit can have four values 5, 6, 7 or 8. If there is exactly one correct password, how many distinct wrong passwords are there?

(a) 63 (b) 80
(c) 81 (d) 64

28. In how many ways can a committee consisting of 3 men and 4 women be formed from a group of 6 men and 7 women?

(a) ${}^6C_4{}^7C_3$ (b) ${}^6C_3{}^7C_5$
(c) ${}^6C_3{}^7C_4$ (d) ${}^7C_5{}^6C_4$

29. Thirty members attended a party. If each person shakes hands with every other person exactly once, then find the number of handshakes made in the party.

(a) ${}^{30}P_2$ (b) ${}^{30}C_2$
(c) ${}^{29}C_2$ (d) ${}^{60}C_2$

30. In how many ways can 6 members be selected from a group of 10 members?

(a) ${}^6C_4$ (b) ${}^{10}C_4$
(c) ${}^{10}C_5$ (d) ${}^{10}P_4$

Level 2

31. In a class there are 20 boys and 25 girls. In how many ways can a pair of a boy and a girl be selected?

(a) 400 (b) 500
(c) 600 (d) 20

32. How many different odd numbers are formed using the digits {2, 4, 0, 6}? (Repetition of digits is not allowed)

(a) 16 (b) 0
(c) 24 (d) 108

33. There are 15 stations from New Delhi to Mumbai. How many first class tickets can be printed to travel from one station to any other station?

(a) 210 (b) 105
(c) 240 (d) 135

34. In how many ways can 3 vowels be selected from the letters of the word EQUATION?

(a) 56 (b) 10
(c) 28 (d) 40

35. In how many ways can 3 consonants and 2 vowels be selected from the letters of the word TRIANGLE?

(a) 25 (b) 13
(c) 30 (d) 20

36. A plane contains 12 points of which 4 are collinear. How many different straight lines can be formed with these points?

(a) 50 (b) 66
(c) 60 (d) 61

37. A plane contains 20 points of which 6 are collinear. How many different triangles can be formed with these points?

(a) 1120 (b) 1140
(c) 1121 (d) 1139

38. Using the letters of the word ENGLISH, how many five letters words can begin with G?

(a) 2520 (b) 360
(c) 180 (d) 1260

39. Twelve teams are participating in a cricket tournament. If every team plays exactly one match with every other team, then the total number of matches played in the tournament is ________.

(a) 132 (b) 44
(c) 66 (d) 88

40. In how many ways can 4 consonants be chosen from the letters of the word SOMETHING?

(a) ${}^9C_4$ (b) ${}^6C_4$
(c) ${}^4C_4$ (d) ${}^4C_3$

41. How many three letter words can be formed using the letters of the word NARESH? (Repetition of letters is not allowed)

(a) 3! (b) ${}^5P_3$
(c) ${}^6P_3$ (d) ${}^6C_3$

PRACTICE QUESTIONS

42. A four digit number is to be formed using the digits 0, 1, 3, 5 and 7. How many of them are even numbers? (Each digit can occur for only one time).

(a) 48 (b) 60
(c) 24 (d) 120

43. How many numbers less than 1000 can be formed using the digits 0, 1, 3, 4 and 5, so that each digit occurs atmost once in each number?

(a) 53 (b) 69
(c) 68 (d) 60

44. There are 15 points in a plane. No three points are collinear except 5 points. How many different straight lines can be formed?

(a) 105 (b) 95
(c) 96 (d) 106

45. There are 12 points in a plane, no three points are collinear except 6 points. How many different triangles can be formed?

(a) 200 (b) 201
(c) 220 (d) 219

46. Twelve points are marked on a plane so that no three points are collinear. How many different triangles can be formed joining the points?

(a) 180 (b) 190
(c) 220 (d) 230

47. How many words can be formed from the letters of the word EQUATION using any four letters in each word?

(a) 840 (b) 1680
(c) 2080 (d) 3050

48. Seventeen points are marked on plane so that no three points are collinear. How many straight lines can be formed by joining these points?

(a) 114 (b) 136
(c) 152 (d) 160

49. The following are the steps involved in solving ${}^nC_2 = 36$ for n. Arrange then in sequential order.

(A) $n^2 - n - 72 = 0$
(B) As $n > 0$, $n = 9$
(C) $n = 9$, $n = -8$
(D) $(n - 9)(n + 8) = 0$
(E) $\dfrac{n(n-1)}{1\times 2} = 36$

(a) EACBD (b) EADCB
(c) EADBC (d) EDCDB

50. The following are the steps involved in finding the value of $\dfrac{n}{r}$ from ${}^nP_r = 1320$. Arrange them in sequential order.

(A) ${}^nP_r = \dfrac{12!}{9!} = \dfrac{12!}{(12-3)!}$

(B) $\Rightarrow \dfrac{n}{r} = \dfrac{12}{3} = 4$

(C) $\Rightarrow {}^nP_r = {}^{12}P_3$

(D) ${}^nP_r = 1320 = 12 \times 11 \times 10$

(a) DACB (b) DABC
(c) DBCA (d) DBAC

Level 3

51. How many 4-digit even numbers can be formed using the digits {1, 3, 0, 4, 7, 5}? (Each digit can occur only once)

(a) 48 (b) 60
(c) 108 (d) 300

52. Using the letters of the word CHEMISTRY, how many six letter words can be formed, which end with Y?

(a) ${}^8P_6$ (b) ${}^9P_6$
(c) ${}^9P_5$ (d) ${}^8P_5$

53. A telephone number has seven digits, no number starts with 0. In a city, how many different telephone numbers be formed using the digits 0 to 6? (Each digit can occur only once)

(a) 6! (b) 6 · 6!
(c) 7! (d) 2 · 7!

54. Using all the letters of the word PROBLEM, how many words can be formed such that the consonants occupy the middle place?

(a) 3000 (b) 4200
(c) 720 (d) 3600

PRACTICE QUESTIONS

55. Using the digits 0, 1, 2, 5 and 7 how many 4-digit numbers that are divisible by 5 can be formed if repetition of the digits is not allowed?

(a) 38 (b) 46
(c) 32 (d) 42

56. If ${}^{2n}C_4 : {}^{n}C_3 = 21 : 1$, then find the value of n.

(a) 4 (b) 5
(c) 6 (d) 7

57. How many three-digit numbers that are divisible by 5, can be formed, using the digits 0, 2, 3, 5, 7, if no digit occurs more than once in each number?

(a) 10 (b) 15
(c) 21 (d) 25

58. In how many ways can we select two vowels and three consonants from the letters of the word ARTICLE?

(a) 12 (b) 14
(c) 18 (d) 22

59. How many 3-digit numbers can be formed using the digits {2, 4, 5, 7, 8, 9}, if no digit occurs more than once in each number?

(a) 80 (b) 90
(c) 120 (d) 140

60. The number of ways of selecting five members to form a committee from 7 men and 10 women is _____.

(a) 5266 (b) 6123
(c) 6188 (d) 8123

61. Twenty points are marked on a plane so that no three points are collinear except 7 points. How many triangles can be formed by joining the points?

(a) 995 (b) 1105
(c) 1200 (d) 1250

62. There are four different white balls and four different black balls. The number of ways that balls can be arranged in a row so that white and black balls are placed alternately is _____.

(a) $(4!)^2$ (b) $2(4!)^2$
(c) 4! (d) $(4!)^3$

63. In a party, there are 10 married couples. Each person shakes hands with every person other than her or his spouse. The total number of handshakes exchanged in that party is _____.

(a) 160 (b) 190
(c) 180 (d) 170

64. How many 4-digit odd numbers can be formed using the digits 0, 2, 3, 5, 6, 8 (each digit occurs only once)?

(a) 64 (b) 72
(c) 86 (d) 96

65. The number of the words that can be formed using all the letters of the word BRAIN such that it starts with R and but does not end with A.

(a) 18 (b) 14
(c) 16 (d) 20

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 720
2. 1
3. whole numbers
4. ${}^nP_r$
5. $\dfrac{n!}{(n-r)!}$
6. 0, 1, 2, 3, 4, 5, 6
7. $\dfrac{n!}{(n-r)!r!}$
8. ${}^nP_r = {}^nC_r r!$
9. ${}^nC_2$; ${}^nC_2 - {}^pC_2 + 1$
10. ${}^nC_3$; ${}^nC_3 - {}^pC_3$
11. $4^3 = 64$.
12. 100
13. 161700
14. 11!
15. ${}^6P_4$.
16. 12
17. (i) ${}^9P_6$ (ii) ${}^8P_5$.
18. 24
19. 1980
20. 4^6
21. 4
22. 18
23. ${}^{15}C_{11}$
24. 840
25. 286
26. 190
27. 170
28. 435
29. 9
30. ${}^{20}C_3$

Short Answer Type Questions

31. 286
32. 108
33. 210
34. ${}^{15}C_5$
35. 56
36. ${}^{16}C_2$
37. ${}^{20}C_4$
38. 20
39. 150
40. 120
41. 4
42. 6
43. 576
44. ${}^{15}C_7$
45. 10

Essay Type Questions

46. 60
47. 286
48. 1130
49. 48
50. 210

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (d)	**20.** (b)
21. (d)	**22.** (b)	**23.** (d)	**24.** (d)	**25.** (b)	**26.** (b)	**27.** (a)	**28.** (c)	**29.** (b)	**30.** (b)

Level 2

31. (b)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (b)
41. (c)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (b)	**50.** (a)

Level 3

51. (c)	**52.** (d)	**53.** (b)	**54.** (d)	**55.** (d)	**56.** (b)	**57.** (c)	**58.** (a)	**59.** (c)	**60.** (c)
61. (b)	**62.** (b)	**63.** (c)	**64.** (d)	**65.** (a)					

CONCEPT APPLICATION

Level 1

1. Use, $^nC_r = \dfrac{n!}{(n-r)!}$.
2. The number of diagonals of a n–sided polygon is $\dfrac{n(n-3)}{2}$.
3. r objects can be arranged out of n objects is ^nP_r ways.
4. $^nP_r = {^nC_r} \times r!$
5. $^nP_n = n!$.
6. $r! = \dfrac{^nC_r}{^nC_r}$.
7. First prize can be distributed in 3 ways similarly 2nd, 3rd, 4th and 5th prizes can be distributed each in 3 ways. Now use the fundamental principle of counting.
8. Take 6 blanks, first blank is filled with B, last blank is filled with P and the remaining blanks can be filled with the remaining letters.
9. r objects can be selected from n objects in ^nC_r ways. Now use the fundamental principle, i.e., task T_1 can be done in m ways and task T_2 can be done in n ways, then the two tasks can be done simultaneously in mn ways.
15. First arrange the consonants in odd places is in 1, 3, 5 and 7 places. Now arrange the vowels in the remaining places and then use the fundamental principle.
16. Select 3 consonants from 21 consonants.
17. Select 3 girls from 5 girls and 2 boys from 8 boys then apply fundamental principle.
18. Total number of stations = 20. Select 2 stations from 20 stations.
20. Find the number of 7 letter words using the 7 letters.
22. (i) Each letter be posted in 3 ways.
    (ii) Now calculate the number of ways in which four letters can be posted by using fundamental theorem of counting.
23. As the letters begin with P and ends with E, four more letters are to be selected from the remaining 5 letters.
24. (i) The units digit of the number must be odd, i.e., it can be done in 3 ways.
    (ii) Now find the number of ways in which the three digits can be filled using four digits.
    (iii) Use the fundamental theorem of counting.
25. (i) If unit digit is 0 or 5, then the number is divisible by 5.
    (ii) If the units digit is 5, then ten thousands digit cannot be zero, now find the number of ways the other four digits can be arranged.
    (iii) Similarly when the units digit is 0, the other 4 digits can be arranged in $^5P_4$ ways. Use the fundamental theorem of counting.
26. (i) The units digit of the required number is 0.
    (ii) Find the number of ways in which the remaining 5 digits can be arranged in three places by using ^nP_r.
27. (i) Each digit of the password can have 4 values, hence first digit can be filled in 4 ways.
    (ii) Find the number of ways in which remaining digits can be filled.
    (iii) Use the fundamental principal of counting and find total number of passwords that are formed.
28. Select 3 men from 6 men and select 4 women from 7 women, then apply the fundamental principle.
29. Select two persons from 30 persons.
30. From a group of n members selecting r members at a time is denoted by ^nC_r.

Level 2

31. (i) The number of ways a boy and a girl can be selected individually is $^{20}C_1$ and $^{25}C_1$.
    (ii) Use the fundamental theorem of counting.
32. All the given digits are even, so no odd number can be formed with the given digits.
33. (i) The total number of stations are 15 (say n).

HINTS AND EXPLANATION

(ii) On a ticket, two stations should be printed.
(iii) The required number of ways $= {}^nC_2 \times 2$.

34. There are 5 vowels and 3 are to be chosen.

35. (i) Find the number of ways in which 3 consonants and 2 vowels can be selected from 5 consonants and 3 vowels.
(ii) Then use fundamental theorem of counting.

36. The number of lines that can be formed from n points in which m points are collinear is ${}^nC_2 - {}^mC_2 + 1$.

37. The number of triangles that can be formed from n points in which m points are collinear is ${}^nC_3 - {}^mC_3$.

38. As Y is filled in last blank, five more letters are to be selected from the remaining 8 letters.

39. Select any two teams from 15 teams.

40. There are 6 consonants and 4 are to be chosen.

41. Arrange 3-letter words out of 6 letters.

44. The number of lines that can be formed from n points in which m points are collinear is ${}^nC_2 - {}^mC_2 + 1$.

45. The number of triangles that can be formed from n points in which m points are collinear is ${}^nC_3 - {}^mC_2$.

46. From the 'n' points (no three points are collinear) in a plane, the number of triangles formed is ${}^nC_3$.

$\therefore$ The required number of triangles

$$= {}^{12}C_3 = \frac{12 \times 11 \times 10}{1 \times 2 \times 3} = 220.$$

47. There are 8 letters in the word EQUATION. The number of words with 4 letters formed with the 8 letters is ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$.

48. From the 'n' points of a plane, the number of straight lines formed is ${}^nC_2$.

$\therefore$ The required number of straight lines $= {}^{17}C_2 = 136$.

49. EADCB is the required sequential order.

50. DACB is the required sequential order.

Level 3

51. (i) If the digit in the units place is an even number, then the number is called even number.
(ii) The units digit of the four digit number must be 0 or 4.

			0

(or)

			4

(iii) If units digit is 4, then thousands digit can be filled in 4 ways and other two digits can be filled in ${}^4P_2$ ways.
(iv) If units digit is 0, then find the number of ways in which the remaining 3 digits can be filled by using 5 digits.

52. (i) Take 6 blanks.
(ii) As Y is filled in last blank, five more letters are to be selected from the remaining 8 letters.

53. (i) The first digit of the number cannot be zero, so it can be filled in 6 ways.
(ii) Now the second digit can be any of the 6 digits., i.e., it can be filled in 6 ways.
(iii) As the digits cannot be repeated, the number of ways the third digit can be filled is 5 ways and so on.
(iv) Now apply the fundamental theorem.

54. (i) There are 5 consonants and middle place can be filled in 5 ways.
(ii) Remaining places can be filled with remaining letters.

55. Unit place is 0 or 5 then the number is divisible by 5.

56. Given, $\dfrac{{}^{2n}C_4}{{}^nC_3} = \dfrac{21}{1}$

$${}^{2n}C_4 = 21\,{}^nC_3$$

$$\frac{(2n)!}{(2n-4)!\,4!} = 21\frac{n!}{(n-3)!\,3!}$$

$$(2n)(2n-1)(2n-2)(2n-3) = 84(n)(n-1)(n-2)$$

$$4(2n-1)(2n-3) = 84(n-2).$$

From the options, $n = 5$ satisfies the above equation.

57. If a number is divisible by 5, then the units digit must be either 0 or 5.

Case 1: If the units digit is 0, then the remaining two places can be filled by remaining 4 digits.

It can be done in ${}^4P_2 = 12$ ways.

Case 2: If the units digit is 5, then the remaining two places can be filled by the remaining 4 digits.

It can be done in $3 \times 3 = 9$ ways.

($\therefore$ The hundreds place can not be filled with 0)

$\therefore$ The required number of 3-digit numbers = $12 + 9 = 21$.

58. There are three vowels and four consonants in the word ARTICLE.

The number of ways of selecting 2 vowels and 3 consonants from 3 vowels and 4 consonants $^3C_2 \times {}^4C_3$ $= 3 \times 4 = 12$.

59. Here, we have 6 digits.

Since we have to find 3-digit numbers, the first digit can be filled in 6 ways, second digit can be filled in 5 and third digit can be filled in 4 ways.

($\because$ No digit is repeated)

$\therefore$ The required number of digits

$= 6 \times 5 \times 4 = 120$.

60. Here, the total number of persons is 17.

The number of ways of selecting 5 members from 17 members is $^{17}C_5 = 6188$.

61. We know that number of triangles formed by 'n' points in which 'm' are collinear is $^nC_3 - {}^mC_3$

The required number of triangles

$= {}^{20}C_3 - {}^7C_3$

$= 1140 - 35 = 1105$.

62. We have to arrange 4 different white balls and 4 different black balls as shown below

(i) WB WB WB WB

(ii) BW BW BW BW

This can be done in $4! \times 4! + 4! \times 4!$ ways, i.e., $2(4!)^2$.

63. There are 20 persons in the party.

Total handshakes $= {}^{20}C_2 = 190$

This includes 10 handshakes in which a person shakes hands with her or his spouse.

$\therefore$ The required number of handshakes

$= 190 - 10 = 180$.

64. If a number is odd, then the units digit of the number must contain any one of the numbers 1, 3, 5, 7 or 9.

Given digits are 0, 2, 3, 5, 6, 8.

Here units digit must contain either 3 or 5, i.e., 2 ways.

First place can be filled in 4 ways (since ten thousand pace can not be filled with 0).

Second place and third digit can be filled in 4 and 3 ways respectively.

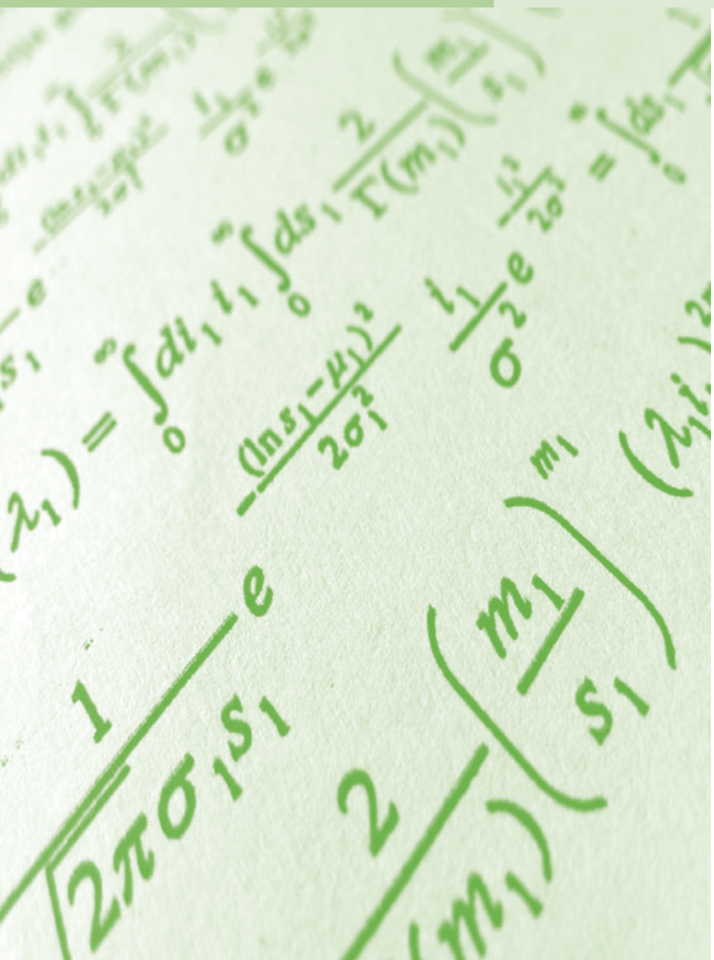
The required number of odd numbers $= 4 \times 4 \times 3 \times 2 = 96$.

65. Since the first place of the word always starts with R and last place is not A. The last place can be filled by any one of the remaining three letters. The remaining 3 places can be filled with 3 letters in 3! ways.

$\therefore$ The total number of words $= 3 \times 3! = 18$.

Chapter 21

Probability



REMEMBER

Before beginning this chapter, you should be able to:

- Recall some basic concepts related to probability.
- Apply the concept in daily life and solve problems.

KEY IDEAS

After completing this chapter, you would be able to:

- Understand about random experiments and events occurring.
- Learn probability of an event and probability of non-occurrence of an event.

INTRODUCTION

In nature, there are two kinds of phenomena—deterministic and in-deterministic. Examples of deterministic phenomena are:

1. The sun rises in the east.
2. If an object is dropped, it falls to the ground.
3. Every organism which takes birth dies.

Examples of in-deterministic phenomena are:

1. The next vehicle that we see on a road going west to east may be headed east or west.
2. If a gas molecule is released in a container, it may head in any direction.
3. A person selected from a population may die before attaining the age of 75 years, when he attains the age of 75 or after he attains the age of 75.

Probability theory is the study of in-deterministic phenomena. While the theory has widespread applications in all walks of life, it is best to confine ourselves to certain simple kind of in-deterministic phenomena in the initial stage. Examples are tossing of coins, rolling dice, picking cards from well-shuffled decks and drawing objects from different containers, containing different objects. All these are examples of **random experiments**—situations in which we do something and we are not sure of the outcome, because there is more than one possible outcome.

Example: Tossing a coin, rolling a dice or drawing a card from a deck.

Sample Space

The set of all possible outcomes of an experiment is called its sample space. It is usually denoted by S.

Examples:

1. Consider a cubical dice with numbers 1, 2, 3, 4, 5 and 6 on its faces.

   When we roll the dice, it can fall with any of its faces facing up. So, the number on each of its faces is a possible outcome.

   Hence, the sample space $(S) = \{1, 2, 3, 4, 5, 6\}$.
2. When we toss an unbiased coin, the result can be a head(H) or a tail(T). So the sample space $(S) = \{H, T\}$.

Event

The outcomes or a combination of the outcomes is called an event. The probability of an event (E) is a measure of our belief that the event will occur. This may be zero, i.e., we do not expect the event to occur at all. For example, if two dice are rolled, the probability that the sum of the numbers which will come up is 1 is zero. The probability may be 1, i.e., we are absolutely certain that the event will occur. For Example, If a coin is tossed the probability that we get a head or a tail is 1. But in general, we may believe that the event may occur but we are not absolutely certain, i.e., $0 < p < 1$.

Examples:

1. In rolling a dice, getting an even number is an event.
2. In tossing a coin, getting a head (H) is an event.

In the general case, when we are not sure of either the occurrence or non-occurrence of an event, how do we assign a number to the strength of our belief?

Consider the case of rolling a dice. It will show up one of the numbers–1, 2, 3, 4, 5 or 6. Each of these outcomes is equally likely, if the dice is an unbiased dice, i.e., well-balanced. So, when a dice is rolled once, there is only one way of getting the outcome 5, out of the six possible outcomes 1, 2, 3, 4, 5 or 6. Therefore, the probability of the number 5 showing up is 1 in 6. In other words, we say that the probability of getting 5 is $\frac{1}{6}$.

We write this as, $P(5) = \frac{1}{6}$.

Also, the probability of getting each of the other numbers is equal to $\frac{1}{6}$.

Similarly, in tossing an unbiased coin, we can say that the probability of getting a head or the probability of getting a tail is $\frac{1}{2}$.

Probability of an Event

Let E be an event of a certain experiment whose outcomes are equally likely. Then, the probability of the event E, denoted by $P(E)$, is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to } E}{\text{Total number of possible outcomes}}.$$

Notes

1. For any event E, $0 \le P(E) \le 1$.
2. If $P(E) = 0$, then E is said to be an impossible event.
3. If $P(E) = 1$, then E is said to be a certain or a sure event.

EXAMPLE 21.1

Find the probability of getting an even number when an unbiased dice is rolled once.

SOLUTION

When a dice is rolled, the total number of possible outcomes is 6.

Let E be the event of getting an even number. Then, the outcomes favourable to E are 2, 4 and 6, i.e., 3 outcomes are favourable.

Hence, $P(E) = \frac{3}{6} = \frac{1}{2}$.

EXAMPLE 21.2

When an unbiased dice is rolled once, what is the probability of getting a multiple of 3?

SOLUTION

When an unbiased dice is rolled, the total number of possible outcomes is 6.

Let E be the event of getting a multiple of 3.

Then, the outcomes favourable to E are 3 and 6, i.e., 2 outcomes are favourable.

Hence, $P(E) = \frac{2}{6} = \frac{1}{3}$.

Probability of Non-occurrence of an Event *E*

Let a random experiment have n possible outcomes – all equally likely. Say m of these are favourable for an event E. Then, there are $(n - m)$ outcomes which are not favourable to the event E.

Let $\bar{E}$ denote the non-occurrence of E.

Then, $P(\bar{E}) = \dfrac{n-m}{n}$,

That is, $P(\text{non-occurrence of } E) = \dfrac{n-m}{n}$.

Now, $P(E) + P(\bar{E})$

$= \dfrac{m}{n} + \dfrac{n-m}{n} = \dfrac{m+(n-m)}{n} = \dfrac{n}{n} = 1.$

That is, $P(E) + P(\bar{E}) = 1$.

Classification of a Pack (or Deck) of Cards

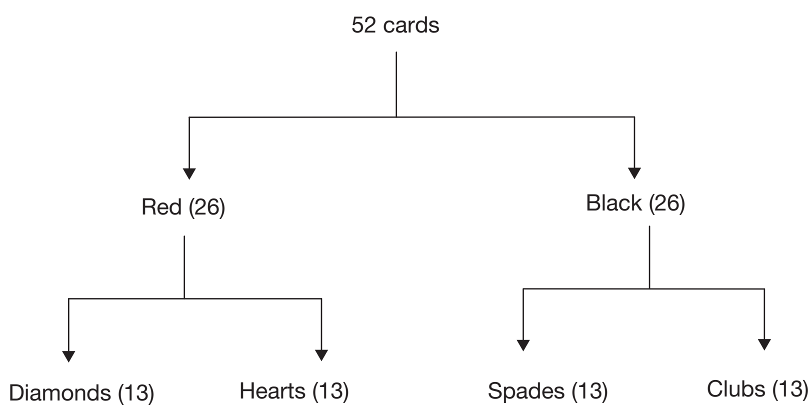


Figure 21.1

The cards in each suit are Ace (A), King (K), Queen (Q), Jack (J), 10, 9, 8, 7, 6, 5, 4, 3 and 2.

The cards A, J, Q and K are called honours and the cards 2, 3, 4, 5, 6, 7, 8, 9 and 10 are called numbered cards. The cards J, Q and K are called face cards.

EXAMPLE 21.3

When a fair dice is rolled, what is the probability of getting a number less than 5?

SOLUTION

When a fair dice is rolled, the total number of possible outcomes is 6.

Let E be the required event. Then, the outcomes favourable to E are 1, 2, 3 and 4, i.e., 4 favourable outcomes.

$$\Rightarrow P(E) = \frac{4}{6} = \frac{2}{3}.$$

Hence, the probability of getting a number less than 5 is $\dfrac{2}{3}$.

EXAMPLE 21.4

When a dice is rolled, what is the probability of getting a number 2 or 3?

SOLUTION

Total number of possible outcomes = 6.

Favourable outcomes are 2 and 3.

That is, 2 favourable outcomes.

$\therefore$ Required probability $= \dfrac{2}{6} = \dfrac{1}{3}$.

EXAMPLE 21.5

A number is chosen randomly from the set of integers from 1 to 20. What is the probability that it will be divisible by 5?

SOLUTION

There are 20 integers from 1 to 20.

So, an integer can be selected from 1 to 20 in 20 ways.

Let E be the required event. Then, the numbers favourable to E are 5, 10, 15 and 20, i.e., 4 favourable numbers.

$$\Rightarrow P(E) = \frac{4}{20} = \frac{1}{5}.$$

Hence, the probability that the number selected is divisible by 5 is $\dfrac{1}{5}$.

EXAMPLE 21.6

A card is selected at random from a pack of cards. What is the probability that it will be a red card?

SOLUTION

There are 52 cards in a pack of cards.

So, a card can be selected from it in 52 ways.

Out of the 52 cards, 26 cards are red coloured.

So, a red card can be selected in 26 ways.

Hence, required probability $= \dfrac{26}{52} = \dfrac{1}{2}$.

EXAMPLE 21.7

When a card is selected at random from a pack of cards, find the probability that it is a king.

SOLUTION

There are 52 cards in a pack of cards.

So, a card can be selected in 52 ways.

Now, there are 4 kings (one in each suit) in the pack.

So, a king can be selected in 4 ways.

$\therefore$ The required probability $= \dfrac{4}{52} = \dfrac{1}{13}$.

EXAMPLE 21.8

A number is selected from the numbers 1 to 20. What is the probability that it will be a prime number?

SOLUTION

Total number of ways of selecting a number from 1 to 20 is 20.

Let E be the event that the number selected is a prime number.

Then the numbers favourable to E are 2, 3, 5, 7, 11, 13, 17 and 19.

That is, 8 favourable numbers.

$$\therefore P(E) = \frac{8}{20} = \frac{2}{5}.$$

Hence, the probability that the number selected will be a prime number is $\dfrac{2}{5}$.

EXAMPLE 21.9

A bag contains 3 blue and 7 red balls. Find the probability that a ball selected at random from the bag will be a blue ball.

SOLUTION

Total number of balls in the bag $= 3 + 7 = 10$.

So, a ball can be selected from the bag in 10 ways.

Now, there are 3 blue balls in the bag.

So, a blue ball can be selected from the bag in 3 ways.

Hence, the required probability $= \dfrac{3}{10}$.

EXAMPLE 21.10

Find the probability of a card that is selected at random from a pack of cards will be a red honour.

SOLUTION

Total number of ways of selecting a card $= 52$.

There are 8 honours in the 26 red cards.

So, a red honour can be selected in 8 ways.

$\therefore$ The required probability $= \dfrac{8}{52} = \dfrac{2}{13}$.

EXAMPLE 21.11

There is a bunch of 10 keys out of which any one of the 4 keys can unlock a door. If a key is selected at random from the bunch and tried on the door, find the probability that the door will be unlocked.

SOLUTION

Total number of ways of selecting a key = 10.

4 keys can unlock the door.

$\Rightarrow$ Favourable outcomes = 4

$\therefore$ The required probability $= \frac{4}{10} = \frac{2}{5}$.

EXAMPLE 21.12

When two unbiased coins are tossed, what is the probability that both will show a head?

SOLUTION

When two coins are tossed together, the possible outcomes are HH, HT, TH or TT,

That is, 4 possible outcomes.

Also, there is only one case where both the coins show heads.

$\therefore$ The required probability $= \frac{1}{4}$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. An experiment in which all the outcomes of the experiment are known in advance and the exact outcome is not known in advance is called a ________.

2. A set of events which have no pair in common are called ________.

3. When a coin is tossed, the probability of getting neither head nor tail is called ________ event.

4. When a dice is thrown, the events {1}, {2}, {3}, {4}, {5} and {6} are called ________ events.

5. When a number is selected from a set of natural numbers, the probability of getting a number which is a multiple of 1 is ________.

6. The set of all possible outcomes of an experiment is called the ________.

7. When a dice is thrown, the probability of getting any one of the numbers from 1 to 6 on the upper face is called ________ event.

8. The probability of any event of a random experiment cannot exceed ________.

9. If the probability of an event of a random experiment is $P(E) = 0$, then the event is called ________.

10. If A is any event in a sample space, then $P(A^1) =$ ________.

11. The probability of getting multiples of 3 when a dice is thrown is ________.

12. The range of probability of any event of a random experiment is ________.

13. If A is an event of a random experiment, then A^c or $\overline{A}$ or A' is called the ________ of the event.

14. The probability of getting 5 when a fair dice is thrown is ________.

15. A dice is thrown once. What is the probability of getting a multiple of 2?

16. If a card is drawn from a well-shuffled pack of 52 cards, then what is the probability of getting either a spade or a diamond?

17. If a dice is thrown once, then find the probability of getting an odd prime number.

18. If one card is drawn from a well-shuffled pack of 52 cards, then what is the probability of getting either a red 6 or a red 8?

19. If two coins are tossed, then find the probability of getting two tails.

20. If three coins are tossed, then find the probability of getting three tails.

21. A bag contains 8 red balls, 4 blue balls and some green balls. The probability of drawing a green ball is the sum of the probabilities of drawing a red ball and a blue ball. Find the number of green balls.

22. If a coin is tossed three times, find the probability that no two successive tosses will show the same face.

23. If two dice are rolled, find the probability that numbers coming up on both the dice will be multiples of 3.

24. Each of two persons a single throw with a dice. What is the probability of getting the same number on both the dice?

25. A cubical die is rolled. Find the probability of getting a composite number.

26. A set contains numbers from 40 to 60. Krishna chooses a number from the set. Find the probability that the number chosen is a prime number.

27. A college offers 8 subjects, including Mathematics, to its intermediate students. Each student has to opt for 3 subjects. If Anand opts for Mathematics, then what is the probability that he will opt for 2 other specified subjects?

28. If n coins are tossed, then find the probability of getting either all heads or all tails.

29. If 8 boys are arranged in a row, what is the probability that 3 particular boys always sit together?

30. If a 5-digit number is formed by using the digits 1, 2, …, 9 (without repetitions), then what is the probability that it will be an even number?

PRACTICE QUESTIONS

Short Answer Type Questions

31. A bag contains 30 balls out of which 15 are red balls, w are white balls and g are green balls. The probability of getting a white ball is two times that of getting a green ball. Find the number of white balls and green balls.

32. A card is selected at random from a pack of cards. What is the probability that it will be an ace?

33. A number is selected from the set of integers from 1 to 20. What is the probability that it will be divisible by both 2 and 3?

34. A number is selected at random from the set of integers 1 to 100. What is the probability that it will be a multiple of 10?

35. What is the probability of arranging the letters of the word CHEMISTRY such that the arrangements start with C and end with Y?

36. When two cards are drawn from a pack of cards, find the probability that both will be diamonds.

37. A number is selected at random from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What is the probability that it will be a root of the equation $x^2 - 2x + 1 = 0$?

38. A speaks truth in 70% cases. What is the probability that A will lie in stating a fact?

39. An urn contains 5 red, 3 blue and 2 green balls. What is the probability of selecting a green or a red ball?

40. A card is selected at random from a pack of cards. What is the probability that it is a numbered card?

41. When a die is rolled, what is the probability of getting a number which is a multiple of both 2 and 3?

42. 50 cards marked with numbers 1 to 50 are placed in a box. If a card is selected randomly from the box, then the probability that the number on the card selected will be a perfect cube is ________.

43. The probability of A winning a game is 0.7. Then the probability of A losing the game is ________.

44. For a cricket team, 11 persons have to be selected out of 20 aspirants. If 6 of the aspirants are definitely selected, and the other are selected at random, then what is the probability that 5 particular aspirants of the remaining 14 will be selected?

45. When two dice are rolled together, what is the probability of getting the same faces on them?

Essay Type Questions

46. What is the probability that date of birth of a person is in the month of January?

47. A card is selected from a pack of cards, what is the probability that it is either spade or diamond?

48. In a six faced dice, two of the faces are painted red, two of the faces are painted black and the other two faces are painted blue. Such two dice are rolled. The probability that both the dice shows same colour is ________.

49. One number is selected from the first 50 natural numbers. What is the probability that it is a root of the inequation?

$$x + \frac{256}{x} > 40?$$

50. A card is drawn from a well shuffled pack of cards, what is the probability that it is either king or heart?

CONCEPT APPLICATION

Level 1

1. What is the probability that a non-leap year has 53 Sundays?

(a) $\frac{6}{7}$ (b) $\frac{1}{7}$

(c) $\frac{5}{7}$ (d) $\frac{2}{7}$

2. A number is selected at random from the integers 1 to 100. What is the probability that it will be a multiple of 4 or 6?

(a) $\frac{8}{25}$ (b) $\frac{33}{100}$ (c) $\frac{17}{50}$ (d) $\frac{41}{100}$

3. Two numbers are selected at once from the set of integers 1 to 20. Find the probability that the product of the numbers will be 24.

(a) $\frac{3}{190}$ (b) $\frac{3}{380}$

(c) $\frac{4}{95}$ (d) $\frac{3}{95}$

4. An urn contains 4 red and 6 green balls. A ball is selected at random from the urn and is replaced back into the urn. Now a ball is drawn again from the bag. What is the probability that the first ball is a red ball and the second is a green ball?

(a) $\frac{6}{25}$ (b) $\frac{8}{25}$

(c) $\frac{5}{6}$ (d) $\frac{4}{15}$

5. A box contains 5 apples, 6 oranges and 'x' bananas. If the probability of selecting an apple from the box is $\frac{1}{3}$, then the number of bananas in the box is ______.

(a) 4 (b) 6

(c) 8 (d) 5

6. Two dice are rolled simultaneously. Find the probability that they show different faces.

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{5}{6}$

7. A number is selected from first 50 natural numbers. What is the probability that it is a multiple of 3 or 5?

(a) $\frac{13}{25}$ (b) $\frac{21}{50}$

(c) $\frac{12}{25}$ (d) $\frac{23}{50}$

8. Two numbers are selected from the set of integers 1 to 25. What is the probability that the product of the numbers will be 36?

(a) $\frac{1}{200}$ (b) $\frac{1}{100}$

(c) $\frac{1}{50}$ (d) $\frac{1}{75}$

9. Tom sold 100 lottery tickets in which 5 tickets carry prizes. If Jerry purchased a ticket, what is the probability of Jerry winning a prize?

(a) $\frac{19}{20}$ (b) $\frac{1}{25}$

(c) $\frac{1}{20}$ (d) $\frac{17}{20}$

10. When two coins are tossed together, what is the probability that neither of them shows up head?

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) 0 (d) $\frac{1}{4}$

11. When two cards are drawn from a well-shuffled pack of cards, what is the probability that both will be aces?

(a) $\frac{1}{221}$ (b) $\frac{2}{221}$

(c) $\frac{1}{13}$ (d) $\frac{1}{231}$

12. Two cards are drawn from a pack of cards one after another so that the first card is replaced before drawing the second card. What is the probability that the first card is an ace and the second is a number card?

(a) $\frac{9}{169}$ (b) $\frac{1}{52}$

(c) $\frac{1}{4}$ (d) $\frac{17}{52}$

13. Two numbers 'a' and 'b' are selected (successively without replacement in that order) from the integers 1 to 10. What is the probability that $\frac{a}{b}$ will be an integer?

(a) $\frac{17}{45}$ (b) $\frac{1}{5}$

(c) $\frac{19}{90}$ (d) $\frac{8}{45}$

14. There are 4 different mathematics books, 5 different physical science books and 3 different biological science books on a shelf. What is the probability of selecting a mathematics book from the shelf?

PRACTICE QUESTIONS

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{5}{12}$

15. A number is selected at random from the integers 1 to 100. What is the probability that it is an even number?

(a) $\frac{1}{2}$ (b) $\frac{1}{50}$

(c) $\frac{1}{3}$ (d) $\frac{49}{100}$

16. A 3-digit number is to be formed using the digits 3, 4, 7, 8 and 2 without repetition. What is the probability that it is an odd number?

(a) $\frac{2}{5}$ (b) $\frac{1}{5}$

(c) $\frac{4}{5}$ (d) $\frac{3}{5}$

17. Two cards are drawn from a pack, what is the probability that the two cards are of different colours?

(a) $\frac{8}{17}$ (b) $\frac{1}{12}$

(c) $\frac{26}{51}$ (d) $\frac{27}{52}$

18. Two numbers '*a*' and '*b*' are selected successively one after another without replacement from the integers 1 to 15. What is the probability that $\frac{a}{b}$ will be an integer?

(a) $\frac{5}{7}$ (b) $\frac{3}{7}$

(c) $\frac{2}{7}$ (d) $\frac{1}{7}$

19. A card is drawn at random from a pack of cards. What is the probability that it is a face card of spades?

(a) $\frac{1}{13}$ (b) $\frac{1}{26}$

(c) $\frac{3}{52}$ (d) $\frac{3}{13}$

20. Two numbers are selected simultaneously from the first 20 natural numbers. What is the probability that the sum of the numbers is odd?

(a) $\frac{10}{19}$ (b) $\frac{1}{19}$

(c) $\frac{1}{10}$ (d) $\frac{19}{20}$

21. Two dice are rolled together, what is the probability that the total score on the two dice is greater than 10?

(a) $\frac{5}{6}$ (b) $\frac{1}{4}$

(c) $\frac{1}{6}$ (d) $\frac{1}{12}$

22. Two numbers are selected simultaneously from the first 25 natural numbers, what is the probability that the sum of the numbers is even?

(a) $\frac{6}{25}$ (b) $\frac{12}{25}$

(c) $\frac{4}{25}$ (d) $\frac{8}{25}$

23. A bag contains 7 red and 7 black coloured balls. A person drawn two balls from the bag, what is the probability that the two balls are the same in colour?

(a) $\frac{6}{13}$ (b) $\frac{2}{7}$

(c) $\frac{4}{13}$ (d) $\frac{1}{13}$

24. Two dice are rolled together, what is the probability that the total score on the two dice is a prime number?

(a) $\frac{5}{12}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{13}{36}$

25. A committee of 4 persons is to be formed from 6 men and 4 women. What is the probability that the committee consists of equal number of men and women?

(a) $\frac{4}{7}$ (b) $\frac{3}{7}$

(c) $\frac{1}{7}$ (d) $\frac{6}{7}$

26. All the cards in an ordinary deck of 52 cards are numbered from 1 to 52. If a card is drawn at

random from the deck, then what is the probability that it will have a prime number?

(a) $\frac{7}{26}$ (b) $\frac{15}{52}$

(c) $\frac{5}{17}$ (d) $\frac{4}{13}$

27. A bag contains 5 pens and 6 pencils. If a boy selects 2 articles from the bag, then what is the probability that the selected articles will be a pen and a pencil?

(a) $\frac{2}{11}$ (b) $\frac{3}{11}$

(c) $\frac{6}{11}$ (d) $\frac{5}{11}$

28. An urn contains 5 red, 3 black and 2 white. If three balls are chosen at random, then what is the probability that they will be of different colours?

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$

(c) $\frac{1}{2}$ (d) $\frac{5}{6}$

29. What is the probability that a leap year has 52 Mondays?

(a) $\frac{2}{7}$ (b) $\frac{4}{7}$

(c) $\frac{5}{7}$ (d) $\frac{6}{7}$

30. If a letter is selected at random from the letters of the word FOUNDATION, what is the probability of the letter selected being a repeated letter?

(a) $\frac{1}{10}$ (b) $\frac{1}{5}$

(c) $\frac{2}{5}$ (d) $\frac{1}{2}$

Level 2

31. Sunny and Bunny go to an icecream parlour where they find 5 different varieties of icecreams. If they order one icecream each, what is the probability that they both order the same variety of icecreams?

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$

(c) $\frac{1}{2}$ (d) $\frac{1}{10}$

32. There are five ₹1 coins, two ₹2 coins and three ₹5 coins. If two coins are selected simultaneously at random, what is the probability of yielding the maximum amount?

(a) $\frac{3}{10}$ (b) $\frac{1}{5}$

(c) $\frac{1}{15}$ (d) $\frac{3}{10}$

33. A dice is rolled twice, what is the probability that the two dice show a different number?

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$

(c) $\frac{5}{6}$ (d) $\frac{1}{2}$

34. A bag contains 3 red, 5 blue and 7 green coloured balls. Find the probability of selecting a blue ball from the bag.

(a) $\frac{3}{15}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

35. 7 coins are tossed simultaneously, what is the probability of getting at least two heads?

(a) $\frac{3}{18}$ (b) $\frac{15}{16}$

(c) $\frac{1}{16}$ (d) $\frac{3}{16}$

36. A bag contains 5 white balls, and 6 green balls. Two balls are drawn from the bag one after another, what is the probability that both the balls are white? (the first ball is replaced before drawing the second ball)

(a) $\frac{2}{11}$ (b) $\frac{1}{11}$

(c) $\frac{3}{121}$ (d) $\frac{25}{121}$

37. A basket contains 12 fruits, of which 7 are not rotten. When one fruit is drawn at random, the probability that it is a rotten fruit is ______.

(a) $\frac{1}{3}$ (b) $\frac{7}{12}$

(c) $\frac{5}{12}$ (d) $\frac{2}{3}$

38. One card is selected from a well shuffled pack of cards, what is the probability that it is a red honored card?

(a) $\frac{2}{13}$ (b) $\frac{3}{13}$

(c) $\frac{11}{13}$ (d) $\frac{9}{13}$

39. A bag contains 5 black balls, and 7 green balls. Two balls are drawn simultaneously at random, what is the probability that both are different in colour?

(a) $\frac{17}{33}$ (b) $\frac{35}{132}$

(c) $\frac{5}{66}$ (d) $\frac{35}{66}$

40. The probability that the month of April has exactly 5 Mondays is ______.

(a) $\frac{4}{7}$ (b) $\frac{5}{7}$

(c) $\frac{3}{7}$ (d) $\frac{2}{7}$

41. Hundred cards marked with numbers 1 to 100 are placed in a box. If a card is selected randomly from the box, then the probability that the number on the card selected will be a perfect square is ______.

(a) $\frac{1}{100}$ (b) $\frac{1}{25}$

(c) $\frac{1}{10}$ (d) $\frac{9}{10}$

42. 20 cards are numbered 1 to 20. If a card is selected at random, then what is the probability that the selected number will be an odd prime?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$

(c) $\frac{3}{10}$ (d) $\frac{7}{20}$

43. A 4-digit number is formed by using the digits 1, 2, 5, 6 and 8 without repetition. What is the probability that it will be an even number?

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$

(c) $\frac{1}{2}$ (d) $\frac{3}{10}$

44. A basket contains 10 fruits of which 3 are rotten. If one fruit is drawn from the basket, then the probability that the fruit drawn is not rotten is ______.

(a) $\frac{4}{5}$ (b) $\frac{4}{5}$

(c) $\frac{7}{10}$ (d) $\frac{3}{10}$

45. An urn contains 6 blue and '*a*' green balls. If the probability of drawing a green ball is double that of drawing a blue ball, then '*a*' is equal to ______.

(a) 6 (b) 18

(c) 24 (d) 12

46. A card is drawn from a well shuffled pack of cards. What is the probability that it will be a black queen?

(a) $\frac{1}{52}$ (b) $\frac{1}{13}$

(c) $\frac{1}{26}$ (d) $\frac{1}{32}$

47. A box contains 30 balls. Among them 10 are black, 12 are blue and the rest are orange. What is the probability that a ball drawn from the box will not be blue?

(a) $\frac{1}{3}$ (b) $\frac{3}{5}$

(c) $\frac{5}{6}$ (d) $\frac{1}{6}$

48. When an unbiased dice is thrown, the probability of getting a prime number is ______.

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

49. If a letter is selected at random from the letters of the word TRIANGLE, then what is the probability that it will be a consonant?

(a) $\frac{2}{7}$ (b) $\frac{3}{8}$

(c) $\frac{5}{8}$ (d) $\frac{1}{8}$

50. If a letter is selected at random from the letters of the word LOGARITHMS, then what is the probability that it will be a consonant?

(a) $\frac{3}{10}$ (b) $\frac{7}{10}$

(c) $\frac{1}{10}$ (d) $\frac{4}{10}$

Level 3

51. If one number is selected from the first 70 natural numbers, the probability that the number is a solution of $x^2 + 2x > 4$ is ______.

(a) $\frac{69}{70}$ (b) $\frac{1}{70}$

(c) 1 (d) 0

52. A man's pocket has seven ₹1 coins, three ₹2 coins and four ₹5 coins. If two coins are selected simultaneously, what is the probability of yielding the minimum amount?

(a) $\frac{3}{13}$ (b) $\frac{6}{13}$

(c) $\frac{3}{26}$ (d) $\frac{6}{43}$

53. Mr Balaram picked a prime number between the integers 1 and 20. What is the probability that it will be number 13?

(a) $\frac{7}{8}$ (b) $\frac{1}{20}$

(c) $\frac{1}{8}$ (d) $\frac{13}{20}$

54. Chandu picks up a letter from the English alphabet and finds it to be a vowel. Find the probability that it is either e or i.

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$

(c) $\frac{1}{5}$ (d) $\frac{4}{5}$

55. A committee of 5 persons is to be formed from 7 men and 3 women. What is the probability that the committee contains 3 men?

(a) $\frac{5}{36}$ (b) $\frac{7}{12}$

(c) $\frac{5}{12}$ (d) $\frac{1}{3}$

56. A purse contains four fifty-paise coins, three two-rupee coins and three five-rupee coins. If three coins are selected at random, then what is the probability of getting the minimum amount?

(a) $\frac{1}{15}$ (b) $\frac{1}{10}$

(c) $\frac{1}{30}$ (d) $\frac{1}{5}$

57. A number is selected from the set {1, 2, 3, 4, 5, 6, 7, 8}. What is the probability that it will be a root of the equation $x^2 - 6x + 8 = 0$?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{3}{4}$ (d) $\frac{1}{4}$

58. An urn contains 9 red balls and p green balls. If the probability of picking a red ball is thrice that of picking a green ball, then p is equal to ______.

(a) 6 (b) 7

(c) 2 (d) 3

59. If two dice are thrown, then find the probability that the total score on the two dice is 5.

(a) $\frac{2}{9}$ (b) $\frac{3}{8}$

(c) $\frac{1}{6}$ (d) $\frac{1}{9}$

60. A bag contains 8 balls numbered 1 to 8. If 2 balls are picked at random, then find the probability of the two balls being 2 and 3.

(a) $\frac{1}{28}$ (b) $\frac{2}{27}$

(c) $\frac{1}{14}$ (d) $\frac{1}{7}$

61. A 4-digit number is formed by using the digits 1, 2, 4, 8 and 9 without repetition. If one number

PRACTICE QUESTIONS

is selected from those numbers, then what is the probability that it will be an odd number?

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

62. If two dice are rolled together, then find the probability that the sum of the numbers on the two dice is less than or equal to 10.

(a) $\frac{11}{12}$ (b) $\frac{1}{12}$

(c) $\frac{7}{12}$ (d) $\frac{1}{9}$

63. Three numbers are chosen from 1 to 15. Find the probability that they are consecutive.

(a) $\frac{1}{35}$ (b) $\frac{1}{13}$

(c) $\frac{2}{13}$ (d) $\frac{3}{35}$

64. One ticket is drawn from a bag containing 70 tickets numbered 1 to 70. Find the probability that it is a multiple of 5 or 7.

(a) $\frac{1}{10}$ (b) $\frac{1}{70}$

(c) $\frac{6}{70}$ (d) $\frac{11}{35}$

65. If a coin is tossed two times, then what is the probability of getting a head at least once?

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$

(c) $\frac{1}{2}$ (d) 1

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. random experiment
2. mutually exclusive
3. impossible event
4. simple or equally likely
5. 1
6. sample space
7. sure event
8. 1
9. impossible event
10. $1 - P(A)$
11. $\frac{1}{3}$
12. [0, 1]
13. complement
14. $\frac{1}{6}$
15. $\frac{1}{2}$
16. $\frac{1}{2}$
17. $\frac{1}{3}$
18. $\frac{1}{13}$
19. $\frac{1}{4}$
20. $\frac{1}{8}$
21. 12
22. $\frac{1}{4}$
23. $\frac{1}{9}$
24. $\frac{1}{6}$
25. $\frac{1}{3}$
26. $\frac{5}{21}$
27. $\frac{{}^7C_2}{{}^8C_3}$
28. $\frac{1}{2^n - 1}$
29. $\frac{3}{28}$
30. $\frac{4}{9}$

Shot Answer Type Questions

31. $\frac{1}{3}$
32. $\frac{1}{13}$
33. $\frac{3}{20}$
34. $\frac{1}{10}$
35. $\frac{1}{72}$
36. $\frac{1}{17}$
37. $\frac{1}{10}$
38. $\frac{3}{10}$
39. $\frac{7}{10}$
40. $\frac{9}{13}$

ANSWER KEYS

41. $\frac{1}{6}$

42. $\frac{3}{50}$

43. 0.3

44. $\frac{1}{2002}$

45. $\frac{1}{6}$

Essay Type Questions

46. $\frac{1}{12}$

47. $\frac{1}{2}$

48. $\frac{1}{3}$

49. $\frac{1}{2}$

50. $\frac{4}{13}$

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (d)
11. (a)	**12.** (a)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (d)	**19.** (c)	**20.** (a)
21. (d)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (c)	**28.** (a)	**29.** (c)	**30.** (b)

Level 2

31. (a)	**32.** (c)	**33.** (c)	**34.** (b)	**35.** (b)	**36.** (d)	**37.** (c)	**38.** (a)	**39.** (d)	**40.** (d)
41. (c)	**42.** (d)	**43.** (a)	**44.** (c)	**45.** (d)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (c)	**50.** (b)

Level 3

51. (a)	**52.** (a)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (a)
61. (b)	**62.** (a)	**63.** (a)	**64.** (d)	**65.** (b)					

CONCEPT APPLICATION

Level 1

1. (i) A non-leap year contains 365 days and there is only one odd day.
   (ii) 52 weeks contains 52 Sundays. Find the probability that odd day is Sunday.
2. P(either multiple of 4 or 6) = P(multiple of 4) + P(multiple of 6) − P(multiple of both 4 and 6).
3. (i) Find the number of pairs a and b such that $a \times b = 24$ and $1 \le a, b \le 20$.
   (ii) Probability of the required event $= \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$.
4. $P(R \text{ and } G) = P(R) \cdot P(G)$.
5. $P(\text{selection apple}) = \dfrac{5}{5+6+x} = \dfrac{1}{3}$, and find x.
6. (i) Find the probability that they show the same faces.
   (ii) P(Showing different faces) = 1 − P(showing same faces).
7. P(either multiple of 3 or of 5) = P(multiple of 3) + P(multiple of 5) − P(multiple 3 and 5).
8. (i) Find the number of pairs (a, b), such that $a \times b = 36$ and $1 \le (a, b) \le 25$.
   (ii) Total number of outcomes = ${}^{25}C_2$.
9. Find the number of favourable outcomes.
10. Both coins should show up tail.
11. There are 4 ace cards in pack of cards.
12. (i) There are 4 ace cards and 36 numbered cards.
    (ii) P(ace and number card) = P(ace card) × P(number card).
13. $b = 1$, $a = 2, 3, \ldots, 10$.
    $b = 2$, $a = 4, 6, \ldots, 10$.
    In this way, find the number of favourable events such that a/b is an in integer.
14. (i) Number of favourable outcomes = 4.
    (ii) Required probability $= \dfrac{\text{Number of mathematics books}}{\text{Total number of books in the shelf}}$.
15. There are 50 even numbers from 1 to 100.
16. (i) If a number ends with an odd number then it is odd number.
    (ii) Find the number of 3-digit odd numbers formed by using the digits 3, 4, 7, 8 and 2.
    (iii) Find the total number of 3-digit numbers formed by using the digits 3, 4, 7, 8 and 2.
17. (i) There are 26 red colour and 26 black colour cards.
    (ii) Select one red card and one black card.
    (iii) Total number of outcomes = ${}^{52}C_2$
18. Let $b = 1$, then $a = 2$ to 15.
    Let $b = 2$, then $a = 4, 6, \ldots, 14$.
    In this way, find numbers a and b such that $\dfrac{a}{b}$ is the integer.
19. Jack, Queen and King are the face cards.
20. (i) One number is odd and other is even then sum is odd.
    (ii) Two numbers from 20 can be selected in ${}^{20}C_2$ ways.
    (iii) There are 10 even numbers and 10 odd numbers.
21. P(greater than 10) = P(sum be 11) + P(sum be 12).
22. (i) (Odd + Odd) = (Even + Even) = Even.
    (ii) There are a total of 13 odd numbers and 12 even numbers.
    (iii) P(sum even) = P(selecting 2 odd numbers) + P(selecting 2 even numbers).
23. (i) Number of ways of drawing 2 red balls = ${}^{7}C_2$, similarly number of ways of drawing 2 black-balls = ${}^{7}C_2$.
    (ii) Total number of ways of drawing 2 balls out of 14 balls = ${}^{14}C_2$.
24. P(sum is prime) = P(sum 2) + P(sum 3) + P(sum 5) + P(sum 7) + P(sum 11).
25. (i) Committee should be formed with 2 men and 2 women.
    (ii) Total number of ways in which the committee be selected is ${}^{(6+4)}C_4$ ways.
26. (i) Find the numbers which are prime from 1 to 52.
    (ii) In total we have to select 1 card out of 52 cards.

HINTS AND EXPLANATION

27. (i) Find the number of ways of selecting one pen and one pencil from the bag.
(ii) In total, the boy has to select 2 out of 11 articles.

28. (i) Choose one ball from each colour.
(ii) In total, we have to select 3 out of 10 balls.

29. (i) Leap year contains 52 weeks and 2 odd days.
(ii) Find the probability that the odd days should not be Monday.

30. Find the number of repeated letters in the given word.

Level 2

31. (i) Let the five different icecreams be a, b, c, d and e.
(ii) Favourable outcomes are (a, a), (b, b), (c, c), (d, d) and (e, e).
(iii) Total number of outcomes = 25.

32. (i) Maximum amount will be yielded, if the selected two coins are of ₹5.
(ii) Required probability

$$= \frac{\text{Number of ways of selecting two ₹5 coins}}{\text{Total number of ways of selecting two coins}}.$$

41. Find the number of favourable outcomes.

42. (i) Find the numbers which are odd primes from 1 to 20.
(ii) In total, we have to select 1 card out of 20.

43. (i) Find the number of 4-digit even numbers by filling the units digit by 2, 6 and 8.
(ii) Find the total number of 4 digit numbers. Required probability

$$= \frac{\text{Number of 4-digit even numbers}}{\text{Total number of 4-digit numbers}}.$$

44. (i) To yield minimum amount, select ₹1 coins.
(ii) Required probability

$$= \frac{\text{Number of ways of selecting two ₹1 coins}}{\text{Total number of ways of selecting two coins}}.$$

45. (i) Total number of balls in the urn = $6 + a$.
(ii) P(Drawing a green ball) = $2P$(Drawing a blue ball).
(iii) Using above condition find the value of a.

46. The total number of cards is 52.

There are two queen cards which are black.

Required probability $= \frac{2}{52} = \frac{1}{26}$.

47. The total number of balls is 30, of them 10 are black, 12 are blue and 8 are orange.

There are 18 balls, which are not blue.

The required probability $= \frac{18}{30} = \frac{3}{5}$.

48. The sample space = $\{1, 2, 3, 4, 5, 6\}$.

The favourable outcomes = $\{2, 3, 5\}$.

Required probability $= \frac{3}{6} = \frac{1}{2}$.

49. There are 8 letters in the word TRIANGLE, of these 5 are consonants.

$\therefore$ Required probability $= \frac{5}{8}$.

50. The total number of letters is 10.

The number of consonants = 7

$\therefore$ Required probability $= \frac{7}{10}$.

Level 3

51. P(blue ball) =
$$\frac{\text{Number of blue balls in the bag}}{\text{Total number of balls in the bag}}.$$

53. (i) Number of favourable events = 1.
(ii) There are 8 prime numbers from 1 to 20.

54. (i) P(either e or i) = $P(e) + P(i)$.
(ii) Total outcomes = 5, i.e., (a, e, i, o, u).

55. (i) Committee should consists of 3 men and 2 women.
(ii) P(committee of 5 persons) = Number of ways of selecting 3 men and 2 women to the number of ways of selecting 5 persons.

56. Total number of coins in the purse = 4 + 3 + 3 = 10.

The total number of ways of selecting 3 coins from 10 coins is ${}^{10}C_3 = 120$.

To get the minimum amount, these three coins must be 50 paise coins. This can be done in ${}^4C_3$ ways = 4 ways.

$\therefore$ Required probability $= \frac{4}{120} = \frac{1}{30}$.

57. There are 8 numbers.

The roots of the equation $x^2 - 6x + 8 = 0$ are 2 and 4.

$\therefore$ The number of favourable outcomes is 2.

$\therefore$ The required probability $= \frac{2}{8} = \frac{1}{4}$.

58. Probability of picking a red ball $= \frac{9}{9+p}$

Probability of picking a green ball $= \frac{p}{9+p}$

Given, $\frac{9}{9+p} = \frac{3p}{9+p}$

$\Rightarrow 3p = 9 \Rightarrow p = 3$.

59. If two dice are rolled together, then the total number of outcomes = $6^2 = 36$.

Favourable outcomes that the sum of the numbers is 5: (4, 1), (1, 4), (2, 3), (3, 2).

$\therefore$ The required probability $= \frac{4}{36} = \frac{1}{9}$.

60. The total number of ways of selecting two balls from 8 balls = ${}^8C_2 = 28$.

Number of favourable cases that one ball is 2 and other is 3, is 1.

Required probability $= \frac{1}{28}$.

61. Total number of outcomes = $5 \times 4 \times 3 \times 2 = 120$

The number of favourable cases = $2(4 \times 3 \times 2)$ = 48 (i.e., odd numbers)

The required probability $= \frac{48}{120} = \frac{2}{5}$.

62. If two dice are rolled together, then total number of outcomes = $6^2 = 36$.

Required probability = 1 – (the probability of getting the sum of the numbers on the two dice is greater than 10)

Favourable cases to get the sum greater than 10: (5, 6), (6, 5) and (6, 6).

Probability of getting the sum of the numbers on the two dice is greater than $10 = \frac{3}{36} = \frac{1}{12}$.

The required probability $= 1 - \frac{1}{12} = \frac{11}{12}$.

63. The total number of ways of selecting three numbers from 15 numbers is = ${}^{15}C_3 = 455$.

The favourable outcomes are

(1, 2, 3), (2, 3, 4)… (13, 14, 15).

The required probability $= \frac{13}{455} = \frac{1}{35}$.

64. Total number of tickets are 70.

Multiples of 5 are 5, 10, 15, …, 70.

There are 14 numbers, which are multiples of 5.

Multiples of 7 are 7, 14, …, 70, i.e.,

There are 10 numbers, which are multiples of 7.

There are two numbers, which are multiple of both 7 and 5. Total number of numbers which are multiples of 5 or 7 = 14 + 10 – 2.

The required probability $= \frac{22}{70} = \frac{11}{35}$.

65. If two coins are tossed, then the total outcomes: {HH, HT, TH, TT}

The favourable outcomes of getting at least one head: {HH, HT, TH}

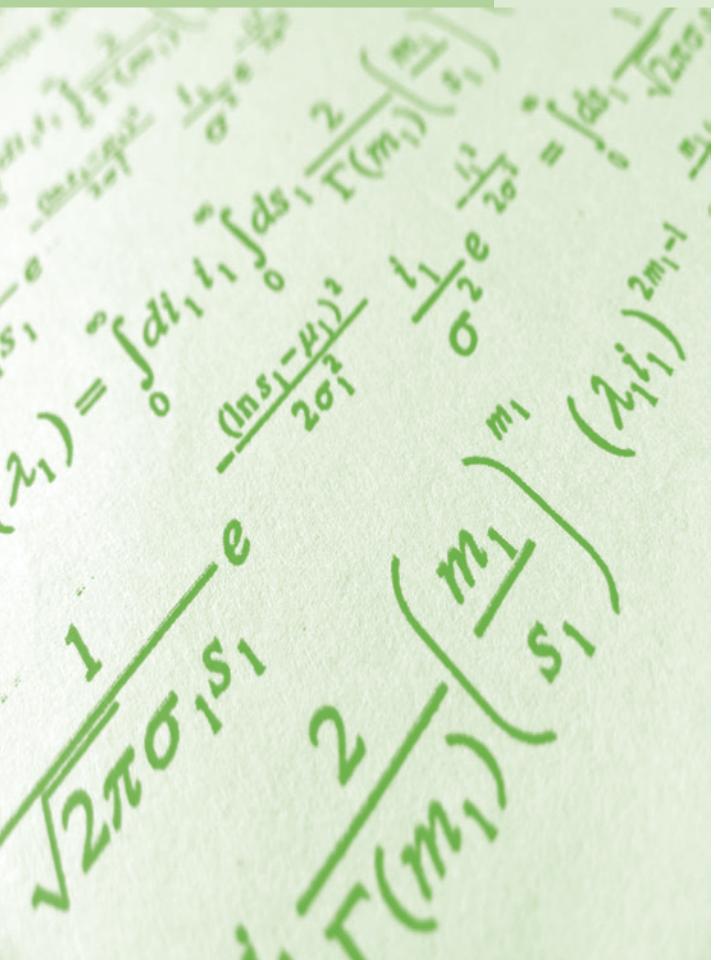
$\therefore$ The required probability $= \frac{3}{4}$.

HINTS AND EXPLANATION

Chapter

22

Banking



REMEMBER

Before beginning this chapter, you should be able to:

- Recall remittance of funds, safe deposit lockers, public utility services.

KEY IDEAS

After completing this chapter, you would be able to:

- Understand deposit accounts—savings bank accounts, current accounts and term deposit accounts and processes related.
- Study about types of cheques and parties dealing with cheques.
- Do calculation of interest on loans.
- Calculate interests on savings accounts in banks.
- Study hire purchase and find the rate of interest on buying in instalment scheme.

INTRODUCTION

Before people began using money, purchase and sale of goods used to be performed through exchange of goods. This system is called 'Barter system'. As there was no uniformity in the valuation of various goods under this system, the values of all goods were converted in terms of money, thus ensuring that a proper value is paid for the goods purchased.

Once the monetary system became a standard method of value exchange, the necessity to ensure the safety of money came into existence. With an intention to safeguard money, and also to facilitate availability of money to everyone in the society, gradually the banking system developed.

Among the various services offered by the banks, taking deposits and providing loans are the basic ones. Apart from these, banks render the following ancillary services:

1. **Remittance of funds:** Banks help in transferring money from one place to another in a safe manner, through the issue of demand drafts, money transfer orders, and telegraphic transfers. Banks also issue travellers cheques to travellers in home currency as well as in foreign currency, by which travellers can minimize the risk of theft or loss of money while travelling to a different place. These travellers' cheques can be easily converted into money. With the advent of technology, nowadays, money transfer has become simple through Internet and phone banking.
2. **Safe deposit lockers:** Banks provide safety lockers to customers to keep safely preserve their valuables. A customer can store their valuables, like gold ornaments, important documents in bank lockers by paying a small amount of rent charged by the bank.
3. **Public utility services:** Through bank accounts, the customers can pay their telephone bills, electricity bills, insurance premium, and other services.

DEPOSIT ACCOUNTS

Deposit accounts offered by the banks are designed in such a way that they cater to various needs of the customers. These come according to their financial capabilities of the customers. At present, the following types of deposit accounts are offered by the banks:

Savings Bank Account

An Indian individual, either resident or non-resident, can open a savings bank account with a minimum balance of ₹500. The minimum balance may vary from bank to bank. A passbook is issued to the customer. It contains all the particulars of the transactions and the balance. Such an account can be opened in joint names also. It is known as a 'joint account'. If one of the joint account holders is a minor, the following guidelines are applicable: A minor who is at least ten-year old can open an account in a bank or a post office. However, the minimum age to open an account and to operate an account differs from bank to bank and post office. In a post office, the minimum age to open and operate a savings account is 10 years. If the minor's minimum age to operate an account is less than his/her minimum age to open an savings account, a guardian can operate the minor's account. A savings bank accounts carry a certain amount of interest compounded half-yearly. The rate of interest varies from bank to bank. It may also vary from time to time. Cheque books are issued to an account-holder against a requisition slip duly filled up and signed by the person. If a customer operates his/her account through cheques, then it is known as **'cheque-operated account'**.

Depositing Money in the Bank Accounts

Money can be deposited in a bank either by cash or through a duly filled pay-in-slip or challan. Pay-in-slips can be used for payment through cash or cheque.

Demand Draft

Money can be deposited through demand drafts (i.e., bank drafts). A person who wants to send money to another person can purchase a bank draft.

A bank draft is an order issued by a bank to its specified branch or to another bank (if there is a tie-up) to make payment of the amount to the party, in whose name the draft is issued.

The purchaser of the draft specifies the name and address of the person to whom the money is being sent, which is written on the bank draft. The payee can encash the drafts by presenting it at the specified branch or bank.

Withdrawing Money From Saving Bank Account

Money deposited in these accounts can be withdrawn by using withdrawal slips or cheques. A specimen of a cheque is given below.

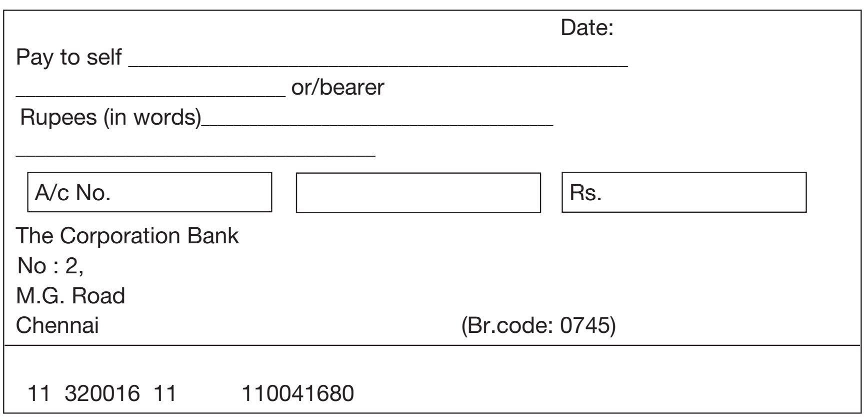


Figure 22.1

Cheque books are issued only to those account-holders who fulfill certain special requirements, such as maintenance of minimum balance, updated information related to the account.

Types of Cheques

A cheque can be classified into two types. These are as follows:

Bearer Cheque

A bearer cheque can be encashed by anyone who possesses the cheque, though the person's name is not written on the cheque. There is a risk of wrong a person getting the payments.

If the word **'bearer'** is crossed-out in the cheque, then the person whose name appears on the cheque can alone encash the cheque. This type of cheque is known as an **'order cheque'**.

Crossed Cheque

If two parallel lines are drawn at the left-hand top corner of a cheque, it is called a **'crossed cheque'**. The words **'A/C payee'** may or may not be written between the two parallel lines. The payee has to deposit the crossed cheque in his/her account. The collecting bank collects the money from the drawer's bank, and it is credited to the payee's account.

Bouncing of Cheques

If an account-holder issues a cheque for an amount exceeding the balance in his account, the bank refuses to make payment. In such an instance, the cheque is said to be **dishonoured** one. This is known as **bouncing of cheque**. If a cheque bounces, the issuer of the cheque is liable for prosecution under the **Negotiable Instruments Act, 1887**.

Safeguards to be taken while Maintaining 'Cheque-Operated Accounts'

1. Immediately after receiving a cheque book, a customer verifies if all the leaves are serially arranged and printed with correct numbers.
2. Blank cheques should not be issued to anybody except the account holder.
3. Any changes, alterations, corrections made while filling a cheque, should be authenticated with full signature.
4. The amount on a cheque has to be written in words and figures legibly.
5. The amount of the cheque should be written immediately after the printed words **'Rupees'** or '₹.' Also, the word **'only'** should be mentioned after the amount in words.
6. A cheque becomes outdated or stale after six months from the date of issue. Hence, it should be presented within six months from the date of issue.

Parties dealing with a cheque

Drawer

The account-holder who writes the cheques and signs on it in order to withdraw money is called 'drawer' of the cheque.

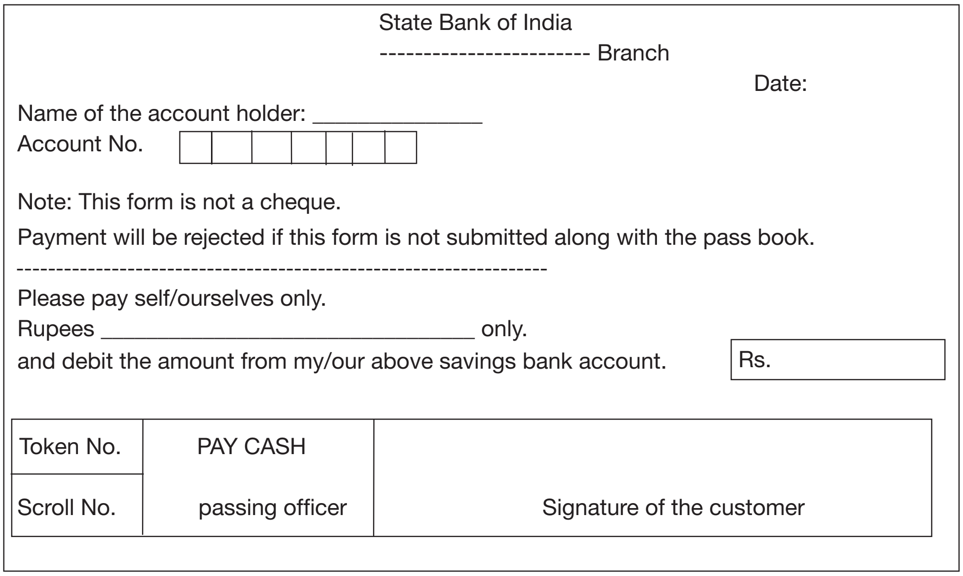
Drawee

The Bank on whom the cheque is drawn is called the 'drawee bank' as they pay the money.

Payee

The party to whom the amount of cheque is payable is called the 'payee'. The payee has to affix his/her signature on the back of the cheque.

Any savings bank account-holder can withdraw money from his/her account using a withdrawal form, a specimen of which is given below:



State Bank of India

------------------------ Branch

Date:

Name of the account holder: ______________

Account No.

Note: This form is not a cheque.

Payment will be rejected if this form is not submitted along with the pass book.

Please pay self/ourselves only.

Rupees ______________________________ only.

and debit the amount from my/our above savings bank account. Rs.

Token No.	PAY CASH	
Scroll No.	passing officer	Signature of the customer

Figure 22.2

Banks impose restrictions on the number of times of withdrawal of money from savings bank accounts. A violation of such restriction attracts a nominal charge. The interest on savings bank accounts are paid half-yearly by taking the minimum balance for each month as the balance for that entire month. **'Minimum balance'** is the least of all the balances left in the account from the 10th to the last day of that month.

Example: The following table shows the particular of the closing balances of a savings account during the month of March, 2006.

Date	Closing Balance
5-3-2006	₹1800.00
10-3-2006	₹2400.00
18-3-2006	₹3500.00
25-3-2006	₹1700.00
31-3-2006	₹2500.00

From the given table, it can be observe that the closing balance on 25th March, i.e., ₹1700, is the minimum closing balance between 10th of March and the last day of March. This is the minimum balance for the month of March.

The monthly minimum balances for every six months are calculated, and on this the interest for six months is calculated. Most of the banks add the interest to the existing balance once in every half year. That is, on June 30th and December 31st.

However, the periodicity of interest calculation differs between banks to post offices.

Calculation of Interest on Savings Accounts in Banks

The monthly minimum balances from January to the end of June are added. This total amount is called the **'product'** in banks. Interest is calculated on this product and added to the opening

balance on July 1st. In the same manner, the interest for the next half year is calculated and added to the opening balance on January 1st.

In savings account, interest is calculated by maintaining the following steps:

1. The least of the balances from the 10th day of a month to the last day of the month is considered as the balance for the month.
2. The sum of all these monthly balances is considered as the product for calculating interest.
3. Interest $= \dfrac{\text{Product} \times \text{Rate of Interest}}{12 \times 100}$

EXAMPLE 22.1

The following is an extract of the savings bank pass book of Mrinalini who has an account with Corporation Bank.

Calculate the interest accrued on the account at the end of June, 2005 at 5% per annum.

Date	Particulars	Amount Withdrawn		Amount Deposited		Balances	
		₹	P	₹	P	₹	P
7-1-2005	Balance B/F					8400	00
10-1-2005	By cash			12,500	00	20,900	00
31-1-2005	To cheque No. 3541	6500	00			14,400	00
15-2-2005	By cash			3500	00	17,900	00
13-3-2005	To cheque No. 3543	2800	00			15,100	00
25-3-2005	By cheque			2000	00	17,100	00
3-4-2005	To cheque No. 3544	1400	00			15,700	00
18-4-2005	To cheque No. 3545	3500	00			12,200	00
21-5-2005	By cash			5400	00	17,600	00
15-6-2005	To cheque No. 3546	6000	00			11,600	00
21-6-2005	To cheque No. 3547	2000	00			9600	00
15-7-2005	By cash			3500	00	13,100	00

SOLUTION

The minimum balance in rupees

for January = 14,400
February = 14,400
March = 15,100
April = 12,200
May = 12,200
June = 9600

77,900

The product is ₹77,900.

$\therefore$ Interest $= \dfrac{\text{Product} \times \text{Rate}}{100 \times 12} = \dfrac{77900 \times 5}{100 \times 12} =$ ₹324.58

Current Account

This account is very convenient for business people, companies, government offices, and various other institutes which need to make frequent and large amounts of monetary transactions. Banks do not give any interest on these accounts, but the operation of these accounts is flexible. There is no restriction on amounts deposited or withdrawn (i.e., on the number of transactions) as the savings bank accounts.

Term Deposit Accounts

These accounts are of two types:

1. Fixed deposit accounts
2. Recurring deposit accounts

Fixed Deposit Accounts

Customers can avail the facility of depositing a fixed amount for a definite period of time. As the time period is fixed, banks give a higher rate of interest on these accounts. If money is withdrawn from these accounts before the fixed time period, banks pay lesser interest than what was agreed upon. As this discourages premature withdrawal, banks rely more on these funds. The rate of interest payable varies with the period for which the money is deposited in these accounts, and it varies from bank to bank. The rates of interest offered by a bank on fixed deposits are as follows:

Time period	Rate of Interest (%) per annum
15 days and upto 45 days	5.25
46 days and upto 179 days	6.50
180 days to, less than 1 year	6.75
1 year to less than 2 years	8.00
2 years to less than 3 years	8.25
3 years and above	8.50

Recurring Deposit Accounts

These accounts help customers to build up large amounts through small deposits. These accounts facilitate depositing a fixed amount per month for a time span of 6 months to 3 years, and above. This time period is called the **maturity period**.

The following table gives an idea about how the principal amounts are taken to calculate the interest in recurring deposit accounts.

Date	Deposit	Principal on which interest is to be paid
1-4-2006	₹3000	₹3000
1-5-2006	₹3000	₹6000
1-6-2006	₹3000	₹9000

Recurring deposit accounts are helpful to those who have low earnings. They can save large amounts through regular and fixed savings. A person who opens this account deposits an initially agreed amount each month. At the end of the maturity period, the cumulative amount with interest, which is called the 'maturity amount', is paid to the account-holder. The rates of interest payable on these accounts are same as those payable on fixed deposit accounts.

The recurring deposit interest is calculated by applying the following formula.

We know that, $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$

If a man deposits ₹k per month, for n months at r% per annum, then simple interest $=₹\left[k\times\dfrac{n(n+1)}{2}\times\dfrac{1}{12}\times\dfrac{R}{100}\right]$.

EXAMPLE 22.2

Govind opened a bank account on 1-4-2006 by depositing ₹3000. He deposited ₹1000 on 11-4-2006 and withdrew ₹500 on 15-4-2006. Compute the interest paid by the bank for the month of April, if the rate of interest is 4% per annum.

SOLUTION

Balance as on 1-4-2006 = ₹3000

as on 11-4-2006 = ₹4000

as on 15-4-2006 = ₹3500

The minimum balance for the month of April = ₹3000

$\therefore$ The interest paid by the bank for the month of April $=\dfrac{3000\times1\times4}{1200}=₹10.$

EXAMPLE 22.3

Rajan makes fixed deposit of ₹8000 in a bank for a period of 2 years. If the rate of interest is 10% per annum compounded annually, find the amount payable to him by the bank after two years.

SOLUTION

The amount of fixed deposit = ₹8000.

R = 10% per annum and n = 2.

$\therefore$ The amount returned by the bank $=P\left[1+\dfrac{r}{100}\right]^n$

$$=8000\left[1+\frac{10}{100}\right]^2$$

$$=8000\,[1+0.1]^2=8000\times1.21=₹9680.$$

EXAMPLE 22.4

Mahesh deposits ₹600 per month in a recurring deposit account for 2 years at 5% per annum. Find the amount he receives at the time of maturity.

SOLUTION

Here, $P = ₹600$,

$N = 2 \times 12$ months and $R = 5\%$ per annum

$$\text{SI} = P \times \frac{n(n+1)}{2} \times \frac{1}{12} \times \frac{R}{100} = 600 \times \frac{24(25)}{2} \times \frac{1}{12} \times \frac{5}{100} = 750$$

$\therefore$ The total amount = (24 × 600) + 750 = 14400 + 750 = ₹15,150.

Loans

The loans given by the banks can be classified into the following three types:

1. Demand loans
2. Term loans
3. Overdrafts (ODs)

Demand Loans

The borrower has to repay the loans on demand. The repayment of the loan has to be done within 36 months from the date of disbursement of the loan. The borrower has to execute a demand promissory note in favour of the bank, promising that he would repay the loan unconditionally as per the stipulations of the bank.

Term Loans

The borrower enters into an agreement with the bank regarding the period of loan and mode of repayment, number of instalments, etc. The repayment period is, generally, more than 36 months. These loans are availed by those who purchases machinery, build houses, etc.

Overdrafts (ODs)

A current account holder enters into an agreement with the bank which permits him to draw more than the amount available in his account, but upto a maximum limit fixed by the bank. These loans are availed by traders.

Calculation of Interest on Loans

Interest on loans is calculated on daily product basis. Once in every quarter the loan amount is increased by that amount.

Daily product = Balance × Number of days it has remained as balance.

$$\text{Interest} = \frac{\text{Sum of daily products} \times \text{Rate}}{100 \times 365}$$

Note If the loan is repaid totally, the date on which it is repaid is not counted for calculation of interest. If the loan is repaid in part, the day of repayment of loan is also counted for calculating the interest.

EXAMPLE 22.5

Ganesh takes a loan of ₹20,000 on 1-4-2005. He repays ₹2000 on the 10th of every month, starting from May 2005. If the rate of interest is 15% per annum, calculate the interest upto 30-6-2005.

SOLUTION

Loan Amount (in ₹)	Loan Period	No. of Days	Daily Product
₹20,000	1-4-2005 to 10-5-2005	40	$40 \times 20{,}000$ = ₹800,000
Repays ₹2000 on 11-5-2005 Balance ₹18,000	11-5-2005 to 10-6-2005	31	$31 \times 18{,}000$ = ₹558,000
Repay ₹2,000 on 11-6-2005 Balance ₹16,000	11-6-2005 to 30-6-2005	20	$20 \times 16{,}000$ = ₹320,000

Total daily product (DP) = ₹1,678,000

$$\therefore \text{ Interest} = \frac{\text{DP} \times \text{Rate}}{100 \times 365}$$

$$= \frac{1{,}678{,}000 \times 15}{100 \times 365} = ₹689.60.$$

Compound Interest

When interest is calculated on principal as well as on interest, it is known as compound interest. The interest is added to the principal at regular intervals, quarterly or half yearly or yearly, and further interest is calculated on the increased principal thus obtained.

The formula to find out the amount payable, when the interest is compounded annually is as follows:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where, p = Principal
r = Rate of interest, and
n = Number of years.

When interest is compounded k times a year, then $A = P\left(1 + \frac{r}{k \times 100}\right)^{n \times k}$

When interest is compounded quarterly $k = \frac{12}{3} = 4.$

When interest is compounded half-yearly $k = \frac{12}{6} = 2,$ and so on.

HIRE PURCHASE AND INSTALMENT SCHEME

When a buyer does not have purchasing capacity, the seller allows the buyer to make part payments in monthly, quarterly, half-yearly or yearly instalments. This scheme is of two types:

1. Hire purchase scheme
2. Instalment scheme

Hire Purchase Scheme

In this scheme, the buyer is called the 'hirer' and the seller is called the 'vendor' enters into an agreement which is known as **'hire purchase agreement'**.

Important Features of Hire Purchase Scheme

1. The hirer pays an initial payment known as 'down payment'.
2. The vendor allows the hirer to take possession of the goods on the date of signing the agreement, but he does not transfer the ownership of the goods.
3. The hirer promises to pay the balances amount in instalments.
4. If the hirer fails to pay the instalments the vendor can repossess the goods.
5. Once goods are repossessed, the hirer cannot ask for repayment of the instalments of money already paid. This money paid will be treated as rent for the period.

Instalment Scheme

Under instalment scheme, the seller transfers the possession as well as the ownership of the goods to the buyer. The buyer has the right to resell, or pledge the goods, but he has to repay the instalments due.

Finding the Rate of Interest on Buying in Instalment Scheme

The formula that is used to calculate the rate of interest on instalment purchase is:

$$R = \frac{2400E}{n[(n+1)I - 2E]}$$

where, R = Rate of Interest

E = Excess amount paid

n = Number of instalments

I = Amount of each instalment

E = Down payment + Sum of instalment amounts – Cash price

EXAMPLE 22.6

A television set is sold for ₹9000 cash on ₹1000 cash down followed by six equal instalments of ₹1500 each. What is the rate of interest?

SOLUTION

$n = 6$

$I = ₹1500$

$E = [1000 + 6 \times 1500 - 9000] = ₹1000$

$$\therefore R = \frac{2400E}{n[(n+1)I - 2E]} = \frac{2400 \times 1000}{6[(6+1)1500 - 2000]} = 47.1\%.$$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Fixed deposit is also known as ________ deposit.

2. Farhan opens a savings bank account on 5-5-2007 in UTI Bank by depositing ₹1000. The interest paid by the bank, if she closed her account on 1-4-2008, is ________. (Rate of interest is 6% per annum)

3. Bhavya opened a savings bank account in a bank on 7-7-2007 with a deposit of ₹500. Since then she neither deposited nor withdrew any amount. The amount, on which she receives interest, if she closed her account on 1-1-2007, is ________.

4. Charan opens a savings bank account on 6-3-2007 depositing ₹750 in Union Bank of India. He deposits ₹1000 on 15-3-2007 and withdraws ₹500 on 23rd March. The sum on which he will earn interest for the month of March is ________.

5. Anil opened a savings bank account with a bank on 12-3-2006 with an initial deposit of ₹1000. He withdrew ₹300 on 18-3-2006. The amount, on which he would receive interest for the month of March 2006 is ________.

6. Banks offer higher rate of interest on savings accounts than fixed deposit accounts. (True/false)

7. Which of the following is a utility service provided by the Banks? (Issuing traveller cheques/Receiving payment for telephone bills)

8. Under a recurring deposit, a depositor is paid a lump sum payment after the period for which the deposit is made. This lump sum payment is called ________ value.

9. In the banks, safe deposit lockers are provided to the customers at free of cost. (True/False)

10. The rate of interest on current account is ________ % per annum.

Short Answer Type Questions

11. Himesh opened a savings bank account with a bank on 3-6-2007 with ₹500. His transactions during June and July were as follows:

    Deposited ₹500 on 8-6-2007.

    Withdrew ₹300 on 11-6-2007.

    Deposited ₹500 on 13-6-2007.

    Withdrew ₹350 on 29-6-2007.

    Deposited ₹500 on 3-7-2007.

    Deposited ₹500 on 12-7-2007.

    Find the amounts qualifying for interest during June, July and August.

Directions for questions 12 to 15:
Hansika opened savings bank account with a bank on 1-2-2007. A page from her pass book is given below. She closed her account on 30-6-2007.

Date	Particulars	Amount Withdrawn (₹)	Amount Deposited (₹)	Balance (₹)
4-1-2007	B/F	–	1500.00	1500.00
23-1-2007	To self	500.00		1000.00
6-2-2007	By cash	–	3500.00	4500.00
15-2-2007	To self	1000.00		3500.00
26-2-2007	By cash	–	6500.00	10,000.00
15-3-2007	By cash	–	2000.00	12,000.00
8-4-2007	To self	3000.00		9000.00
15-4-2007	By cash	–	7000.00	16,000.00

12. Find the amount for which Hansika receives interest.

13. If the bank pays interest at 6% per annum, find the interest she receives while closing her account.

14. If the bank pays interest 8% per annum, then find out total amount Hansika will receive on closing her account.

15. Calculate the total interest earned by Hansika till 3-6-2007, if the bank pays
(i) 4.8% per annum till 31-3-2007
(ii) 7.2% per annum from 1-4-2007 to 30-6-2007

Essay Type Questions

16. Javed makes a fixed deposit of ₹100,000 in a bank for one year. If the rate of interest is 6% per annum, compounded half-yearly, then find the maturity value.

17. Srija makes a fixed deposit of ₹125,000 with a bank. The bank pays interest at 8% per annum compounded annually, and she received ₹157,464 at the time maturity. Find the time period for which she held account.

18. Tilak opened a recurring deposit account with a bank and deposited ₹600 per month for one year. Find the interest that Tilak will receive, if the bank pays 6% per annum.

19. David makes a fixed deposit of ₹50,000 in a bank for $1\frac{1}{2}$ years. If the interest is compounded half-yearly and the maturity value is ₹66,550, then find the rate of interest per annum.

20. Tushar opens a recurring deposit account with a bank and deposits ₹500 per month for $1\frac{1}{2}$ years. If he receives ₹570 as interest, then find the rate of interest offered by the bank.

CONCEPT APPLICATION

Level 1

1. Ramesh opens a savings bank account in a bank on 16-6-2007 with a deposit of ₹700. He deposited ₹1,500 on 7-7-2007. Find the amount on which he would receive the interest at the end of July 2007.
(a) ₹700 (b) ₹1500
(c) ₹2200 (d) ₹800

2. Akshit opened a savings bank account in a bank on 13-2-2007 with a deposit of ₹1000. He again deposited ₹1000 on 9-3-2007. Find the amount on which he receives interest, if he closes his account on 31-3-2007.
(a) ₹1000 (b) ₹2000
(c) ₹3000 (d) None of these

3. Madhu makes a fixed deposit of ₹15,000 in a bank, for two years. If the rate of interest is 10% per annum, compounded annually, then find the maturity value.
(a) ₹3150 (b) ₹17,500
(c) ₹16,750 (d) ₹18,150

4. Dinesh makes a fixed deposit of ₹50,000 in a bank, for one year. If the rate of interest is 12% per annum, compounded half-yearly, then find the maturity value.
(a) ₹66,125 (b) ₹56,180
(c) ₹57,500 (d) ₹63,250

5. Karthik makes a fixed deposit of ₹15,000 in a bank, for 219 days. If the rate of interest is 9% per annum, then what amount will he receive on the maturity of the fixed deposit?
(a) ₹15,810 (b) ₹16,320
(c) ₹15,430 (d) ₹16,610

6. Prabhu deposits ₹600 per month in a recurring deposit account for 1 year at 8% per annum. Find the interest received by Prabhu.
(a) ₹424 (b) ₹312
(c) ₹360 (d) ₹450

7. Kamal deposits ₹550 per month in a recurring deposit account for $1\frac{1}{2}$ year at 8% per annum. Find the interest that Kamal will receive at the time of maturity.

(a) ₹550 (b) ₹627
(c) ₹230 (d) ₹346

8. Sneha opened a cumulative time deposit account with a bank. She deposits ₹500 per month for $1\frac{1}{4}$ years. If she receives ₹300 as interest, find the rate of interest per annum.

(a) 6% (b) 8%
(c) 7.5% (d) 10%

9. Vijay makes a fixed deposit of ₹10,000 in a bank for 2 years under compound interest. If the maturity value is ₹11,664, find the rate of interest per annum compounded annually.

(a) 4% (b) 5%
(c) 8% (d) 10%

10. Subhash makes a fixed deposit of ₹25,000 in a bank for 146 days. If the rate of interest is 7.5% annum, then what amount would he receive on the maturity of the fixed deposit?

(a) ₹27,500 (b) ₹25,750
(c) ₹26,500 (d) ₹28,450

Level 2

11. Find the amount received by Prakash on closing of his account, if the bank pays interest at 6% per annum.

(a) ₹65,010 (b) ₹11,810
(c) ₹62,310 (d) ₹12,100

12. Varsha opened a recurring deposit account with Oriental Bank of Commerce and deposited ₹800 per month at an interest rate of 4% per annum. If she receives ₹800 as interest, then find out the total time for which the account was held. (in years)

(a) $1\frac{1}{2}$ (b) 2
(c) $1\frac{3}{4}$ (d) $2\frac{1}{4}$

13. Susheel has a cumulative time deposit account of ₹800 per month at 6% per annum. If he receives ₹1,300 as interest, then find out the total time for which the account was held. (in months)

(a) 26 (b) 25
(c) 24 (d) 28

14. Vishal has a recurring deposit account in a finance company for 1 year at an interest rate of 8% per annum. If he receives ₹9390 at the time of maturity, then what amount, per month, was invested by Vishal?

(a) ₹650 (b) ₹700
(c) ₹750 (d) ₹800

Directions for questions 15 to 17:
A page from the pass book of Noel is given below. He closes his account on 3-12-2006.

Date	Particulars	Withdrawn (₹)	Deposited (₹)	Balance (₹)
3-7-2006	B/F	–	–	5000.00
12-7-2006	By cash	–	3000.00	8000.00
15-8-2006	To self	2500.00	–	5500.00
6-10-2006	By cash	–	5000.00	10,500.00
8-11-2006	To self	1500.00	–	9000.00
15-11-2006	By cash	–	6000.00	15,000.00

15. Find the amount on which he receives interest on closing his account.

(a) ₹41,500 (b) ₹35,500
(c) ₹44,500 (d) ₹33,500

16. The interest received by Noel on closing his account, if the bank pays interest at 6% per annum, is ________.

(a) ₹177.50 (b) ₹207.50
(c) ₹222.50 (d) ₹167.50

17. If the bank pays 6% per annum, find the amount received by Noel on closing his account.

(a) ₹41,722.50 (b) ₹35,677.50
(c) ₹15,177.50 (d) ₹9177.50

Directions for questions 18 to 20: Answer these questions based on the information provided.
A page from Giri's pass book is given below. He closed his account on 2-7-2007. Assume that there were no transactions involving his account after 18-5-2007.

Date	Particular	Withdrawn	Deposited	Balance
2-1-2007	B/F	–	–	4000
14-1-2007	By cash	–	5000	9000
14-2-2007	To self	3000	–	6000
7-4-2007	By cash	–	2000	8000
8-5-2007	To self	5500	–	2500
18-5-2007	By cash	–	6500	9000

18. If the bank paid interest to Giri at 8% per annum, then find the interest received by him on closing his account. (in ₹)

(a) 226 (b) 280
(c) 237 (d) 240

19. Find the sum on which Giri received interest on closing his account (in ₹) from January 2007–June 2007. (in ₹)

(a) 32,500 (b) 33,500
(c) 34,500 (d) 35,500

20. Using the information as provided in the previous question, find out the amount received by Giri on closing his account (in ₹) from January 2007–June 2007?

(a) 9237
(b) 2737
(c) 35,847
(d) 35,737

Directions for question 21 to 25: Select the correct alternative from the given choices.

21. Anil opened a savings account in a bank on 12-11-2008. His first deposit to his account was ₹800 on that day. His second deposit to his account was ₹1200 on 9-12-2008. Find the sum on which he would receive interest at the end of December 2008. (in ₹)

(a) 800 (b) 1200
(c) 400 (d) 2000

22. Chetan deposited ₹1200 per month in a recurring deposit account for one year at 6% per annum Find the interest received by him. (in ₹)

(a) 384 (b) 426
(c) 468 (d) 492

23. Bala made a fixed deposit of ₹30,000 in a bank for two years at R% per annum under compound interest. The maturity value is ₹35,643. Find R.

(a) 7 (b) 9 (c) 11 (d) 13

24. Bhuvan made a fixed deposit of ₹15,000 in a bank on 1-1-2007 for 219 days under 12% per annum simple interest. Find out the maturity value. (in ₹)

(a) 15,840 (b) 15,960
(c) 16,080 (d) 16,140

25. Amar made a fixed deposit of ₹20,000 in a bank, for two, years at 8% per annum under compound interest. Find out the maturity value. (in ₹)

(a) 22,896 (b) 23,328
(c) 24,124 (d) 24,312

Level 3

Directions for questions 26 to 28:

A page from Richa's pass book is given below. Answer the following questions by finding the missing entries. She closes her account on 30-6-2007.

Date	Particulars	Amount With drawn (₹)	Amount deposited (₹)	Balance (₹)
5-1-2007	By Cash		500.00	500.00
23-1-2007	By Cash		6000.00	6500.00
8-2-2007	By Cash	(missing entry)		8000.00
13-2-2007	To self	(missing entry)		5000.00
18-2-2007	By Cash		2000.00	(missing entry)
9-3-2007	By Cash		5000.00	12,000.00
15-3-2007	To self	(missing entry)		9000.00
11-4-2007	To self	(missing entry)		5000.00
5-5-2007	By Cash	(missing entry)		10,050.00

26. If the bank pays 4% per annum, then find the interest received by Richa on closing her account.

(a) ₹98.5 (b) ₹115
(c) ₹132 (d) ₹133

27. Find the amount on which she will receive interest on closing her account.

(a) ₹29,550 (b) ₹34,500

(c) ₹39,600 (d) ₹36,900

28. If the bank pays interest 8% per annum from 1-1-2007 to 30-4-2007 and 6% per annum from 1-5-2007 to 30-6-2007, then find the total interest received by Richa.

(a) ₹230.50 (b) ₹247

(c) ₹196.50 (d) ₹188

Directions for questions 29 to 35: Select the correct alternative from the given choices.

29. Arun made a fixed deposit in bank A at R% per annum, for T days. Bala made a fixed deposit in bank B at $\frac{R}{2}$% per annum for $2T$ days. Charan made a fixed deposit in bank C at $2R$% per annum for $\frac{T}{2}$ days. All of them deposited equal sums of money at simple interest on 1-1-2005. Name the person whose deposit earned the highest maturity value?

(a) Arun

(b) Bala

(c) Charan

(d) All deposits had equal maturity values

30. Charan opened a cumulative time deposit account with a bank. For $1\frac{1}{2}$ years, he deposited ₹800 per month. He received an interest of ₹1140. Find the rate of interest. (in % per annum)

(a) 10 (b) 9

(c) 8 (d) 12

31. In the above problem, if the bank paid Charan interest at 9% per annum, then find the interest he would have received on closing his account. (in ₹)

(a) 202.5 (b) 216

(c) 229.5 (d) 243

32. Ashwin opened a savings account in a bank on 4-1-2006 with a deposit of ₹2000. On the 9th of every odd month of that year, he deposited ₹500 to his account. On the 9th of every even month of that year, he withdrew ₹500 from his account. He closed his account on 1-2-2007. Find the sum on which he would have received interest on closing the account. (in ₹)

(a) 22,500 (b) 24,000

(c) 25,500 (d) 27,000

33. Dinesh opened a recurring deposit account with State Bank of India. He deposited ₹900 per month at 7.5% per annum. He received ₹1687.5 as interest. Find the time period of his deposit. (in years)

(a) 3 (b) 3.5

(c) 2.5 (d) 2

34. Ganesh makes a fixed deposit of ₹40,000 in a bank for a year at 20% per annum simple interest. Harish makes a fixed deposit of an equal sum for the same period and at the same rate of interest, interest being compounded half-yearly. Find the difference between maturity values of investment of Harish and Ganesh. (in ₹)

(a) 400 (b) 360

(c) 440 (d) 300

35. Ramu deposited ₹400 per month in a recurring deposit account for 2 years at 9% per annum. Somu deposited ₹400 per month in a recurring deposit account for $2\frac{1}{2}$ years at 12% per annum. Which of the following can be concluded about the interests to be paid to them on maturities of their deposits?

(a) Ramu must be paid an interest of ₹860 less than that of Somu.

(b) Ramu must be paid an interest of ₹960 less than that of Somu.

(c) Ramu must be paid an interest of ₹916 less than that of Somu.

(d) Ramu must be paid an interest of ₹816 less than that of Somu.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. term deposit
2. ₹40.
3. ₹1500
4. ₹750
5. ₹0
6. False
7. Receiving payment for telephone bills
8. Maturity
9. False
10. 0

Short Answer Type Questions

11. ₹700, ₹1350 and ₹1850
12. ₹55,500
13. ₹277.50
14. ₹16,370
15. ₹304

Essay Type Questions

16. ₹106,090
17. 3 years
18. ₹234
19. $r = 20\%$ per annum
20. 8% per annum

CONCEPT APPLICATION

Level 1

1. (c) **2.** (b) **3.** (d) **4.** (b) **5.** (a) **6.** (b) **7.** (b) **8.** (a) **9.** (c) **10.** (b)

Level 2

11. (a) **12.** (b) **13.** (b) **14.** (c) **15.** (b) **16.** (a) **17.** (c) **18.** (c) **19.** (d) **20.** (a)
21. (d) **22.** (c) **23.** (b) **24.** (c) **25.** (b)

Level 3

26. (c) **27.** (c) **28.** (a) **29.** (d) **30.** (a) **31.** (a) **32.** (d) **33.** (d) **34.** (a) **35.** (b)

CONCEPT APPLICATION

Level 1

1. Find the minimum balance from 10th of month to end of the month.

2. Calculate the minimum balances from the 10th to end of the each month and the total minimum balance.

3. Use, $A = P\left(1+\frac{r}{100}\right)^n$

4. (i) $A = P\left(1+\frac{r}{100}\right)^n$

   (ii) Given, $P =$ ₹50,000,

   $n = 2$ half-years,

   $R = 6\%$ per half-year.

   (iii) Use, $A = \left(1+\frac{R}{100}\right)^n$ to find the maturity value (A).

5. (i) Use, SI $= \frac{P\times d\times R}{36500}$, where

   $d =$ number of days.

   (ii) $P =$ ₹1500,

   $T = \frac{219}{365} = \frac{3}{5}$ years and $R = 9\%$ per annum

   (iii) Use, $A = P + \frac{PRT}{100}$ to find the maturity value (A).

6. Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$

7. (i) Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$

   (ii) Given $P =$ ₹550, $n = 18$ months, $R = 8\%$

   (iii) Use, $I = \frac{p\times n(n+1)}{2}\times\frac{R}{1200}$ to find interest revised.

8. Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$.

9. Use, $A = P\left(1+\frac{r}{100}\right)^n$.

10. (i) SI $= \frac{P\times d\times R}{100\times 365}$, where

    $d =$ Number of days.

    (ii) Total $= P +$ SI.

12. Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$

13. (i) Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$

    (ii) Given $P =$ ₹800, $I = 1300$, $R = 6\%$

    (iii) Use, $I = \frac{p\times n(n+1)}{2}\times\frac{R}{1200}$ to find n.

14. (i) Use, SI $= \frac{P\times n(n+1)}{2}\times\frac{R}{1200}$.

    (ii) Let the amount deposited per month be ₹P.

    Amount $A =$ ₹9390, SI $= 9390 - P$.

    (iii) Use, $9390 = P + \frac{p\times n(n+1)}{2}\times\frac{R}{1200}$ to find P.

15. Calculate the minimum balance available from the 10th to the end of each month and the total minimum balances.

16. Interest is calculated on the minimum balance for the entire period.

17. Use, SI $= \frac{P\times T\times R}{100}$.

18. Interest received by Giri (in ₹)

    $= (35500)\left(\frac{1}{12}\right)\left(\frac{8}{100}\right) = \frac{710}{3}$

    $=$ ₹237 (approximately).

19. Minimum value from the 10th January to the end of January = ₹4000

    Minimum balance from the 10th February to the end of February = ₹6000

    Minimum balance from the 10th March to the end of March = ₹6000

    Minimum balance from the 10th April to the end of April = ₹8000

    Minimum balance from 10th May to the end of May = ₹2500

    Minimum balance from the 10th June to the end of June = ₹9000

The sum of minimum balances = ₹35,500

∴ Giri, on closing his account, will receive interest on ₹35,500.

20. Amount received by Giri (in ₹) = 9000 + 237 = ₹9237.

21. Minimum balance from the 10th November to the end of November = ₹0

Balance in December = ₹(800 + 1200) = ₹2000

∴ Minimum balance from the 10th December to the end of December ₹2000. Anil will receive interest on the sum of minimum balance, i.e., ₹2000 at the end of December 2008.

22. Time period = 1 year = 12 months

Interest received by him (in ₹)

$$= 1200\frac{(12)(13)}{2}\left(\frac{1}{12}\right)\left(\frac{6}{100}\right)$$

$$= (12)\left(\frac{13}{2}\times 6\right) = 156\times 3 = 468.$$

23. $30,000\left(1+\frac{R}{100}\right)^2 = 35,643$

$$\left(1+\frac{R}{100}\right)^2 = \frac{35,643}{30,000} = \frac{11,881}{10,000}$$

$$\left(1+\frac{R}{100}\right)^2 = \left(\frac{109}{100}\right)^2 \Rightarrow R = 9.$$

24. Time period = 219 days $= \frac{219}{365}$ years

$= \frac{3}{5}$ years

Maturity value (in ₹)

$$= 15,000\left(1+\frac{\left(\frac{3}{5}\right).12}{100}\right)$$

$$= 15,000 + (150)\left(\frac{3}{5}\right)(12)$$

$$= 15,000 + 1080 = 16,080.$$

25. Maturity value (in ₹) $= 20,000\left(1+\frac{8}{100}\right)^2$

$= 20,000(1.08)^2 = 23,328.$

26. (i) Interest is calculated on minimum balance for the entire period.

(ii) Use, SI $= \frac{P\times R\times T}{100}$, where P = sum of all minimum balances, r = 4% and $T = \frac{1}{12}$ year.

27. (i) Calculate the minimum balances from the 10th of the month to the end of each month.

(ii) Find the sum of the minimum balances of all the months.

28. (i) Calculate the total minimum balance from 1-1-2007 to 30-4-2007, and also the rate of interest given.

(ii) Calculate the total minimum balance from 1-5-2007 to 30-6-2007 and the SI with the rate of interest given.

(iii) Apply the new rates of interest and the corresponding interest received by Richa.

29. In this problem too, all deposits are placed under SI.

∴ Time periods of Arun, Bala and Charan where

$\frac{T}{365}$ years, $\frac{2T}{365}$ years and $\frac{\frac{T}{2}}{365}$ years, respectively.

Maturity value of Arun's investment (in ₹)

$$= P\left(1+\frac{\frac{T}{365}(R)}{100}\right)$$

Maturity value of Bala's investment (in ₹)

$$= P\left(1+\frac{\frac{2T}{365}\cdot\left(\frac{R}{2}\right)}{100}\right) = P\left(1+\frac{\frac{T}{365}(R)}{100}\right)$$

Maturity value of Charan's investment (in ₹)

$$= P\left(1+\frac{\frac{\frac{T}{2}}{365}(2R)}{100}\right) = P\left(1+\frac{\frac{T}{365}R}{100}\right)$$

All deposits had equal maturity value.

30. Let the rate of interest be R% per annum time period $= 1\frac{1}{2}$ years = 18 months.

Interest received by him (in ₹) = 800

$$\frac{(18)(19)}{2}\left(\frac{1}{12}\right)\left(\frac{R}{100}\right) = R\left[(8)\left(\frac{9\times 19}{12}\right)\right]$$

$$= R\left(\frac{72\times 19}{12}\right) = 114R \Rightarrow 114R = 1140$$

(given) $R = 10$.

31. Interest received by him (in ₹) $= \frac{9}{100}$

$$(27{,}000)\left(\frac{1}{12}\right)$$

$$= \frac{2430}{12} = 202.50.$$

32. His minimum balances in the year 2006 (in ₹) are 2500, 2000, 2500, 2000, 2500, 2000, 2500 and 2000.

The sum on which Ashwin would have received interest (in ₹)

$= 6(2500 + 2000) = 27{,}000.$

33. Let the time period be T years.

T years = 12 T months.

Interest received by him

$$= (900)\frac{(12T)(12T+1)}{2}\left(\frac{1}{12}\right)\left(\frac{7.5}{100}\right)$$

$$= 1687.5$$

$$\frac{9(T)(12T+1)(7.5)}{2} = 1{,}687.5$$

$$T(12T+1) = \frac{(1687.5)(2)}{(9)(7.5)} = 50 \qquad (1)$$

$12T^2 + T - 50 = 0$

$12T^2 - 24T + 25T - 50 = 0$

$12T(T - 2) + 25(T - 2) = 0$

$(12T + 25)(T - 2) = 0$

$$T = \frac{-25}{12} \text{ or } 2$$

But $T > 0$

$\therefore T = 2.$

34. Maturity value of Ganesh's investment (in ₹)

$$= 40{,}000\left(1+\frac{20}{100}\right) = 40{,}000(1.2) = 48{,}000$$

Maturity value of Harish's investment (in ₹)

$$= 40{,}000\left(1+\frac{20}{2(100)}\right)^2$$

$= 40{,}000(1.1)^2$

$= 40{,}000(1.21)$

$= 48{,}400$

The required excess amount (in ₹)

$= 48{,}400 - 48{,}000 = 400.$

35. Time periods of Ramu's account and Somu's account are 2 years, i.e., 24 months and $2\frac{1}{2}$ years, i.e., 30 months, and respectively.

Interest to be paid to Ramu (in ₹)

$$= \frac{400(24)(25)}{2}\left(\frac{1}{12}\right)\left(\frac{9}{100}\right)$$

$$= 4(300)\left(\frac{3}{4}\right) = 900$$

Interest to be paid to Somu (in ₹)

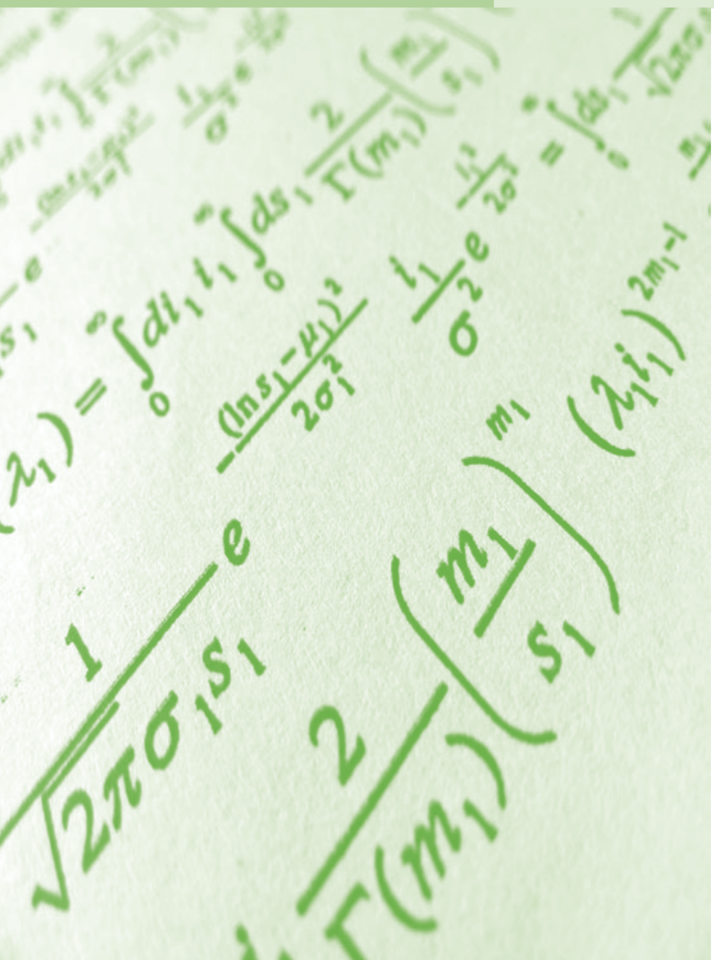
$$= \frac{400(30)(31)}{2}\left(\frac{1}{12}\right)\left(\frac{12}{100}\right) = 1{,}860$$

Interest to be paid to Ramu is less than that to be paid to Somu by ₹960.

Chapter

23

Taxation



REMEMBER

Before beginning this chapter, you should be able to:

- Study about sales tax.
- Know what is value added tax (VAT).
- Understand cost of living index.

KEY IDEAS

After completing this chapter, you would be able to:

- Understand concepts related to income tax.
- Learn about rebates, funds and solve numerical problems related to it.
- Study sales tax and do calculation of sales tax.

INTRODUCTION

The government of a country requires performing many social and economic functions, for which it needs money that comes from both domestic and foreign sources. For a government, the most important source of money is taxation. Tax is a type of fee that a government charges on various economic activities and the wealth that is created by such activities or for providing legal safeguards.

Taxes can thus be classified on the basis of the economic activities or the kind of legal safeguards provided. Taxes can also be classified as direct or indirect types, based on the manner in which they are collected. If taxes are collected directly from the person who is paying, then those are direct taxes. If taxes are paid by a person other than the person upon whom it is levied, such as sales tax—those fall in the category of indirect taxes. The most important type of direct tax is income tax. Here, will shall discuss about income tax and sales tax.

INCOME TAX

The following discussion is only an indicative of the general nature of the computation of income tax. The actual details vary from one financial year to another. A financial year begins from 1st April of a calendar year and ends on 31st March of the subsequent calendar year. The year next to a financial year is called the assessment year for that financial year. For the financial year 2004–2005, the assessment year is 2005–2006. People who earn above a certain limit are liable to pay income tax. The tax imposed on an individual is called personal income tax. A certain part of the income, called standard deduction, is not liable to be taxed.

As and example, the following table (Table 23.1(a)) shows of standard deduction. The symbol S stands for the annual salary of an individual.

Table 23.1(a)

Salary (₹)	Standard Deduction (₹)
$S \leq 90,000$	$S/3$
$900,000 < S \leq 150,000$	30,000
$150,000 < S \leq 350,000$	25,000
$350,000 < S \leq 500,000$	20,000
$500,000 < S$	Nil

In addition, deduction from income is allowed on certain specified donations as detailed below.

Table 23.1(b)

Donation	Part Exempted
PM's National Relief Fund	100%
National Defence Fund	100%
Medical Research	100%
Charitable Trusts, Educational Bodies, Hospitals and Orphanages	50%

Such donations (i.e., the indicated rates) are deducted from income. However, this deduction is subject to a maximum of 10% of the total income. To explain this, if more than 10% is donated, the excess amount is subject to tax at the applicable rate.

After allowing for these deductions (D) (i.e., standard deduction and specified donations) the net taxable income ($TI = S - D$) is taxed at the following rate(s). (Table 23.2)

Table 23.2

Taxable Income (₹)	Rate of Tax	Rate of Surcharge
TI ≤ 50,000	Nil	0
50,000 < TI ≤ 60,000	10% of (TI – 50,000)	0
60,000 < TI ≤ 150,000	1000 + 20% of (TI – 60,000)	5%
150,000 < TI	19,000 + 30% of (TI – 150,000)	5%

By using the above table, we can compute the tax payable. Senior citizens (Aged 65 years or more), women and citizens who have invested in specified funds are allowed certain rebates on this tax. Therefore, this amount is referred to as Tax Before Rebate (TBR).

Note The third column in the above table gives the rate of surcharge on tax. Surcharge is calculated after deducting the rebate as illustrated in the examples.

These rebates are tabulated in the following tables.

Table 23.3(a) Senior Citizens

TBR	Rebate
TBR ≤ 15,000	TBR
15,000 < TBR	15,000

Table 23.3(b) Women (less than 65 years old)

TBR	Rebate
TBR ≤ 5000	TBR
5000 < TBR	5000

All citizens (including senior citizens and women) who save in the following funds are allowed rebate at the mentioned rates as shown below.

1. Contributory Provident Fund (CPF)
2. General Provident Fund (GPF)
3. Public Provident Fund (PPF)
4. Life Insurance Premium (LIC)
5. National Savings Certificates (NSC)
6. Certain Infrastructure Bonds (CIB)

Table 23.3(c)

Salary (S) (₹)	Rate of Rebate
$S \leq 150{,}000$	20% of savings
$150{,}000 < S \leq 500{,}000$	15% of savings
$500{,}000 < S$	Nil

We note that while Tables 23.3(a) and (b) are applicable to different classes of people, Tables 23.3(a) and (c) or (b) and (c) could apply to the same person. An individual who pays income tax is known as an 'assessee'. On applying every assessee is given a Permanent Account Number (PAN) by the Income Tax Department.

The tax after rebate on TAR is computed from Tables 23.3(a), (b), and (c).

Finally, the surcharge (the rate given in Table 23.2) is computed on the TAR and the total tax payable (TTP) is computed. The computation of income tax is illustrated with the following examples.

EXAMPLE 23.1

The monthly salary of Madhuri is ₹15,000. She contributes ₹5000 to the PM's National Relief Fund and ₹2500 to a hospital. She also pays an annual premium of ₹2000 towards her LIC policy. Compute her income tax.

SOLUTION

We have to compute the TI, TBR, TAR and TTP.

All amounts are in rupees.

Annual salary (S) = 12(15,000) = 180,000

Standard deduction = 25,000 (Table 23.1(a))

Donation (PMNRF) = 5000 (Table 23.1(b))

Donation (Hospital) = 2500 (Table 23.1(b))

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Total deduction (D) = 32,500

=======

$$TI = S - D = 147{,}500$$

TBR = 1000 + 20% of (147,500 − 60,000)

$$= 1000 + \frac{20}{100}(87500) \quad \text{(Table 23.2)}$$

= 1000 + 17,500 = 18,500

Rebate for women = 5000 (Table 23.3(b))

Rebate for LIC premium = 20% of 2000 (Table 23.3(c)) = 400

Total rebate = 5400

∴ TAR = TBR − Rebate = 18,500 − 5400 = 13,100

Surcharge = 5% of 13,100 $= \frac{13{,}100}{20} = 655$

∴ TTP = TAR + Surcharge = 13,100 + 655 = ₹13,755.

EXAMPLE 23.2

Raghav's monthly salary is ₹20,000. He donates ₹15,000 per annum for cancer research and ₹1000 to the NDF. He contributes ₹50,000 to PPF. Compute his income tax.

SOLUTION

We compute the TI, TBR, TAR and TTP as follows:

Annual salary, S = 20,000 × 12 = ₹240,000

Standard Deduction = ₹25,000 (Table 23.1(a))

Donation to NDF = ₹10,000

Donation to Cancer Research = ₹15,000

As the total donation is 25,000, only 10% of Raghav's income, i.e., 24,000 can be deducted.

$\therefore$ Total deduction = (25,000 + 24,000) = ₹49,000

$TI = S - D = 240{,}000 - 49{,}000 =$ ₹191,000

TBR = 19,000 + 30% of (191,000 − 150,000)

$$= 19{,}000 + \frac{3}{10}(41{,}000) = ₹31{,}300$$

He contributes 50,000 to PPF.

$\therefore$ He receives a rebate of 15% on 50,000 = ₹7500

$\therefore$ TAR = 31,300 − 7500 = ₹23,800

He has to pay a surcharge of 5%.

Surcharge $= \frac{5}{100}(23{,}800) =$ ₹1190

$\therefore$ TTP = 23,800 + 1190 = ₹24,990.

EXAMPLE 23.3

Somu earned a monthly salary of ₹17,500. He contributed ₹5250 per month towards LIC premium. Find the income tax paid by him. (in ₹)

(a) 20,094.50 **(b)** 21,016.50 **(c)** 21,052.50 **(d)** 21,078.50

SOLUTION

The annual salary (in ₹) = (17,500)(12) = 210,000

Standard deduction = ₹25,000

Net taxable income ₹210,000 − ₹25,000 = ₹185,000

The tax that Somu has to pay for different parts of his income are tabulated below.

Range of T1 (in ₹)	Rate	Tax
0 to 50,000	0	0
50,000 to 60,000	10	1000
60,000 to 150,000	20	11,000
150,000 to 185,000	30	10,500
		29,500

But Somu invested in LIC premiums.

His annual investment (in ₹) = (5250)(12) = 63,000.

$\therefore$ He would have received a rebate of 15% on savings, i.e., $\frac{15}{100}$(₹63,000) = ₹9450

Surcharge rate = 5%

$\therefore$ Surcharge (in ₹)

$$= \frac{5}{100}(29{,}500 - 9450) = \frac{1}{20}(20{,}050) = 1002.50.$$

$\therefore$ The amount of tax to be paid by him = ₹20,050 + ₹1002.50 = ₹21,052.50.

EXAMPLE 23.4

Ashwin's annual salary was ₹150,000. He contributed ₹5000 to Public Provident Fund and ₹10,000 to National Savings Certificate. Find the income tax paid by him. (in ₹)

(a) 10,200 **(b)** 10,500 **(c)** 10,800 **(d)** 11,200

SOLUTION

Annual salary (in ₹) = 150,000.

Standard deduction = ₹30,000.

Net taxable income = ₹150,000 – ₹30,000 = ₹120,000.

The tax to be paid if Ashwin had no investments (in ₹)

$$= 1000 + \frac{20}{100}(120,000 - 60,000)$$

$$= 1000 + 12,000 = 13,000$$

Contribution to PPF = ₹5000.

Contribution to NSC = ₹10,000.

Total contribution = ₹15,000.

That is, he would have received a rebate of 20% on his savings, i.e., $\frac{20}{100}$ (₹15,000) = ₹3000

Surcharge rate = 5%

∴ Surcharge (in ₹)

$$= \frac{5}{100}(13,000 - 3000)$$

$$= \frac{5}{100}(10,000) = 500.$$

∴ The amount of tax to be paid by him = ₹10,000 + ₹500 = ₹10,500.

EXAMPLE 23.5

Ganesh's salary is ₹25,000 per month. He contributes ₹25,000 to Contributory Provident Fund and ₹20,000 to Public Provident Fund. He donates ₹13,000 to the PM's National Relief Fund and ₹15,000 to Medical Research. Find his net taxable income. (in ₹)

(a) 243,000 **(b)** 247,000 **(c)** 251,000 **(d)** 25,500

SOLUTION

Annual salary (in ₹) = (25,000)(12) = 300,000

Standard deduction = ₹25,000

Donation to PM's NRF = ₹13,000

Donation to MR = ₹15,000

Total donation = ₹28,000

10% of annual salary = ₹30,000

Total donation is less than this.

The total donation can be completely deducted.

$\Rightarrow$ Total deduction = ₹28,000 + ₹25,000 = ₹53,000.

$\therefore$ Net taxable income = ₹300,000 − ₹53,000 = ₹247,000.

SALES TAX

Sales tax is the tax levied on the sale of goods within a state.

Central sales tax is the tax levied by the Union Government when goods produced in one state are sold in another state.

The proceeds under sales tax are credited to the government's account. Hence, sales tax is not included in the selling price.

Calculation of Sales Tax

1. When no discount is given, the marked prices of articles become the sale price, and sales tax is calculated on it.
2. When a certain discount is given, then sales tax is calculated on the reduced price of the article after the discount.

EXAMPLE 23.6

Rakesh bought a radio for ₹1296. This price includes a discount of 20% offered on the marked price and 8% sales tax on the remaining amount. Find the marked price of the radio.

SOLUTION

Let the marked price of the radio be ₹x.

$$\Rightarrow \text{Discount offered} = ₹(20\% \text{ of } x) = ₹\left(\frac{20}{100} \times x\right) = ₹\frac{x}{5}$$

$$\text{The price of the radio, after discount} = ₹\left(x - \frac{x}{5}\right) = ₹\frac{4x}{5}$$

$$\text{Sales tax charged} = ₹\left(8\% \text{ of } \frac{4x}{5}\right) = ₹\left(\frac{8}{100} \times \frac{4x}{5}\right)$$

$$\text{The cost of the radio, inclusive of sales tax} = ₹\left[\frac{4x}{5} + \frac{8}{100} \times \frac{4x}{5}\right] = ₹\left[\frac{27}{25} \times \frac{4x}{5}\right]$$

Given that Rakesh paid ₹1296 for the radio.

$$\Rightarrow \frac{27}{25} \times \frac{4x}{5} = 1296$$

$$\Rightarrow x = \frac{1296 \times 25 \times 5}{108} = ₹1500.$$

EXAMPLE 23.7

The list price of an article is ₹2160 and sales tax applicable on the article is 8%. If a customer asked the shopkeeper to give a certain discount on its list price such that he pays ₹2160 inclusive of sales tax, then find the per cent of discount offered.

SOLUTION

Let us assume that the reduced price of the article after discount to be ₹x.

Sales tax charged = ₹(8% of x) $= ₹\left(\frac{8}{100}\times x\right) = ₹\frac{2x}{25}$

The selling price of the article, inclusive of sales tax $= ₹\left(x+\frac{2x}{25}\right) = ₹\frac{27x}{25}$

Given that the customer pays ₹2160 for the article, inclusive of taxes

$$\Rightarrow \frac{27x}{25} = 2160$$

$$\Rightarrow x = \frac{2160\times 25}{27} = ₹2000$$

$$\therefore x = ₹2,000.$$

⇒ Discount offered = ₹(2160 − 2000) = ₹160

⇒ The rate of discount $= \frac{160}{2160}\times 100 = 7\frac{11}{27}\%$

∴ The discount per cent offered $= 7\frac{11}{27}$.

EXAMPLE 23.8

A fridge has a listed price of ₹16,000. Successive discounts of 10% and 20% are given on its listed price. Sales tax is then charged at 30%. Ravi bought it for ₹S which includes the sales tax. Find S.

(a) ₹16,000 **(b)** ₹15,848 **(c)** ₹16,168 **(d)** ₹14,976

SOLUTION

List price of the fridge = ₹16,000

Price of the fridge after deducting the discounts (in ₹)

$= 16,000\left(1-\left(\frac{10}{100}\right)\right)\left(1-\left(\frac{80}{100}\right)\right)$

$= 16,000(0.9)(0.8)$

$= ₹11,520$

Sales tax $= \frac{30}{100}(11,520) = ₹3456$

∴ The final price of the fridge = ₹11,520 + ₹3456 = ₹14,976.

EXAMPLE 23.9

Amit bought a TV for ₹17,280 at a discount of 20% followed by a 20% sales tax. Had a 10% discount been offered instead, followed by a 10% sales tax, he would have bought it at a price which, including the sales tax, would have been _____.

(a) ₹170 less **(b)** ₹140 less **(c)** ₹280 more **(d)** ₹540 more

SOLUTION

Let the list price of the TV be ₹k.

The final price of the TV = ₹17,280.

$$\therefore ₹k\left(k - \frac{20}{100}\right)\left(1 + \frac{20}{100}\right) = ₹17,280$$

$$k = \frac{17,280}{(0.8)(1.2)} = \frac{17,280}{0.96}$$

$$= \frac{17,280}{96}(100) = 1800$$

In the hypothetical case, the final price of the TV (in ₹) would have been

$$= 18,000\left(1 - \frac{10}{100}\right)\left(1 + \frac{10}{100}\right)$$

$$= 18,000(0.9)(1.1)$$

= 16,200(1.1) = 17,820. This is ₹540 more than the actual price paid.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. The tax charged on the sale of goods that are moved from one state to other is called _______.
2. If the marked price of an article is ₹100 and sales tax is 12%, then the selling price is ₹_______.
3. The tax imposed on an individual or group of individuals which affects them directly, and is paid to the government directly is known as _______.
4. The tax imposed on an individual or a group of individuals on their annual incomes is known as _______.
5. A financial year begins from _______.
6. Kumar paid ₹220 to buy an article including 10% sales tax. Then, the selling price of the article is ₹ _______.
7. Exemption rate on donation to Prime Minister's National Relief Fund is _______.
8. While assessing income tax, the year next to a financial year is called _______ year for that financial year.
9. Ajay paid ₹150 to buy an article whose selling price is ₹120. The amount of sales tax paid by Ajay is ₹_______.
10. Every assessee is expected to file a statement of the previous year's income to the income tax department in a prescribed form which is known as _______.
11. The tax paid by the first person is less than the tax paid by the second person. If neither of them saved any amount, then the second person has more income. (True/False)
12. Rate of surcharge if annual taxable income exceeds ₹60,000 is _______.
13. Anand paid ₹30 as sales tax on a bottle of mineral water with a marked price of ₹400. Calculate the rate of sales tax.
14. The Annual Union Budget is usually presented in the Lok Sabha every year, on _______.
15. A washing machine is available for ₹7950, including sales tax. If the rate of sales tax is 6%, find the list price of the washing machine.

Short Answer Type Questions

16. Find the list price of a bicycle which costs ₹1595, inclusive of sales tax. The rate of sales tax is 10%.
17. The marked price and the selling price of an article are ₹1500 and ₹1800, respectively. If there is no discount, find the rate of sales tax.
18. Madhu bought the following items from a super market:

    (i) Soaps worth ₹220,

    (ii) Cosmetics worth ₹580 and

    (iii) Vegetables worth ₹650.

    If sales tax is charged at the rate of 5% on soaps, 10% on cosmetics, and 2% on vegetables, then find the total amount paid by Madhu.
19. Prasad has a total salaried income of ₹160,000 per annum. Calculate the amount of income tax he has to pay.
20. Rijwana's monthly salary is ₹25,000. She contributes ₹600 per month towards GPF and pays ₹7000 towards annual LIC premium. Find the amount of income tax she has to pay for the last month if she paid ₹4000 per month towards income tax for 11 months.

Essay Type Questions

21. Ramakanth draws a salary of ₹15,000 per month. He contributes ₹5000 per month towards PPF. He also donates ₹1250 per month towards medical research. Calculate the income tax Ramakanth has to pay.
22. Khalique's monthly salary is ₹20,000. He donates ₹1500 per month for medical research (100%, relief), contributes ₹2000 per month to PF, and pays annual LIC premium of ₹3000.

PRACTICE QUESTIONS

Calculate the income tax he has to pay for the financial year.

23. Chakradhar buys a TV marked at ₹14,500 after receiving successive discounts of 15% and 20% and paying 10% sales tax. He spends ₹2000 on it and sells the TV for ₹12,000. Find his gain or loss per cent.

24. Subiksha earns ₹12,500 per month. She donates ₹3300 per month towards Prime Minister National Relief Fund (100% relief). Find the amount of tax she has to pay.

25. Laxmikala, a senior citizen receives a pension of ₹12,000 per month. Calculate the income tax to be paid by her.

CONCEPT APPLICATION

Level 1

1. Ramu purchased a motorcycle at a price of ₹37,800 which includes sales tax. If the rate of sales tax is 8%, then what is the list price of the motorcycle?

(a) ₹31,000 (b) ₹33,000

(c) ₹35,000 (d) ₹37,000

2. Rohan buys a pair of shoes marked at ₹2500. He receives a rebate of 8% on it. After receiving the rebate, sales tax is charged at the rate of 5%. What is the amount Rohan will have to pay for the pair of shoes?

(a) ₹2400 (b) ₹2415

(c) ₹2430 (d) ₹2445

3. Raju purchased a car for ₹155,750, inclusive of sales tax. He paid ₹5750 as sales tax. What is the rate of sales tax?

(a) $3\frac{2}{3}\%$ (b) $3\frac{3}{4}\%$

(c) $3\frac{4}{5}\%$ (d) $3\frac{5}{6}\%$

4. Amar wants to buy a shirt which is listed at ₹378. The rate of sales tax is 8%. He requested the shopkeeper to reduce the list price to such an extent that he has to pay not more than ₹378 including sales tax. What is the minimum reduction needed in the list price of the shirt?

(a) ₹24 (b) ₹26

(c) ₹28 (d) ₹30

5. Ramesh purchased a bag listed at ₹550. If the rate of sales tax is 8%, then what is the amount of sales tax paid by him?

(a) ₹42 (b) ₹43

(c) ₹44 (d) ₹45

6. Ismail gets a monthly salary of ₹12,500. He contributes ₹3500 per month towards PF. Calculate the income tax paid by him.

(a) ₹3000 (b) ₹5600

(c) ₹4830 (d) ₹3300

7. Rakesh purchased a car which was quoted at ₹256,000. The dealer charged sales tax on it at the rate of 12%. As Rakesh wanted to take the car outside the state, the dealer further charged 3% extra as central sales tax. What is the amount he had to pay for the car?

(a) ₹295,123.60

(b) ₹286,720

(c) ₹294,400

(d) None of these

8. Satish earns an annual salary of ₹150,000 and the standard deduction applicable to him is 40% of the salary or ₹30,000, whichever is less. Then his net taxable income is _______.

(a) ₹30,000 (b) ₹120,000

(c) ₹60,000 (d) ₹90,000

9. Ajay purchased a computer for ₹34,650 which includes 12% rebate on the marked price and 5% sales,tax on the remaining price. What is the marked price of the computer?

(a) ₹35,000 (b) ₹37,500

(c) ₹40,000 (d) ₹42,500

Direction for questions 10 and 11: These questions are based on the following data.

Dinakar's salary is ₹30,000 per month. He contributes ₹27,000 towards GPF and ₹30,000 towards LIC.

PRACTICE QUESTIONS

He donates ₹8,000 to a charitable trust (50%), ₹10,000 towards National Relief Fund (100% relief).

10. Calculate taxable income.

(a) ₹198,000 (b) ₹280,000
(c) ₹326,000 (d) ₹380,000

11. Find the rebate amount on his savings.

(a) ₹15,350 (b) ₹10,250
(c) ₹8550 (d) ₹20,000

12. If the marked price of an article is ₹M, then find the rate at which sales tax is charged if the person pays ₹M inclusive of sales tax. Discount allowed is 10%.

(a) 9% (b) 10%
(c) $11\frac{1}{9}\%$ (d) $9\frac{1}{11}\%$

13. The list price of a TV is ₹15,000, and the shopkeeper allows a discount of 20% and 10% successively on list price. On the remaining amount, he charges 20% as sales tax. If the buyer paid ₹x, then by how much amount will the list price exceed ₹x?

(a) ₹3000 (b) ₹4200
(c) ₹2040 (d) ₹5000

14. Saritha's annual salary is ₹160,000. She contributes ₹6000 towards GPF and pays an LIC annual premium of ₹5000. Calculate the income tax she will have to pay in the year.

(a) ₹16,117.50 (b) ₹15,117.50
(c) ₹17,117.50 (d) ₹16,017.50

15. Ranjit purchased a refrigerator for the price of ₹8910 which includes 10% rebate on marked price and 10% sales tax on the remaining price. If the sales tax is increased to 20% without allowing the 10% rebate on the marked price, how much more will the customer pay for a refrigerator?

(a) ₹1850 (b) ₹1890
(c) ₹1860 (d) ₹1840

Level 2

16. The list price of an article is 50% more than its original cost price. The shopkeeper allowed a discount of 20% and charged a sales tax of 20% on it. Finally, the buyer paid ₹2880. What is the cost price of the article?

(a) ₹2500 (b) ₹3500
(c) ₹3000 (d) ₹2000

17. If the tax to be paid is ₹12,700 and surcharge is calculated as 10% of the tax payable, then find the net tax payable.

(a) ₹13,970 (b) ₹14,690
(c) ₹12,690 (d) ₹13,500

18. Mr Ranvir Patnikar earns an annual salary of ₹270,000. If his employer deducts ₹3000 every month from his salary for the first 11 months, then calculate the amount he has to pay towards tax in the last month of the financial year.

Standard deduction is 40% of the salary or ₹30,000, whichever is less.

The income tax on his earnings is calculated based on the data given below.

Slabs for income tax:

(i) Upto ₹50,000–Nil

(ii) From ₹50,000–10% of the amount ₹100,000 exceeding 50,000

(iii) From ₹100,001–₹5000 + 20% of the ₹200,000 amount exceeding ₹1,00,000

(iv) Above ₹200,000–₹25,000 + 30% of the amount exceeding ₹200,000

(a) ₹3000 (b) ₹4000
(c) ₹5000 (d) ₹6000

19. The annual salary of Mr Ravi Teja is ₹178,500. He donates ₹750 per month towards the National Defence Fund (eligible for 100% exemption). If standard deduction is 30% of the gross salary income or ₹30,000, whichever is less, and then find his net taxable income.

(a) ₹148,000
(b) ₹147,250
(c) ₹139,500
(d) ₹150,000

PRACTICE QUESTIONS

20. Siri's total income is ₹16,500. Of this, ₹5000 is free from tax. Find the net income remaining with her after she paid the income tax at 5%. (in ₹)

(a) 10,925 (b) 15,675

(c) 15,925 (d) 14,750

21. Bala bought a motorbike at ₹49,050 which included sales tax. The rate of sales tax was 9%. Find the,listed price of the motorbike. (in ₹)

(a) 44,000 (b) 45,000

(c) 46,000 (d) 47,000

22. David earns an annual salary of ₹160,000. If the standard deduction applicable to him is 30% of his gross salary or ₹50,000 whichever is less, find his net taxable income. (in ₹)

(a) 110,000 (b) 118,000

(c) 112,000 (d) 108,000

23. Giri bought an auto for ₹89,880 which included a sales tax of ₹5880. Find the rate of the sales tax.

(a) 5% (b) 8%

(c) 9% (d) 7%

24. Hari bought a raincoat marked at ₹550. He received a 15% discount on it, but he had to pay 10% sales tax on the discounted price. Find the price which he paid for the raincoat (in ₹) inclusive of the sales tax.

(a) 508.25 (b) 514.25

(c) 520.25 (d) 526.25

25. Amar bought a book listed at ₹350. The rate of sales tax was 6%. Find the sales tax paid by Amar. (in ₹)

(a) 14 (b) 17.50

(c) 21 (d) 24.50

Level 3

26. Manish's annual income is ₹132,000. There is no income tax on the money donated to charity. On the remaining amount, he pays ₹4480 as income tax at 4%. What amount did he denote to the charity?

(a) ₹24,500 (b) ₹20,000

(c) ₹30,000 (d) ₹18,250

27. Amrit Raj bought a mobile handset for ₹5040 which includes 10% discount on the market price and 12% sales tax on the remaining price. Find the marked price of the mobile phone.

(a) ₹4672 (b) ₹5124

(c) ₹5000 (d) ₹4830

28. Laxmi's total income is ₹22,500. Of this, ₹7000 is free from tax. Find the net income remaining with her after she paid income tax at the rate of 8%. (in ₹)

(a) 21,375 (b) 21,260

(c) 20,675 (d) 22,105

29. Rajnesh's annual income is ₹180,000. He pays no income tax on the money invested in premiums. On the remaining amount, he pays ₹10,920 as income tax at 7%. What amount does he invest in premiums?

(a) ₹18,000 (b) ₹22,500

(c) ₹24,000 (d) ₹27,400

30. Karan bought a TV for ₹9350 which includes 15% discount on the marked price, and then 10% sales tax on the remaining price. Find the marked price of the TV. (in ₹)

(a) 9000 (b) 9750

(c) 10,200 (d) 10,000

31. Govind earned an annual salary of ₹330,000. His employer deducted ₹3000 per month from his salary for the first 11 months of the financial year. Find the amount of tax he paid (in ₹) in the last month of that year using the following information.
If standard deduction is 45% of salary or ₹150,000, whichever is less. Income tax on taxable income is calculated in the following manner.

(i) Less than or equal to ₹50,000 – Nil

(ii) From ₹50,000 to ₹100,000: 20% of the amount exceeding ₹50,000

(iii) From ₹100,001 to ₹150,000: ₹10,000 + 30% of the amount exceeding ₹100,000

(iv) Above ₹150,000: ₹25,000 + 40% of the amount exceeding ₹150,000

(a) 6400 (b) 5600
(c) 4800 (d) 4600

32. Eswar earned an annual salary of ₹102,000. The standard deduction was ₹30,000. Donations upto 10% of gross salary were exempt from tax. He donated ₹1050 per month to the National Defence Fund. Find his net taxable income. (in ₹)

(a) 59,400 (b) 61,800
(c) 60,200 (d) 60,800

33. An article was marked at ₹m. A 10% discount was given on it. The sales tax was then charged at y%. A person bought it for at ₹p (where $p \geq m$) which included the sales tax. The rate of y must be at least _______.

(a) 10 (b) $9\frac{1}{11}$
(c) $11\frac{1}{9}$ (d) $12\frac{1}{2}$

34. The sales tax to be paid for an article is ₹13,500. A surcharge of 8% of the sales tax was also payable. Find the total amount of tax payable. (in ₹)

(a) 14,580
(b) 12,420
(c) 12,960
(d) 14,040

35. An article was marked at ₹m. A discount of x% was given on it. A sales tax of x% was then charged. A person bought it for ₹s which included the sales tax. Which of the following can be concluded?

(a) $s < m$
(b) $s = m$
(c) $s > m$
(d) None of the above

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Central Sales Tax
2. 112
3. Direct Tax
4. Income Tax
5. 1st April
6. ₹200
7. 100%
8. Assessment
9. 30
10. Income tax return
11. False
12. 5% of income tax
13. 7.5%
14. 28th February
15. ₹7500

Short Answer Type Questions

16. ₹1450
17. 20%
18. ₹1532
19. ₹16,800
20. ₹7839

Essay Type Questions

21. ₹8400
22. ≈ ₹30,503.
23. Percentage of loss = 6.58%
24. ₹5250.
25. ₹0

CONCEPT APPLICATION

Level 1

1. (c) **2.** (b) **3.** (d) **4.** (c) **5.** (c) **6.** (c) **7.** (c) **8.** (b) **9.** (b) **10.** (c)
11. (c) **12.** (c) **13.** (c) **14.** (a) **15.** (b)

Level 2

16. (d) **17.** (a) **18.** (b) **19.** (c) **20.** (c) **21.** (b) **22.** (c) **23.** (d) **24.** (b) **25.** (c)

Level 3

26. (b) **27.** (c) **28.** (b) **29.** (c) **30.** (d) **31.** (d) **32.** (b) **33.** (c) **34.** (a) **35.** (a)
36. (c)

CONCEPT APPLICATION

Level 1

1. 108% of list price = cost price.
2. Find 105% of 92% of ₹2500.
3. Rate of sales tax is calculated on list price, and hence comes the list price.
4. (i) Apply the concept of percentages.
   (ii) Let the price after reduction be ₹P.
   (iii) Now, $P + \frac{8P}{100} = 378$, find P.
   (iv) Discount needed = ₹$(378 - P)$
5. Tax is 8% of MRP.
6. Deduct the savings (as per the norms) and standard deduction from the annual salary.
7. (i) Add the sales tax and then add the central sales tax to the list price.
   (ii) First, calculate 12% sales tax and add to list price.
   (iii) Calculate 3% central sales tax and add to the above price.
8. Taxable income = Annual salary – standard deduction. (in ₹)
9. 105% of 88% of MP = ₹34,650.
10. (i) Taxable income is obtained when standard deduction, donations and savings are deducted.
    (ii) Find the standard deduction.
    (iii) Taxable income = Annual salary – (SD + GPF + LIC + NRP + 50% of donations).
11. (i) Refer the introduction for slab on rebate amount on savings.
    (ii) As his salary falls between ₹150,000 and ₹500,000, rate of rebate is 15% of his savings.
    (iii) Calculate total savings and find 15% of it.
12. What per cent of $\frac{9M}{10}$ is M.
13. (i) Apply the concept of percentages.
    (ii) Calculate 20% and 10% successive discounts on list price.
    (iii) Calculate 20% sales tax on the amount after giving discounts.
    X = Sales tax + amount after giving discount.
    (iv) Required amount = ₹$(15,000 - X)$.
14. Find the taxable income after deducting the eligible deductions and based on slab, calculate the amount of tax.
15. (i) Calculate by using $x \times 0.9 \times 1.1 = ₹8910$
    (ii) Now, use the value of x and calculate $x \times 1.2 = y$
    (iii) Find the value of ₹$(y - 8910)$.

Level 2

16. (i) Assume the cost price as ₹100 and proceed.
    (ii) Let the cost price be ₹x, then
    List price $= ₹\frac{3x}{2}$
    (iii) Price after discount $= ₹\frac{3x}{2}\left(1 - \frac{20}{100}\right)$
    (iv) Selling price with tax
    $$= \frac{3x}{2}\left(1 - \frac{20}{100}\right)\left(1 + \frac{20}{100}\right) = ₹2880.$$
17. Apply the concept of percentages.
18. Find the taxable income, and then find the net tax payable from the given slabs.
19. Taxable income is obtained when both the standard deduction and total donation are deducted.
21. Let the list price of the motorbike be ₹m.
    Sales tax $= \frac{9}{100}(₹m) = ₹0.09m$.
    The buying price of the bike = ₹$(m + 0.09m)$ = ₹$1.09m$.
    $1.09m = 49,050$
    $$m = \frac{49050}{1.09} = \frac{49050}{109}(100)$$
    $$= (450)(100) = ₹45,000.$$
22. Gross salary = ₹160,000
    30% of ₹160,000 = ₹48,000 which is < ₹50,000

$\therefore$ Standard deduction = ₹48,000

The net taxable income = ₹160,000 − ₹48,000 = ₹112,000.

23. List price of the auto = Buying price − Sales tax = ₹89,880 − ₹5880 = ₹84,000.

$$\text{Rate of sales tax} = \frac{5880}{84,000} \times (100) = 7\%.$$

24. List price of the raincoat = ₹550

$$\text{Rebate} = \frac{15}{100}(₹550) = ₹82.50$$

After deducting the rebate, its price (in ₹) = 467.50

$$\text{Sales tax} = \frac{10}{100}(₹467.50) = ₹46.75$$

Buying price of the raincoat = ₹467.50 + ₹46.75 = ₹514.25.

25. $\text{Sales tax} = \frac{6}{100}(₹350) = ₹21.$

Level 3

31. Gross salary = ₹330,000

$$45\% \text{ of } ₹330,000 = \frac{45}{100}(₹330,000)$$

$$= \frac{90}{200}(₹330,000)$$

$$= \frac{1}{2}(0.9(₹330,000))$$

= ₹148,500 which is less than ₹150,000.

$\therefore$ Standard deduction = ₹148,500.

Taxable income ($T1$) = ₹330,000 − ₹148,500 = ₹181,500.

The applicable tax rates and the tax are tabulated below:

Range of T1 (in ₹)	Rate	Tax
0 to 50,000	0%	0
50,000 to 100,000	20%	10,000
100,000 to 150,000	30%	15,000
150,000 to 181,500	40%	12,600
		37,600

Total tax deducted by the employer (in ₹)

= (3,000)(11) = 33,000

Tax paid in the last month = ₹37,600 − 33,000 = ₹4600.

32. Gross salary = ₹102,000

Standard deduction for a salary of ₹102,000 is ₹30,000.

$\therefore$ Standard deduction = ₹30,000

The total amount donated (in ₹)

= (12)(1050) = 12,600

But, 10% of the total salary is ₹10,200 which is less than ₹12,600.

Net taxable income

= ₹102,000 − ₹(30,000 + 10,200) = ₹61,800.

33. List price of the article = ₹m

$$\text{Rebate} = \frac{10}{100}(₹m) = ₹0.1\ m$$

Price of the article after deducting the discount = ₹$0.9m$

$$\text{Sales tax} = \frac{y}{100}(₹0.9\ m)$$

The final price of the article

$$= ₹0.9\ m + \frac{y}{100}(₹0.9\ m)$$

$$= ₹0.9\ m\left(1 + \frac{y}{100}\right)$$

Which was at least ₹m

$$\therefore\ 0.9\ m\left(1 + \frac{y}{100}\right) \geq m$$

$$1 + \frac{y}{100} \geq \frac{1}{0.9}$$

$$y \geq \left(\frac{1}{0.9} - 1\right)100 = \frac{100}{9} = 11\frac{1}{9},$$

i.e., y must be at least $11\frac{1}{9}$.

34. Sales tax = ₹13,500

$$\text{Surcharge} = \frac{8}{100}(₹13,500) = ₹1080$$

Tax payable = ₹(13,500 + 1080) = ₹14,580.

35. The discount was x% of the marked price, m.

The sales tax was x% of the discounted price (which is less than M).

$\therefore$ The final price is less than m.

PRACTICE QUESTIONS

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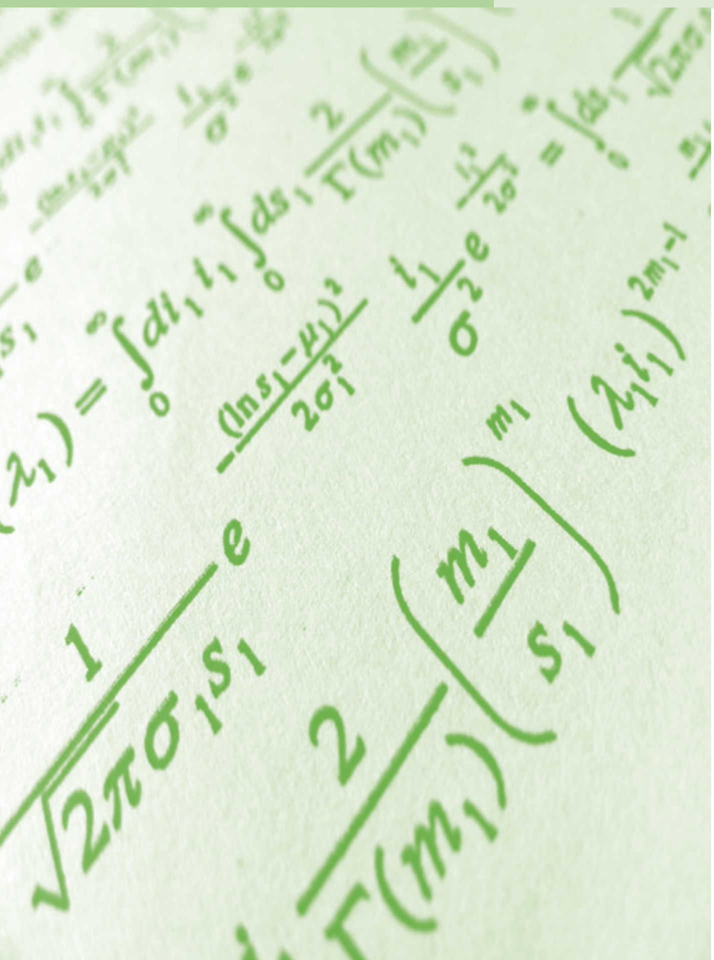
Chapter

24

Instalments



REMEMBER

Before beginning this chapter, you should be able to:

- Recognise terms such as cost price, selling price, etc.
- Have a basic concept on interest and time period.

KEY IDEAS

After completing this chapter, you would be able to:

- Learn about cash price, initial payment, etc.
- Understand calculation of interests, instalments.
- Study repayment of loan.
- Gain knowledge of important features of hire purchase scheme.

INTRODUCTION

Sometimes, a customer cannot buy an article, if he does not have enough money. In such cases, the trader offers the article on instalment basis.

Under this facility, the customer need not pay the entire amount at the time of purchasing the article. He pays only part of the amount at the time of purchasing the article and agrees to pay the balance in small amounts over a period of time. The payment of sale price in small amounts periodically is known as instalment scheme. The small amounts are known as instalments. These instalments may be either monthly, quarterly or yearly.

Usually in this scheme, the customer pays more than the sale price of the article, because the seller charges an interest on the sale price. Generally the interest charged is simple interest unless otherwise mentioned.

Cash Price

The price at which the article is offered to the customer, in case he wants to pay the entire amount right away.

Initial Payment or Down Payment

The amount which a customer has to pay as part payment at the time of purchasing an article is called down payment or initial payment.

EXAMPLE 24.1

A bag is available for ₹900 cash down or for ₹500 down payment followed by a payment of ₹440 after 5 months. Find the rate of interest under the instalment plan.

SOLUTION

Cash price = ₹900

Down payment = ₹500

Balance to be paid by instalments = ₹900 – ₹500 = ₹400

The instalment to be paid at the end of 5 months = ₹440

∴ The interest charged on ₹400 = ₹440 – ₹400 = ₹40 for 5 months.

Let R% be the rate of interest per annum.

$$\therefore I = \frac{PTR}{100}$$

$$40 = \frac{(400)(5)(R)}{(100)(12)} \quad \left(\because 5 \text{ months} = \frac{5}{12} \text{ years}\right)$$

$$\Rightarrow \frac{(40)(100)(12)}{(400)(5)} = R$$

$$\Rightarrow R = 24\%.$$

∴ The rate of interest = 24% per annum.

EXAMPLE 24.2

A scooter is offered for ₹28,000 cash down or for ₹12,000 down payment, followed by two monthly instalments of ₹8200 each. Calculate the rate of interest under the instalment plan.

SOLUTION

Cash price = ₹28,000

Down payment = ₹12,000

Balance to be paid by instalments = ₹28,000 – ₹12,000 = ₹16,000.

Let R be the rate of interest per annum.

After two months ₹16,000 will amount to $16,000+\frac{16,000R(2)}{100(12)}=16,000+\frac{80R}{3}$ (1)

The customer has to pay ₹8200 each month.

$\therefore$ The first instalment will amount to $8200+\frac{8200R}{100(12)}=8200+\frac{41R}{6}$ after one month.

$\therefore$ The second instalment is ₹8200.

$\therefore$ The total value of the two instalments is $8200+8200+\frac{41R}{6}=16,400+\frac{41R}{6}$ (2)

The value of the loan (₹16,000) at the end of 2 months is equal to the total value of the two instalments at the end of the two months.

$$\therefore 16,000+\frac{80R}{3}=16,400+\frac{41R}{6}$$

$$\Rightarrow \frac{80R}{3}-\frac{41R}{6}=16,400-16,000$$

$$\Rightarrow \frac{160R-41R}{6}=400$$

$$\Rightarrow \frac{119R}{6}=400$$

$R=\frac{400(6)}{119}\approx 20.17\%$.

$\therefore$ The rate of interest is 20.17%.

EXAMPLE 24.3

A cycle is offered for ₹1200 cash down or ₹600 down payment, followed by 4 equal monthly instalments. If the rate of interest charged by the dealer is 10% per annum, find the amount of each instalment.

SOLUTION

Cash price = ₹1200

Down payment = ₹600

Balance amount = ₹1200 – ₹600 = ₹600

Rate of interest = 10% per annum.

$\therefore$ After 4 months, the amount ₹600 will be equal to $600+\frac{(600)(10)(4)}{100(12)}$

= 600 + 20 = ₹620. (1)

Let each instalment be ₹x

The first instalment of ₹x will amount to ₹$\left(x+\dfrac{x(3)(10)}{100(12)}\right)$

That is, ₹$\left(x+\dfrac{30x}{1200}\right)$ after 3 months.

The second instalment of ₹x will amount to ₹$\left(x+\dfrac{x(2)(10)}{100(12)}\right)$

That is, ₹$\left(x+\dfrac{20x}{1200}\right)$ after 2 months.

The third instalment of ₹x the will amount to ₹$\left(x+\dfrac{x(10)}{100(12)}\right)$

That is, ₹$\left(x+\dfrac{10x}{1200}\right)$ after 1 month.

$\therefore$ The 4th instalment is ₹x. The total value of the 4 instalments is

$$= \left(x+\frac{30x}{1200}\right)+\left(x+\frac{20x}{1200}\right)+\left(x+\frac{10x}{1200}\right)+x$$

$$= 4x+\frac{60x}{1200}.$$

This is equal to the value of ₹600 after 4 months.

$$\therefore \quad \frac{81x}{20} = 620 \quad \text{[from Eq. (1)]}$$

$$\Rightarrow x = \frac{620(20)}{81} = 153.09.$$

$\therefore$ Each instalment = ₹153 (approximately).

Repayment of Loan

In the above problems, the instalment payment does not extend for more than one year and the interest is calculated at simple interest. But when the amount is very large (e.g., housing loans, etc.) the instalments are payable yearly or half yearly and the interest is calculated at compound interest.

EXAMPLE 24.4

A man borrows money from a finance company and has to pay it back in two equal half-yearly instalments of ₹5,115 each. If the interest charged by the finance company is at the rate of 20% per annum, compounded semi-annually, find the sum borrowed.

SOLUTION

Each instalment = ₹5,115

Rate of interest = 20% per annum = 10% per half yearly.

The amount of ₹5115 paid as an instalment at the end of the first six months includes the principal and interest on it at the rate of 10% half yearly.

$$\therefore \text{ Principal } = 5115 \div \left(1+\frac{10}{100}\right) = 5115 \times \left(\frac{100}{110}\right) = ₹4650.$$

Similarly, the value which amounts to ₹5115 after 1 year at the rate of 10% compounded semi-annually is

$$= 5115 \div \left(1+\frac{10}{100}\right)^2 = 5115\left(\frac{100}{110}\right)^2.$$

$\approx$ ₹4227.27.

$\therefore$ Sum borrowed = 4650 + 4227 = ₹8877 (approximately).

Hire Purchase Scheme

In this scheme the buyer, called the hirer and the seller, called the vendor enter into an agreement which is known as Hire Purchase Agreement.

Important Features of Hire Purchase Scheme

1. The hirer pays an initial payment known as down payment.
2. The vendor allows the hirer to take possession of the goods on the date of signing the agreement, but he does not transfer the ownership of the goods.
3. The hirer promises to pay the balances amount in instalments.
4. If the hirer fails to pay the instalments the vendor can repossess the goods.
5. When goods are repossessed, hirer cannot ask for the repayment of the instalments of money already paid. This money paid will be treated as rent for the period.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If an article is bought under an instalment scheme, the amount a customer has to pay as part payment of the selling price of the article at the time of its purchase is called ____________.
2. The scheme of buying an article by making the part payments periodically is called ____________.
3. If a bicycle is available for ₹1200 cash down or ₹700 down payment followed by five equal monthly instalments of ₹120 each, then the interest charged is ____________.
4. If an article is bought under an instalment scheme, the amount paid in addition to the cash price of the article is known as ____________.
5. Under hire purchase scheme, the ownership of the goods lies with the ____________ until the repayment of all instalments is made.
6. Under hire purchase scheme, if the hirer fails to pay the instalments, the vendor can repossess the goods. The money already paid by the hirer is treated as ____________.
7. A car is available for ₹200,000 cash down or ₹50,000 down payment followed by 16 equal monthly instalments. If the interest charged due to this scheme at the end of 16 months is ₹10,000, then each monthly instalment is ____________.
8. A water filter is available for ₹500 down payment followed by two instalments of ₹600 each. If the total interest paid is ₹100, then the cash down price of the water filter is ____________.
9. An electric generator is offered for ₹30,000 cash down or 22% down payment under instalment scheme. If a buyer has chosen the instalment scheme, then the buyer has to pay ₹____________ as down payment.
10. A mobile phone is available for a cash price of ₹1200 or for a certain down payment followed by three equal instalments of ₹300 each. The total interest paid is ₹100 when bought under the instalment scheme. The down payment for the mobile phone is ____________.

Short Answer Type Questions

11. An article can be sold for ₹20,000 cash down or for ₹12,000 down payment and 4 equal monthly instalments of ₹2200 each. Find the interest paid.
12. A washing machine is available for ₹9000 cash down or 45% down payment and 3 equal monthly instalments. Each instalment is 20% of the cash down payment. If the interest is calculated at simple interest, what is the approximate annual rate of interest?
13. A sum is lent at compound interest, under instalment scheme, interest compounded annually at 10% per annum. If after one year, ₹4400 is repaid and after another year, the balance ₹4300 is repaid, find approximate present value of the instalments.
14. A computer is sold by a company for ₹20,000 cash down or ₹8000 down payment followed by 5 equal monthly instalments of ₹2500 each. Find the total principal on which the interest is charged to realize the total interest in a month.
15. A man borrows ₹10,500 from a finance company and repays it in two equal annual instalments. If the rate of interest being compounded annually is 10% per annum, find the value of each instalment.

Essay Type Questions

16. A loan has to be repaid in three equal half yearly instalments. If the rate of interest, compounded semi-annually, is 16% per annum and each instalment is ₹3500 then find (approximately) the sum borrowed.
17. A plasma TV is available for ₹42,000 cash down or ₹8900 down payment and three equal quarterly instalments of ₹12,000 each. Find the interest charged under instalment plan.

18. A table is available for a down payment of ₹1590 and 3 equal half-yearly instalments of ₹1331 each. If a shop owner charges interest compounded semi-annually at the rate of 20% per annum, then find the cash down price of the table.

19. Prasad borrowed some money under compound interest and repaid it in 3 equal instalments of ₹8470 each. If the rate of interest is 10% per annum, then what is the present value of the instalment paid at the end of the second year?

20. A TV set is available for ₹36,000 or some amount of down payment and two equal annual instalments of ₹12,100 each. If a shop keeper charges interest, compounded annually, at the rate of 10% per annum find the down payment.

CONCEPT APPLICATION

Level 1

1. A fan is sold for ₹900 cash down or ₹200 down payment followed by two equal monthly installments of each ₹375. The annual rate of interest is _______ (approximately).

(a) 25% (b) 30%
(c) 54% (d) 59%

2. A briefcase available for ₹500 cash down or for a certain down payment followed by a payment of ₹312 after 4 months. If the rate of interest is 12% per annum, find the down payment.

(a) ₹198 (b) ₹205
(c) ₹195 (d) ₹200

3. A typewriter is available for ₹970 cash down or for a certain down payment followed by 3 equal monthly instalments of ₹260 each. If the rate of interest is 16% per annum, then find the down payment for purchasing it. (Approximately)

(a) ₹200 (b) ₹210
(c) ₹205 (d) ₹220

4. A pressure cooker is sold for ₹600 cash down or ₹300 down payment followed by ₹310 after one month. The annual rate of interest is _______.

(a) 40% (b) 30%
(c) 33% (d) 20%

5. A ceiling fan is available at a certain cash down price or for ₹250 down payment together with ₹305 to be paid after two months. If the rate of interest is 10% per annum, then find the price of the fan?

(a) ₹545 (b) ₹540
(c) ₹550 (d) ₹535

6. A man borrows ₹6500 from a finance company and has to return it in two equal annual instalments. If the rate of interest is 8% per annum, interest being compounded annually, then each instalment is (in ₹)

(a) 3645 (b) 2916
(c) 2542 (d) 1980

7. A book is available for ₹800 cash down or for ₹250 down payment followed by 3 equal monthly instalments of ₹200 each. Find the principal for the third month.

(a) ₹550 (b) ₹350
(c) ₹150 (d) ₹250

8. An article is available for ₹6000 cash down or for ₹1275 down payment and 5 equal monthly instalments. If the rate of interest is 4% per month, then each monthly instalment is (in ₹ approximately).

(a) 1200 (b) 954
(c) 1050 (d) 875

9. An article is available for ₹24,000 cash down or for a certain down payment followed by six equal instalments of ₹2800 each. If the rate of interest is 12% per annum, find the down payment for purchasing it (approximately).

(a) ₹7500 (b) ₹7605
(c) ₹7755 (d) ₹7800

10. A sum of ₹64,890 is to be paid back in 3 equal annual instalments. If the interest is compounded annually at the rate of $6\frac{2}{3}\%$ per annum, then each instalment is (in ₹)

(a) 26,476 (b) 25,326
(c) 22,600 (d) 24,576

Level 2

11. A loan has to be returned in two equal annual instalments. If the rate of interest is 16% per annum, interest being compounded annually and each instalment is ₹6728, then the total interest is (in ₹)

(a) 4000 (b) 3250
(c) 3600 (d) 2656

12. A loan of ₹15,580 is to be paid back in two equal half-yearly instalments. If the interest is compounded half-yearly at 10% per annum, then the interest is (in ₹)

(a) 1200 (b) 1500
(c) 1178 (d) 1817

13. A loan has to be returned in two equal annual instalments. If the rate of interest is 15% per annum, interest being compounded annually and each instalment is ₹3703, then the principal of the loan is (in ₹)

(a) 7090 (b) 6020
(c) 5090 (d) 8040

14. A refrigerator is available at a certain price on full payment or for ₹1400 down payment and five equal monthly instalments of ₹1030 each. If the rate of interest is 12%, find the cost of the refrigerator approximately.

(a) ₹6000 (b) ₹9009
(c) ₹8008 (d) ₹6403

15. A sofa set is available for ₹50,000 cash down or for ₹30,000 down payment followed by 4 equal monthly instalments of ₹6000 each. Find the principal for the second month.

(a) ₹20,000 (b) ₹18,000
(c) ₹5000 (d) ₹14,000

16. A mobile phone is available for ₹2180 cash down or for a certain down payment followed by three equal monthly instalments of ₹600 each. The annual rate of interest is $57\frac{1}{7}\%$. Find the down payment to be paid (in ₹) if the interest paid is at simple interest.

(a) 480 (b) 530
(c) 580 (d) 640

17. A loan of ₹3640 is to be repaid in three equal half-yearly instalments. The rate of interest is 40% per annum, interest being compounded semi-annually. Find each instalment. (in ₹)

(a) 1440 (b) 1728
(c) 15,334 (d) 1487

18. Anil borrowed a certain sum from a bank at R% per annum compound interest, interest being compounded annually. He repaid it in two equal annual instalments. The values of the first and second instalments, when the second instalment was paid, were in the ratio 5 : 4. Find R.

(a) 20 (b) 15
(c) 25 (d) 30

19. A TV is available at a certain price on full payment or for ₹2000 down payment and two equal quarterly instalments of ₹8640 each. The shopkeeper charged interest at 20% per quarter, interest being computed quarterly. Find the cost of the TV. (in ₹)

(a) 15,200 (b) 15,800
(c) 16,400 (d) 16,800

20. A gold ring is available for ₹10,220 cash payment or ₹3600 down payment and three equal annual instalments. The shopkeeper charged interest at 10% per annum, interest being compounded annually. Find each instalment. (in ₹)

(a) 2684 (b) 2618
(c) 2574 (d) 2662

Level 3

21. A car is available for ₹200,000 cash or for ₹50,000 cash down payment followed by 4 equal half-yearly instalments of ₹52,000 each. Find the total interest charged.

(a) ₹58,000
(b) ₹48,000
(c) ₹44,000
(d) ₹50,000

22. A microwave oven is available for ₹4500 cash down or for ₹2100 down payment followed by three equal monthly instalments. If the shop keeper charges interest at the rate of 10% per annum, compounded every month, find the total of the present values of the three instalments.

(a) ₹1800 (b) ₹2400
(c) ₹4500 (d) ₹2100

23. A gold chain is available for ₹10,500 cash down or for ₹4000 down payment and three equal instalments. If the shopkeeper charges interest at the rate of 10% per annum, compounded annually, find the total of the present value of the three instalments.

(a) ₹8000 (b) ₹10,000
(c) ₹6500 (d) ₹4000

24. Alok borrowed some money from Prasad which is to be paid back in 3 equal annual instalments of ₹322 each. The present value of the first installment paid at the end of first year is ₹280. Find the rate of interest if the interest is being compounded annually.

(a) 12% (b) 18%
(c) 15% (d) 20%

25. Find the principal if it is lent at 5% per annum, interest being compounded annually, in two parts. One part amounts to ₹1050 at the end of the 1st year and the other part amounts to ₹1323 at the end of the 2nd year.

(a) ₹1900 (b) ₹2000
(c) ₹1800 (d) ₹2200

26. Each of Ganesh and Harish bought a car. Each bought it by paying two half-yearly instalments under the same rate of compound interest, interest being compounded half-yearly. Ganesh's first instalment was equal to Harish's second instalment and vice-versa. The present values of Ganesh's instalments were the same. Which person's car had the lower cash price?

(a) Harish's
(b) Ganesh's
(c) Both cars had the same price
(d) Cannot say

27. A double cot is available for ₹13,000 cash down or for ₹1300 down payment followed by four equal monthly instalments of ₹3000 each. Find the principal for the third month, if interest charged at simple interest. (in ₹)

(a) 5200 (b) 4900
(c) 6200 (d) 5700

28. In the previous question, the difference in Ganesh's and Harish's first instalments was ₹4410 and the rate of the interest was 10% per annum. Find the difference in the cost of the two cars.

(a) 240 (b) 400
(c) 200 (d) 320

29. A car is available for a certain price or four yearly instalments of ₹146,410 each. The rate of interest is 10% per annum, interest being compounded annually. Find the cost of the car. (in lakhs)

(a) 4.641 (b) 4.583
(c) 4.727 (d) 4.799

30. Bhavan borrowed a certain sum of money. He had to repay it in three equal annual instalments at a fixed rate of compound interest. The present values of his first and second instalments were ₹16,900 and ₹13,000 respectively. Find the present value of his third instalment. (in ₹)

(a) 8000 (b) 19,000
(c) 10,000 (d) 11,000

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. down payment
2. instalment scheme
3. ₹100
4. interest
5. vendor/seller.
6. rent.
7. ₹10,000
8. ₹1600
9. ₹6600
10. ₹400

Short Answer Type Questions

11. ₹800
12. 57.14% per annum
13. ₹7554.
14. ₹35,000
15. ₹6050

Essay Type Questions

16. ₹9020
17. ₹2900
18. ₹4900
19. ₹7000
20. ₹15,000

CONCEPT APPLICATION

Level 1

1. (d) **2.** (d) **3.** (b) **4.** (a) **5.** (c) **6.** (a) **7.** (c) **8.** (b) **9.** (c) **10.** (d)

Level 2

11. (d) **12.** (c) **13.** (b) **14.** (d) **15.** (d) **16.** (b) **17.** (b) **18.** (c) **19.** (a) **20.** (d)

Level 3

21. (a) **22.** (b) **23.** (c) **24.** (c) **25.** (d) **26.** (b) **27.** (d) **28.** (c) **29.** (a) **30.** (c)

CONCEPT APPLICATION

Level 1

1. Calculate the interest and principle amount for 2 months.
   (i) Assume each installment amount as ₹x.
   (ii) Calculate the balance, interest and the principle.

2. (i) Let the down payment amount be ₹x.
   (ii) Calculate balance and interest.
   (iii) Let the down payment be ₹x.
   Balance = ₹$(970 - x)$.
   (iv) Interest paid under instalment scheme $= ₹[3 \times 260 - (970 - x)] = ₹[x - 190]$.
   (v) Principal for the 1st month = ₹$(970 - x)$. Principal for the 2nd month = ₹$(970 - x - 260)$. Principal for the 3rd month = ₹$(970 - x - 260 - 260)x$.
   (vi) Use SI $= \dfrac{PRT}{100}$, (P = total principal).

4. Find the interest paid under instalment scheme.

5. (i) Let the cost of ceiling fan in cash be ₹x.
   (ii) Calculate balance, interest and the principle amount.
   (iii) Let the cash price be ₹x.
   (iv) Balance = ₹$(x - 250)$.
   (v) Interest = ₹$[350 - (x - 250)]$.
   (vi) Use, SI $= \dfrac{PRT}{100}$, P = Balance,
   $R = 10\%$ and $T = \dfrac{2}{12}$ years.

6. (i) Principal $= \dfrac{\text{Installment}}{\left(1+\dfrac{r}{100}\right)^n}$.
   (ii) Find the principal value for both the instalments and equate to the total sum.

7. (i) Calculate the balance.
   (ii) Calculate the principle for the first, 2nd and the third months successively.
   (iii) Use, Total principal = Each instalment value $\left[\dfrac{1}{\left(1+\dfrac{R}{100}\right)} + \dfrac{1}{\left(1+\dfrac{R}{100}\right)^2} + \dfrac{1}{\left(1+\dfrac{R}{100}\right)^3}\right]$ and evaluate R.

8. (i) Let the cash down payment be ₹x.
   (ii) Down payment = 45% of 9000 and each instalment = 20% of 9000.
   (iii) Calculate the amount to be paid in the each instalment.
   (iv) Equate the above value to the amount that has to be paid and find R.

9. (i) Let the cash down payment be ₹x.
   (ii) Use, Total present value = Each instalment value $\left[\dfrac{1}{\left(1+\dfrac{R}{100}\right)} + \dfrac{1}{\left(1+\dfrac{R}{100}\right)^2}\right]$ and evaluate R.

10. Refer to the hint of Q. No. 4.

Level 2

11. (i) Refer to the hint of Q. No. 6.
    (ii) First of all amount paid in instalment is to be found.
    (iii) Calculate the amount to be paid in the each instalment.
    (iv) Equate the above value to the amount that has to be paid and find R.

12. Refer to the hint of Q. No. 4.

13. Refer to the hint of Q. No. 4.

14. Let the cost of refrigerator in full payment be ₹x.

16. Let the down payment to be paid be ₹x cash price = ₹2180. The balance is ₹$2180 - x$.
    The instalments are paid at the end of the first, second and third months.
    The values of instalments at the end of the first, the second and the third month are (in ₹)
    $600\left(1+\dfrac{(2)(R)}{1200}\right)$, $600\left(1+\dfrac{(1)(R)}{1200}\right)$ and 600 respectively,

HINTS AND EXPLANATION

where $R = 57\frac{1}{7}$.

$$600\left(1+\frac{2R}{1200}\right)+600\left(1+\frac{R}{1200}\right)+600$$

$$=(2180-x)\left(1+\frac{3R}{1200}\right)$$

$$\Leftrightarrow \frac{1800+1.5R}{1+\frac{R}{400}}=(2180-x)$$

But $R = 57\frac{1}{7} = \frac{400}{7}$

$$\therefore \frac{1800+1.5\left(\frac{400}{7}\right)}{1+\frac{400}{7}\left(\frac{1}{400}\right)} = 2180-x \; \frac{\frac{13200}{7}}{\frac{8}{7}} = 2180-x$$

$1650 = 2180 - x$

$x = 530.$

17. Let each instalment be ₹i

Rate of interest = 40% per annum = 20% half-yearly.

The present values of the first, the second and the third instalments are ₹$\frac{i}{1+\frac{20}{100}}$, ₹$\frac{i}{\left(1+\frac{20}{100}\right)^2}$ and ₹$\frac{i}{\left(1+\frac{20}{100}\right)^3}$ respectively.

$$\frac{i}{1.2}+\frac{i}{(1.2)^2}+\frac{i}{(1.2)^3}=3640$$

$$\frac{(1.2)^2+1.2+1}{(1.2)^3}i=3640$$

$$\frac{(1.44+1.2+1)}{1.728}i=3640$$

That is, $\frac{3.640}{1.728}i = 3640$

$i = 1728.$

18. Let each instalment be ₹i

Present value of the first instalment =₹$\frac{i}{1+\frac{R}{100}}$

Present value of the second instalment

$$=₹\frac{i}{\left(1+\frac{R}{100}\right)^2}$$

$$\frac{i}{1+\frac{R}{100}} : \frac{i}{\left(1+\frac{R}{100}\right)^2} = 5:4$$

$$1+\frac{R}{100}=\frac{4}{5} R = 25.$$

19. Let the cash price be ₹x

Down payment = ₹2000

Balance to be paid by instalments = ₹(x – 2000)

Present value of the first instalment

$$=₹\frac{8640}{1+\frac{20}{100}}=₹\frac{8640}{1.2}=₹7200$$

Present value of the second instalment

$$=₹\frac{8640}{\left(1+\frac{20}{100}\right)^2}=₹\frac{8640}{1.44}=₹6000$$

Total of the present values = ₹13,200

x – 2000 = 13,200

x = 15,200.

20. Cash price = ₹10,220

Down payment = ₹3600

Balance to be paid by instalments (in ₹) = 10,220 – 3600 = 6620

Let each instalment be ₹i

Present value of the first instalment $=\frac{i}{1+\frac{R}{100}}$

Present value of the second instalment

$$=\frac{i}{\left(1+\frac{R}{100}\right)^2}$$

Present value of the third instalment $=\frac{i}{\left(1+\frac{R}{100}\right)^2}$

$$\frac{i}{1.1}+\frac{i}{(1.1)^2}+\frac{i}{(1.1)^3}=6620$$

$$\frac{i(1.1^2+1.1+1)}{(1.1)^3}=6620$$

$$\frac{i(1.21+1.1+1)}{1.331}=6620,$$

$$\text{i.e., } \frac{i(3.31)}{1.331}=6620$$

$$i = 2662.$$

Level 3

21. (i) Interest = (Total amount paid in installments) – (Principal).

(ii) Use, Total present value

$$= \frac{4400}{\left(1+\frac{R}{100}\right)} + \frac{4300}{\left(1+\frac{R}{100}\right)^2} \text{ and evaluate } R.$$

22. (i) Refer to the hint of Q. No. 6.

(ii) Use, Present value at the end of the second year

$$= \frac{\text{Each instalment value}}{\left(1+\frac{R}{100}\right)^2} \text{ and evaluate } R.$$

23. Interest = (Total amount paid in installments) – (Principal)

24. (i) Principal $= \dfrac{\text{Installment}}{\left(1+\frac{r}{100}\right)^n}$

(ii) Use, Total present value = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)} + \frac{1}{\left(1+\frac{R}{100}\right)^2}\right] \text{ and evaluate } R.$$

25. Use, Total principal = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)} + \frac{1}{\left(1+\frac{R}{100}\right)^2} + \frac{1}{\left(1+\frac{R}{100}\right)^3}\right] \text{ and}$$

evaluate R.

26. Ganesh's (G) and Harish's (H) first and second instalments are tabulated below

	1	2
G	Q	Q(1 + r)
H	Q(1 + r)	Q

As the present values of G's instalments are equal, if his first instalment are equal if his first instalment is Q, the second is Q(1 + r), where r is the half-yearly rate of interest.

The total value of G's instalments (when the second is paid) is $Q(1 + r) + Q(1 + r) = Q(2 + 2r)$

The total value of H's instalment is $Q(1 + r)^2 + Q = Q(2 + 2r + r^2)$

∴ The cost of Ganesh's car is lower.

27. Cash price = ₹13,000

Down payment = ₹1300

Balance to be paid by instalment (in ₹)

= 13,000 – 1300 = 11,700.

∴ Principal for the first month = ₹11,700

Principal for the second month (in ₹) = 11,700 – 3000 = 8700

Principal for the third month (in ₹) = 8700 – 3000 = 5700.

28. The difference in G's and H's first instalment is Qr.

The cost of G's car is $\dfrac{2Q}{(1+r)}$

The cost of H'S car is

$$Q + \frac{Q}{(1+r)^2} = \frac{Q}{(1+r)^2}[2+2r+r^2]$$

∴ The difference is

$$\frac{Q}{(1+r)^2}\,[(2 + 2r + r^2) - (2 + 2r)]$$

$$= Qr\left[\frac{r}{(1+r)^2}\right] = ₹4410\frac{(0.05)}{(1.05)(1.05)} = ₹200.$$

29. Let the price of the car be ₹p and each of the 4 instalments be Q. (Q = ₹146,410)

$$P(1 + r)^4 = Q[(1 + r)^3 + (1 + r)^2 + (1 + r) + 1]$$

where $r = 0.1$ per annum = 10% per annum

$$\therefore P = Q\frac{[1+1.1+1.21+1.331]}{1.4641}$$

$$= 146,410\left(\frac{4.641}{1.4641}\right) = 464,100.$$

30. Let the each instalment be ₹Q and let the annual rate of interest be r (or $100r$ % per annum).

The value of the first instalment (at the time of borrowing) $= \dfrac{Q}{1+r} = 16,900$ (1)

The value of the second (at the time of borrowing)

$$= \frac{Q}{(1+r)^2} = 13,000 \quad (2)$$

The value of the third is $\dfrac{Q}{(1+r)^3}$.

Eq. (1) ÷ Eq. (2)

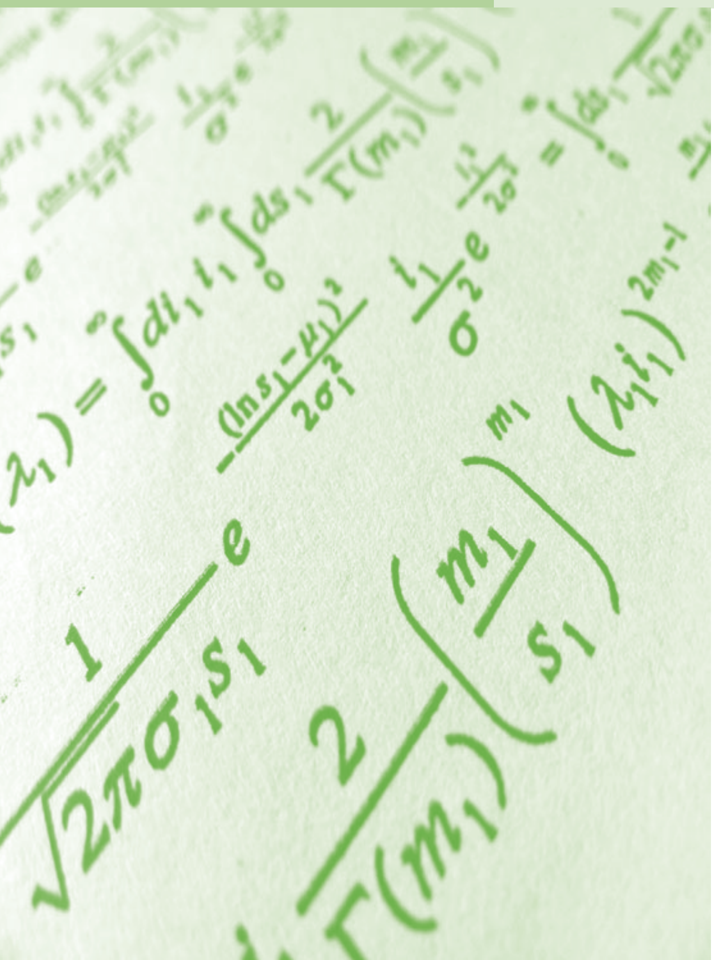
$\Rightarrow 1 + r = 1.3$

$$\therefore \frac{Q}{(1+r)^3} = \frac{Q}{(1+r)^2}\,\frac{1}{(1+r)} = \frac{13000}{1.3} = 10,000.$$

Chapter

25

Shares and Dividends



REMEMBER

Before beginning this chapter, you should be able to:

- Recall some basic concepts related to shares and dividends.
- Understand the term dividend.

KEY IDEAS

After completing this chapter, you would be able to:

- Study classification of share capital.
- Learn about nominal value and market value.
- Understand about dividend in detail.
- Find return on investments.

INTRODUCTION

The total amount of money required to start a business enterprise or a company is called the **'capital'**. A business enterprise may be a sole proprietary concern, a partnership firm, or a company.

The capital received to start a significantly big company runs into several crores of rupees. A group of entrepreneurs called **'promoters'** start a company, but they cannot possibly invest huge amount capital. Hence, the whole capital is divided into many small and equal units. These units are called 'shares'. Later, the company invites the public to invest money in its business by buying these shares. All companies cannot raise capital from the public. Only public limited companies can exercise this right.

The money arranged and used for setting up a company or a business enterprise is called the **'investment'**. Those who arrange this money are called **'investors'**.

A public limited company is a company limited by shares with no restriction on the maximum number of shareholders, transfer of shares, and acceptance of public deposits. The minimum number of shareholders is seven.

CLASSIFICATION OF SHARE CAPITAL

Authorised capital is the capital limit authorised by the registrar of companies. It is the specified amount up to which shares can be issued to the members or public. The paid-up share capital is the paid portion of the capital subscribed by the shareholders. The minimum limit of authorised capital for a public limited company is ₹500,000.

In other words, **authorised capital** is the capital with which a company is registered. The company can issue shares up to the maximum level of its authorized capital. Paid-up capital means the amount of capital actually paid by the shareholders in respect of shares allotted to them.

The following example gives a clear idea of the division of capital into shares.

Example: Suppose the capital requirement of a company is ₹15 crores.

Then, the whole capital can be divided into:

1. 15 lakh shares of ₹100 each or
2. 30 lakh shares of ₹50 each or
3. 60 lakh shares of ₹25 each.

For every investment, the company issues a share certificate reflecting the value of each share and the number of shares held by the investors.

A person who is allotted shares or who buys shares is called a 'shareholder'. After the shares are allotted, the shareholder can sell those in the stock market.

Nominal Value of a Share (NV)

Nominal value of a share is the value printed on the share certificate. It is also known as the 'face value' (FV) or 'par value'.

Market Value of a Share (MV)

Market value is the value of a share at which it is bought or sold in the stock market.

Notes A share is said to be:

1. At premium or above par, if MV > FV.
2. At discount or below par, if MV < FV.
3. At par, if MV = FV.

Dividend

Dividend is a part of the annual profit distributed among the shareholders.

Dividend is paid on per share or as a percentage. If it is paid as a percentage, it is reckoned on the face value of the share.

Notes

1. The face value of a share always remains the same.
2. The market value of a share changes from time to time.
3. Dividend is always paid on the face value of a share.
4. The number of shares held by a person

$$= \frac{\text{Total income}}{\text{Income from each share}} = \frac{\text{Total face value}}{\text{Face value of each share}}.$$

Return on Investment

$$\text{Rate of return} = \frac{\text{Annual income}}{\text{Total investment}} \times 100$$

EXAMPLE 25.1

Find the market value of ₹200 share bought at a premium of ₹50.

SOLUTION

Face value of the share = ₹200

The share was bought at a premium of ₹50.

∴ Market value of the share = ₹(200 + 50) = ₹250.

EXAMPLE 25.2

If a ₹100 share is available at a discount of ₹10, then find the market value of 270 such shares.

SOLUTION

Face value of the share = ₹100

The share is available at a discount of ₹10

Market value of each share = 100 − 10 = ₹90

∴ Market value of 270 such shares = 90(270) = ₹24,300.

EXAMPLE 25.3

Satish invests a certain amount of money in 200, ₹60 shares of a company by paying a dividend of 8%. Find his annual income from the investment.

SOLUTION

Face value of the share = ₹60

Number of shares bought = 200

Rate of dividend paid = 8%

Annual income from each share $= \frac{8}{100}(60) = \frac{480}{100} =$ ₹4.80.

$\therefore$ Annual income Satish earned from 200 shares = 4.8 × 200 = ₹960.

EXAMPLE 25.4

Karan invests ₹18,000 by buying ₹100 shares of a company available at a discount of ₹10. If the company pays a dividend of 10%, then find the number of shares bought by Karan, and the rate of return on his investment.

SOLUTION

Total investment = ₹18,000

Face value of each share = ₹100

The shares are available at a discount of ₹10.

Market value of each share = 100 − 10 = ₹90.

Number of shares bought $= \frac{\text{Total investment}}{\text{Market value of each share}} = \frac{18,000}{90} = 200$

$\Rightarrow$ Number of shares bought by Karan = 200

Rate of dividend = 10%

His annual income from each share = 10% (100) = ₹10

$\Rightarrow$ His annual income from 200 shares = 200 (10) = ₹2000

$$\Rightarrow \text{The rate of return on his investment} = \frac{\text{Annual income}}{\text{Total investment}}(100) = \frac{2000}{18,000}(100)$$

$$= \frac{100}{9}\% = 11\frac{1}{9}\%$$

$\therefore$ Karan receives a return of $11\frac{1}{9}\%$ per annum on his investment.

Alternative solution:

We have, FV × Rate of dividend = MV × Rate of return.

100 × 10% = 90 × Rate of return

$\Rightarrow$ Rate of return $= 11\frac{1}{9}\%$.

EXAMPLE 25.5

Which is a better investment: ₹200 shares at ₹220 which pay a dividend of 10%, or ₹100 shares at ₹120 which pay a dividend of 12%?

SOLUTION

Case 1: ₹200 shares at ₹220 paying a dividend of 10%

$$\text{Rate of return} = \frac{\text{FV} \times \text{Rate of dividend}}{\text{MV}} = \frac{200 \times 10\%}{220} = 9\frac{1}{11}\%.$$

Case 2: ₹100 shares at ₹120 paying a dividend of 12%

$$\text{Rate of return} = \frac{\text{FV} \times \text{Rateof dividend}}{\text{MV}}$$

$$= \frac{100 \times 12\%}{120} = 10\%.$$

∴ 12%, ₹100 shares at ₹120 is a better investment.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. The whole capital divided into small and equal units are called _______.
2. The value of a share printed on the share certificate is called its _______.
3. Dividend is always calculated on the _______ of a share.
4. _______ is the annual profit distributed among the shareholders.
5. The value at which a share is bought or sold in the market is called its _______.
6. A person who is allotted shares or who buys shares is called a _______ of the company.
7. Does the face value of a share remain constant all the time?
8. Is the market value of a share always constant?
9. If a ₹180 share is quoted at a premium of ₹27, then the market value of each share is _______.
10. If a ₹100 share is available at par, then its market value is _______.

Short Answer Type Questions

11. If Anil bought a ₹150 share at a premium of ₹20 while Raju bought the same kind of share at a discount of ₹20 of the same company, then the dividend earned by Anil is _______ the dividend earned by Raju. [equal to/less than/more than]
12. Find the market value of 300, ₹150 shares bought at a premium of ₹30.
13. Find the annual income derived from an investment of ₹18,000 in ₹150 shares available at ₹180 of a company paying 11% dividend.
14. ₹60,000 is invested in buying ₹120 shares of a company which are available at a premium of 25%. Find the number of shares bought and the annual rate of return on the investment, if dividend is paid at the rate of 10% per annum.
15. An investment of ₹36 on a share earns the investor a return of 10%. If dividend is 12% on each share, then what is its face value?
16. A man invests ₹20,000 in ₹80 shares of a company available at a premium of ₹20. If the company pays a dividend of 8%, then find the rate of return on the investment.
17. If Satish bought 120, ₹200 shares of a particular company at a premium of ₹24, paying 9% dividend, then find Satish's annual income.
18. If an investment of ₹42,000 in ₹300 shares of a company, paying a dividend of 7%, results in an annual income of ₹2100, then find the market value of each share.
19. A person invested ₹18,000 in buying ₹150 shares of a company which are available at a premium of ₹50. If the company pays a 9% dividend, then find the number of shares bought by him and also the annual income he earned from the investment.
20. Which is a better investment, (A) ₹60 shares at ₹75 paying a dividend of 10% or (B) ₹100 share at ₹120 paying a dividend of 12%?

Essay Type Questions

21. Naresh invested a certain amount of money in ₹100 shares at a discount of ₹20 paying a dividend of 7%, while Sahil invested an equal amount in ₹90 shares at a discount of ₹10 paying a dividend of 8%. Who earns more?
22. How much should a man invest in 12%, ₹150 shares of a company available at a premium of ₹30, if the annual income earned is to be ₹3,600?
23. Rupesh invests ₹45,000 partly in ₹150 shares at ₹180 for 8% and partly in ₹75 shares at ₹135 for 12%. If the annual incomes from the investments are in the ratio 2 : 3 respectively, then find the investments made by Rupesh in the two types of shares.

24. Rahul invested a certain amount in buying ₹25 shares of a company, which pays a dividend of 12%. If he earns 10% per annum on his investment, then find the market value of each share.

25. Kavya bought 300, ₹50 shares paying a dividend of 8%. If she sold them when the price rose to ₹90 and invested the proceeds in 10%, ₹50 shares at ₹30, then find the change in her annual income.

CONCEPT APPLICATION

Level 1

1. If a ₹200 share is bought at a discount of ₹50 and the dividend paid is 9%, then the rate of return is ______ per annum.

(a) 18% (b) 9%

(c) 10% (d) 12%

2. A person invests ₹20,000 at 20%, ₹150 shares at a premium of ₹50. The income from these shares in is ₹ ______.

(a) 1,000 (b) 3,000

(c) 1,500 (d) 2,000

3. If 200, ₹100 shares are bought at a premium of ₹20, then the total investment made is ₹ ______.

(a) 20,000 (b) 40,000

(c) 24,000 (d) 48,000

4. A person invests ₹15,000 in ₹50 shares of a company paying 10% dividend. If the company pays a dividend of ₹6250, the market value of each share is ______.

(a) ₹8 (b) ₹10

(c) ₹12 (d) ₹7

5. A man buys 24 shares at ₹150 per share having the par value of ₹100. If the dividend is 7.5% per annum, then the ratio of total annual income to his total investment is______.

(a) 1 : 10 (b) 10 : 1

(c) 1 : 20 (d) 20 : 1

6. The total investment made in buying ₹X shares at a premium of 25% is ₹125X. The number of shares bought is ______.

(a) 500 (b) 250

(c) 125 (d) 100

7. Praveen invests ₹24,000 in ₹100 shares of a company paying 10% dividend. If his annual income from these shares is ₹1200, then the market value of each share is ₹______.

(a) 100 (b) 200

(c) 300 (d) 400

8. Which of the following is/are true for the statement: '9%, ₹100 shares at ₹120'?

(a) Dividend on 1 share = ₹9

(b) Rate of return is 7.5%

(c) Both (a) and (b)

(d) None of the above

9. A man has 62 shares of ₹100 each, dividend is 7.5% per annum. If he wants to increase his annual income by ₹150, then he should buy ______ more shares.

(a) 20 (b) 40

(c) 10 (d) 30

10. The market value of x, ₹50 shares, at a discount of ₹10 is ₹4000, then x = ______.

(a) 10 (b) 100

(c) 50 (d) 75

Level 2

11. A man buys 500, ₹10 shares at a premium of ₹3 on each share. If the rate of dividend is 12%, then the rate of interest received by him on his money is ________ (approximately).

(a) 6.5% (b) 7.5%

(c) 9.23% (d) 5%

12. Rishi bought ₹50 shares of a company for ₹75 each. The company pays a dividend of 12% per annum.

The effective rate of return on his investments is ________.

(a) 6% (b) 8%

(c) 9% (d) 10%

13. If ₹56,000 is invested in buying ₹120 shares at a discount of ₹$16\frac{2}{3}$%, then find the number of shares bought is ________.

(a) 2,800 (b) 560

(c) 280 (d) 400

14. The market value of a share is ₹90, its face value is ₹80, and the dividend is 8%. The rate of return is __________.

(a) $8\frac{1}{3}$% (b) $7\frac{1}{9}$%

(c) $6\frac{2}{3}$% (d) $5\frac{2}{3}$%

15. Ankit invests ₹18,000 in buying ₹270 shares in a company at a premium of ₹30. If the dividend paid is 10% per annum, then Ankit's annual income from these shares is __________.

(a) ₹162 (b) ₹1620

(c) ₹16.20 (d) ₹16,200

16. Which is a better investment: '15%, ₹220 shares at ₹240 of Company A' or '18%, ₹200 shares at ₹240 of Company B'?

(a) Company A

(b) Company B

(c) Both (a) and (b)

(d) Cannot say

17. Mr Gellard invested ₹16,000 in 7%, ₹200 shares at ₹160. He sold the shares at ₹150 each. He incurred a ________.

(a) loss of ₹1300

(b) loss of ₹1000

(c) profit of ₹1000

(d) profit of ₹1200

18. Priya invested ₹27,000 in ₹180 shares at ₹150 at the beginning of a financial year. The company paid a dividend of 12%. Priya sold all her shares at the end of the year. Over this period, her net earnings were ₹1828. She sold the shares for ________each.

(a) ₹160.15 (b) ₹181.75

(c) ₹185 (d) ₹138.56

19. Teja brought ₹80 shares of a company which pays 15% dividend. If the rate of return is 12%, then the market price is ________.

(a) ₹90 (b) ₹80

(c) ₹100 (d) ₹99

20. Randeep invests ₹25,200 in buying shares of face value of ₹40 each at 5% premium. The dividend on these shares is 10% per annum. Find the dividend he receives annually.

(a) ₹2520 (b) ₹2400

(c) ₹4200 (d) ₹3680

21. Mr Prudhvi invested ₹16,900 in buying shares worth ₹100 at a premium of ₹30. Find the number of shares bought by him.

(a) 169 (b) 150

(c) 130 (d) 129

22. The dividend received from 10% ₹125 shares at ₹150, is ₹1875. Find the number of shares.

(a) 125 (b) 150

(c) 270 (d) 135

23. Find the market value of 200 shares worth ₹120, each bought at a discount of ₹20. (in ₹)

(a) 28,000 (b) 24,000

(c) 26,000 (d) 20,000

24. Ms Meenakshi invested ₹32,400 in buying certain shares of a company. If the amount of dividend received by her is ₹4860, then find the rate of return.

(a) 15% (b) 12%

(c) 14% (d) 13%

25. Ms Shriya bought 1000 shares worth ₹500 each at 12% premium. Find the market value of each share.

(a) ₹612 (b) ₹560

(c) ₹760 (d) ₹840

PRACTICE QUESTIONS

Level 3

26. Which of the following is a better investment?

(A) 10%, ₹100 shares at ₹120.

(B) 9%, ₹100 shares at ₹110.

(a) A

(b) B

(c) Both (A) and (B)

(d) Data insufficient

27. Vamsi invested some amount buying ₹150 shares of a company which pays dividend at the rate of 8% per annum. If he gets back 10% per annum on his investment, then the value at which he bought the shares is ______.

(a) ₹100 (b) ₹120

(c) ₹180 (d) ₹210

28. Mr Sirkar bought ₹80 shares of a company at a premium of ₹10. If the company pays a dividend at the rate of x% per annum, and Mr Sirkar earns at the rate of y% per annum on his investment, then $x : y$ is ______.

(a) 1 : 8 (b) 8 : 1

(c) 8 : 1 (d) 9 : 8

29. Sachin invests ₹15,000 on buying ₹100 shares of a company at a premium of ₹50 and gets $6\frac{2}{3}$% per annum on his investment. Then the rate at which the company pays the dividend is ______.

(a) 5% (b) 8%

(c) 9% (d) 10%

30. Praveen made an investment of ₹54,000 in 10%, ₹100 shares at ₹90, while Vijay made an investment of ₹60,000 in 8%, ₹150 shares at ₹120. If both of them sold their shares at the end of the year for ₹110 each, then their individual earnings are ______, respectively.

(a) ₹12,000 and ₹12,000

(b) ₹18,000 and ₹1000

(c) ₹12,000 and ₹1000

(d) ₹18,000 and ₹1200

31. If ₹25,000 more invested, then 400 more shares can be purchased. Find the market value of each share.

(a) ₹62.50

(b) ₹77.50

(c) ₹85.00

(d) Cannot be determined

32. Which of the following is/are true for the statement: '15%, ₹120 shares at ₹200'?

(a) Face value of each share is ₹120.

(b) Market value of each share is ₹200.

(c) Dividend from each share is ₹18.

(d) All of these

33. Find the rate of return from 'd% ₹f at ₹m'. (in %)

(a) fd/m

(b) md/f

(c) fm/d

(d) None of these

34. Shares worth ₹150 each were bought at ₹180. Which of the following is necessarily true?

(a) Each share was bought at ₹30 discount.

(b) Each share was bought at ₹30 premium.

(c) Each share was bought at 30% discount.

(d) Each share was bought at 30% premium.

35. Mrs Pravallika has 100 shares worth ₹100 at a premium of ₹100. If the rate of dividend is 10%, then find the total dividend received by her at the end of a year. (in ₹)

(a) ₹4000 (b) ₹3000

(c) ₹2000 (d) ₹1000

36. Mr Anil bought ₹110 shares of a company at a discount of ₹10. If the company pays a dividend at the rate of x% per annum, and Mr Anil earns at the rate of y% per annum on his investment, then $x : y$ is ______.

(a) 10 : 9 (b) 9 : 10

(c) 11 : 10 (d) 10 : 11

37. Mr Balu invested ₹15,000 in 10%, ₹100 shares at ₹120, and also invested ₹24,000 in 12%, ₹120 shares at ₹100. The total earnings from the two investments is _____.

(a) ₹4500 (b) ₹4706

(c) ₹4700 (d) ₹4509

38. Mrs Apoorva invests ₹12,000 in buying ₹200 shares in a company at a premium of ₹50. If the dividend paid is 10% per annum, then Apoorva's annual income from these shares is _____.

(a) ₹1,020 (b) ₹960

(c) ₹840 (d) ₹780

39. Mr Krish made an investment of ₹27,000 in 10%, ₹100 shares at ₹90 while Mr Ram made an investment of ₹30,000 in 8%, ₹150 shares at ₹120. If both of them sold their shares at the end of a year for ₹110 each, then their individual earnings are _____, respectively.

(a) ₹2000 and ₹1200

(b) ₹1800 and ₹1000

(c) ₹9000 and ₹500

(d) ₹1800 and ₹1200

40. Mrs Karuna invested some amount in buying ₹120 shares of a company which pays a dividend at the rate of 10% per annum. If she gets back 8% per annum on her investment, then the value at which she bought the shares is _____.

(a) ₹100 (b) ₹150

(c) ₹180 (d) ₹210

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. shares
2. face value (or) nominal value (or) par value
3. face value
4. Dividend
5. market value
6. shareholder
7. Yes
8. No
9. ₹207
10. ₹100

Short Answer Type Questions

11. equal to
12. ₹54,000
13. ₹1650
14. 400; 8%
15. ₹30
16. 6.4%
17. ₹2160
18. ₹420
19. 90; ₹1215
20. ₹100 shares at ₹120 paying a dividend of 12%

Essay Type Questions

21. Sahil
22. ₹36,000
23. ₹18,000 and ₹27,000.
24. ₹30
25. ₹3300

CONCEPT APPLICATION

Level 1

1. (d) **2.** (b) **3.** (c) **4.** (c) **5.** (c) **6.** (d) **7.** (b) **8.** (c) **9.** (a) **10.** (b)

Level 2

11. (c) **12.** (b) **13.** (b) **14.** (b) **15.** (b) **16.** (b) **17.** (b) **18.** (d) **19.** (c) **20.** (b)
21. (c) **22.** (b) **23.** (d) **24.** (a) **25.** (b)

Level 3

26. (a) **27.** (b) **28.** (d) **29.** (d) **30.** (b) **31.** (a) **32.** (d) **33.** (a) **34.** (b) **35.** (d)
36. (d) **37.** (b) **38.** (b) **39.** (c) **40.** (b)

ANSWER KEYS

CONCEPT APPLICATION

Level 1

1. Rate of dividend × FV = Rate of return × MV
2. (i) First of all, the annual income from one share is to be found.
   (ii) MV = FV + Premium.
   (iii) Find the number of shares.
   (iv) $\text{Income} = \dfrac{(\text{Number of shares} \times \text{FV}) \times (\text{Rate of dividend})}{100}$.
3. Investment = MV × (Number of shares)
4. (i) $\text{Rate of return} = \dfrac{\text{Annual income}}{\text{Total investment}} \times 100$.
   (ii) By using the following formula find MV.
   Use, FV × Rate of dividend = MV × Rate of return.
5. (i) Require rates = Annual income = MV of 1 share.
   (ii) $\text{Income from 1 share} = \dfrac{(\text{FV} \times \text{Rate of dividend})}{100}$.
   (iii) Then find the total income from 24 shares.
   (iv) MV = (150 + 100)
   Total investment = MV × 24.
6. $\text{Number of share} = \dfrac{\text{Total investment}}{\text{MV}}$.
7. (i) $\text{MV} = \dfrac{\text{Total investment}}{\text{Number of shares}}$
   (ii) $\text{Rate of return} = \dfrac{\text{Annual income}}{\text{Total investment}} \times 100$.
   (iii) By using the following formula find MV.
   Use, FV × Rate of dividend = MV × Rate of return.
9. Find annual income from one share.
11. (i) Rate of dividend × FV = MV × Rate of return
   (ii) In the given data, rate of interest is same as the rate of return.
   (iii) FV × Rate of dividend = MV × Rate of return.

Level 2

12. (i) Same as Question No. 6
   Use the following formula to find the rate of return.
   FV × Rate of dividend = MV × Rate of return.
13. $\text{Number of shares} = \dfrac{\text{Amount invested}}{\text{MV}}$.
14. MV × Rate of return = Rate of dividend × FV.
15. First of all, the number of shares is to be found.
16. (i) Find the dividends of the two companies.
   (ii) Find the rate of return in two cases by using the following formula.
   FV × Rate of dividend = MV × Rate of return.
   (iii) The one with greater rate of return is better.
17. (i) $\text{Number of shares} = \dfrac{\text{Total investment}}{\text{MV}}$.
   (ii) $\text{Number of shares} = \dfrac{\text{Total investment}}{\text{MV of each share}}$.
   (iii) SP of total shares = ₹150 × Number of shares.
18. (i) $\text{Number of shares} = \dfrac{(\text{Total investment})}{(\text{MV of each share})}$.
   (ii) Find the dividend paid on all the shares.
   (iii) $\text{SP} = \dfrac{(27,000 + 1828 - \text{Dividend})}{\text{Number of shares}}$.
19. MV × Rate of return = Rate of dividend × FV.
21. Total investment = ₹16,900
   FV = ₹100
   Premium = ₹30
   MV = FV + Premium = ₹130
   The number of shares
   $= \dfrac{\text{Total investment}}{\text{MV}} = \dfrac{16,900}{130} = 130.$

HINTS AND EXPLANATION

22. FV = ₹125

MV = ₹150

Rate of dividend = 10%

Dividend from each shares = 10% of ₹125

= ₹12.5

Given, total dividend = ₹1875.

The number of shares $= \frac{1875}{12.5} = 150.$

23. The number of shares = 200

Face value (FV) of each share = ₹120

Discount on each share = ₹20

Market value of each share = FV − Discount

= ₹120 − ₹20

= ₹100

∴ Market value of 200 shares (in ₹) = 200 × 100 = ₹20,000.

24. Total investment = ₹32,400

Dividend = ₹4860

Rate of return $= \frac{4860}{32400} \times 100\% = 15\%.$

25. The number of shares = 1000

FV = ₹500

Premium = 12% of ₹500 = ₹60

MV = FV + Premium = ₹560.

Level 3

27. (i) Same as Question No. 6

Use the following formula to find the MV.

FV × Rate of dividend = MV × Rate of return.

28. MV × Rate of return = Rate of dividend × FV.

29. (i) MV = FV + Premium.

(ii) Use the following formula to find the rate of dividend.

FV × Rate of dividend = MV × Rate of return.

30. (i) Find the dividends received by two persons.

(ii) Number of shares $= \frac{\text{Total investment}}{\text{MV of each share}}.$

(iii) Find their dividends by using FV and rate of dividend.

(iv) Find their profit/loss obtained by selling each at ₹110.

31. From the given data, 400 shares can be purchased with an investment of ₹25,000.

∴ Market value of each shares (in ₹)

$= \frac{25,000}{4000} = 62.50.$

32. FV = ₹120

MV = ₹200

Rate of dividend = 15%

Dividend from each share (in ₹) = 15% of 120 = 18

All the given options are true.

33. FV = ₹f

MV = ₹M

Rate of dividend = d%

Rate of return $= \frac{\text{FV} \times \text{Rate of dividend}}{\text{MV}} = \frac{fd}{m}\%.$

34. FV = ₹150

MV = ₹180

Premium = MV − FV = ₹30 is true.

35. FV = ₹100

Rate of dividend = 10%

Number of shares = 100

Dividend from each share (in ₹) = 10% of 100 = 10

Total dividend from 100 shares (in ₹) = 100 × 10 = 1000.

36. Dividend × Face value = Market value × Rate of return

$\Rightarrow 110 \times x = 100 \times y \Rightarrow \frac{x}{y} = \frac{10}{11}.$

37. On the first Investment:

Number of shares $= \frac{15,000}{120} = 125$

Dividend earned $= \left(\frac{10}{100} \times 125\right) 125 = ₹1250$

On the second Investment:

Number of shares $= \frac{24,000}{100} = 240$

$\Rightarrow$ Dividend earned $= 240 \times \frac{12}{100} \times 120 = ₹3456$

Total earnings = ₹4706.

38. Market value = Face value + Premium = 200 + 50 = ₹250

Number of shares purchased

$= \frac{\text{Total investment}}{\text{Market value}} = \frac{12,000}{250} = 48$

Dividend is 10% of ₹200 $= \frac{10}{100} \times 200 = ₹20$

$\Rightarrow$ Total income from 48 shares = 48 × 20

= ₹960.

39. Number of shares bought by Krish $= \frac{27,000}{90} = 300$

$\Rightarrow$ Dividend earned on 300 shares

$= 300 \times \frac{10}{100} \times 100 = ₹3000$

Money received by selling these 30 shares = 300 × 110 = ₹33,000

$\Rightarrow$ Earnings received by Krish = 33,000 + 3000 − 27,000 = ₹9000

Number of shares bought by Vijay

$= \frac{\text{Total Investment}}{\text{Market Value}} = \frac{30,000}{120} = 250$

$\Rightarrow$ Total dividend earned on 500 shares

$= 250 \times \frac{8}{10} \times 150 = ₹3000$

Money received by selling these 500 shares = 250 × 110 = ₹27,500

$\Rightarrow$ Earnings received by Vijay = 27,500 + 3000 − 30,000 = ₹500.

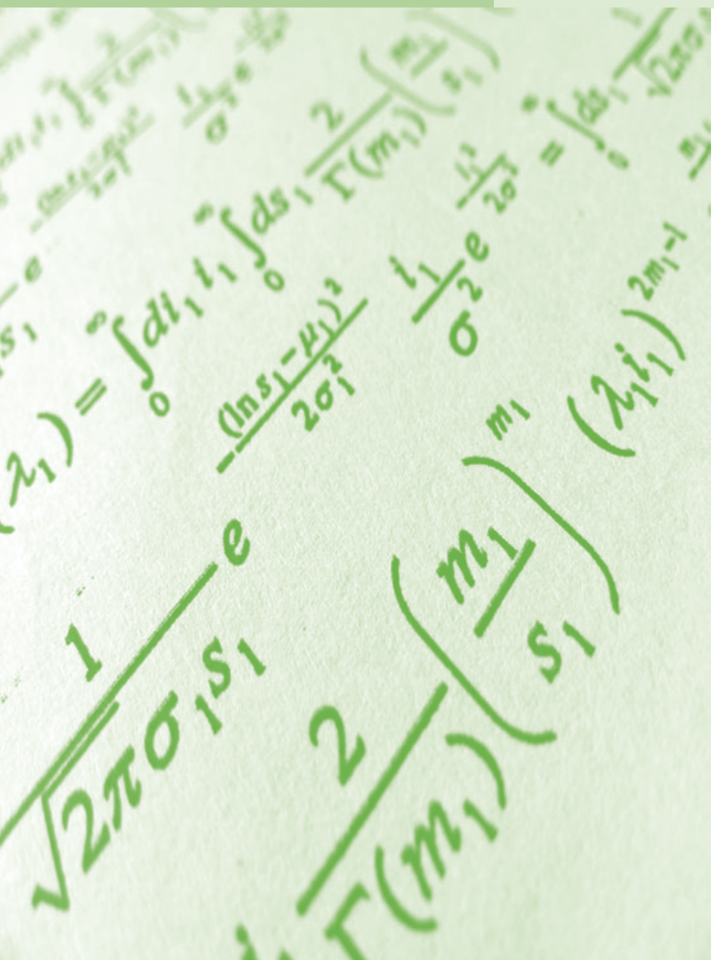
40. Rate of dividend × FV = Rate of return × MV

10 × 120 = 8 × Market value

∴ Market value = ₹150.

Chapter

26 Partial Fractions



REMEMBER

Before beginning this chapter, you should be able to:

- State expressions and equations
- Have knowledge of polynomials

KEY IDEAS

After completing this chapter, you would be able to:

- Learn about rational fractions and partial fractions
- Study various methods to resolve polynomial equations to partial fractions

INTRODUCTION

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$, where $a_0, a_1, \ldots, a_n$ are real numbers and $a_0 \neq 0$ is called a polynomial of degree n.

The quotient of any two polynomials $f(x)$ and $g(x)$ where $g(x) \neq 0$ is called a rational fraction.

In a rational fraction $\frac{f(x)}{g(x)}$, if the degree of $f(x)$ is less than the degree of $g(x)$, then the rational fraction $\frac{f(x)}{g(x)}$ is called a proper rational fraction or simply a proper fraction.

Example: $\frac{2x+3}{3x^2-5x+2}, \frac{3}{x-4}$ are some examples of proper fractions.

If the degree of $f(x)$ is greater than or equal to the degree of $g(x)$, then the rational fraction $\frac{f(x)}{g(x)}$ is called an improper fraction.

Example: $\frac{x^2+2x-1}{2x^2-5x+1}, \frac{3x^3+5x+7}{x^2+6x+5}, \frac{x^2+x+1}{2x+1}$ are some examples of improper fractions.

Note The sum of two proper rational fractions is a proper fraction.

Example: $\frac{2}{x+3}$ and $\frac{5}{2x+1}$ are two proper fractions.

Consider, $\frac{2}{x+3} + \frac{5}{2x+1} = \frac{2(2x+1)+5(x+3)}{(x+3)(2x+1)} = \frac{9x+17}{2x^2+7x+3}$, which is a proper fraction.

That is, a proper fraction may be split into sum of two or more proper fractions, which are also called partial fractions.

The process of expressing a proper fraction as the sum of two or more proper fractions is called resolving it into partial fractions.

Note

If an improper fraction $\frac{f(x)}{g(x)}$ is to split into partial fractions, first we divide $f(x)$ by $g(x)$ till we obtain a remainder $r(x)$, which is of lower degree than $g(x)$ and then we express the given fraction as $\frac{f(x)}{g(x)} = \text{Quotient} + \frac{r(x)}{g(x)}$. Then we resolve the proper fraction $\frac{r(x)}{g(x)}$ into partial fractions.

Now let us discuss about the different methods of resolving a given proper fraction $\frac{f(x)}{g(x)}$ into partial fractions.

Method 1

When $g(x)$ contains non-repeated linear factors only: For every non-repeated linear factor of the form $ax + b$ of $g(x)$, there exists a corresponding partial fraction of the form $\frac{A}{ax+b}$.

EXAMPLE 26.1

Resolve $\dfrac{3x+5}{(x+2)(3x-1)}$ into partial fractions.

SOLUTION

In the given fraction, the denominator has two linear, non-repeated factors.

$\therefore$ The given fraction can be written as the sum of two partial fractions.

Let $\dfrac{3x+5}{(x+2)(3x-1)} = \dfrac{A}{x+2} + \dfrac{B}{3x-1}$

$$\Rightarrow 3x+5 = A(3x-1) + B(x+2) \tag{1}$$

Put $x = -2$ in Eq. (1),

$\Rightarrow 3\ (-2) + 5 = A[3(-2) -1] + B(-2 + 2) -1 = -7A \Rightarrow A = \dfrac{1}{7}$.

Comparing the constant terms on both sides of Eq. (1), we have

$5 = -A + 2B$

$5 = -\dfrac{1}{7} + 2B$ or $B = \dfrac{18}{7}$.

$$\therefore \frac{3x+5}{(x+2)(3x-1)} = \frac{\frac{1}{7}}{x+2} + \frac{\frac{18}{7}}{3x-1} = \frac{1}{7}\left[\frac{1}{x+2} + \frac{18}{3x-1}\right].$$

EXAMPLE 26.2

Resolve $\dfrac{2x^2+5x-1}{x^2-3x-10}$ into partial fractions.

SOLUTION

Given, $\dfrac{2x^2+5x-1}{x^2-3x-10}$

We can clearly notice that the given fraction is an improper fraction. So dividing the numerator by the denominator, we can express $\dfrac{2x^2+5x-1}{x^2-3x-10}$ as $2 + \dfrac{11x+19}{x^2-3x-10}$.

Let $\dfrac{11x+19}{(x+2)(x-5)} = \dfrac{A}{x+2} + \dfrac{B}{x-5}$

$$\Rightarrow 11x + 19 = A(x - 5) + B(x + 2) \tag{1}$$

Put $x = 5$ in Eq. (1) $\Rightarrow 55 + 19 = A(0) + B(7)$

$\therefore B = \dfrac{74}{7}$.

Again put $x = -2$ in Eq. (1) $\Rightarrow -22 + 19 = A(-7) + B\ (0) \Rightarrow A = \dfrac{3}{7}$.

$$\therefore \frac{2x^2+5x-1}{x^2-3x-10} = 2 + \frac{3}{7(x+2)} + \frac{74}{7(x-5)}.$$

Method 2

When $g(x)$ contains some repeated linear factors and the remaining are non-repeated linear factors: Let $g(x) = (px + q)^n (a_{1x} + \alpha_1)(a_{2x} + \alpha_2) \ldots (a_nx + \alpha_n)$, then there exist fractions of the form $\frac{A_1}{Px+q}, \frac{A_2}{(Px+q)^2}, \cdots, \frac{A_n}{(Px+q)^n}$ corresponding to every repeated linear factor and fractions of the form $\frac{B_1}{a_1x+\alpha_1}, \frac{B_2}{a_2x+\alpha_2}, \cdots, \frac{B_n}{a_nx+\alpha_n}$ corresponding to every non-repeated linear factors where $A_1, A_2, \ldots, A_n$, and $B_1, B_2, \ldots, B_n$ are real numbers.

EXAMPLE 26.3

Resolve $\frac{2x^2-5x+7}{(x+1)^2(x+3)(2x+1)}$ into partial fractions.

SOLUTION

Let $\frac{2x^2-5x+7}{(x+1)^2(x+3)(2x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3} + \frac{D}{2x+1}$

$\Rightarrow 2x^2 - 5x + 7 = A(x + 1)(x + 3)(2x + 1) + B(x + 3)(2x + 1) + C(x + 1)^2 (2x + 1)$
$+ D(x + 1)^2 (x + 3)$ (1)

Substituting $x = -1$ in Eq. (1), we have

$2 (-1)^2 - 5 (-1) + 7 = A(0) + B(-1 + 3)(-2 + 1) + C(0) + D(0)\ 14 = -2B$

$B = -7.$

Substituting $x = -3$ in Eq. (1), we get

$40 = A(0) + B(0) + C(-20) + D(0)\ -20C = 40$

$C = -2.$

Substituting $x = -\frac{1}{2}$ in Eq. (1), we have

$10 = \frac{5}{8}D \Rightarrow D = 16.$

Substituting $x = 0$, we have

$7 = 3A + 3B + C + 3D$

Substituting the values of B, C, D in the above equation, we get $A = -6$.

$$\therefore \frac{2x^2-5x+7}{(x+1)^2(x+3)(2x+1)} = \frac{-6}{x+1} + \frac{-7}{(x+1)^2} + \frac{-2}{x+3} + \frac{16}{2x+1}.$$

Method 3

When $g(x)$ contains non-repeated irreducible factors of the form $px^2 + qx + c$: Corresponding to every non-reducible non-repeated quadratic factors of $g(x)$, there exists a partial fraction of the form $\frac{Ax+B}{px^2+qx+c}$, where p, q, A, and B are real numbers.

EXAMPLE 26.4

Resolve $\dfrac{2x-5}{(x+2)(x^2-x+5)}$ into partial fractions.

SOLUTION

Let $\dfrac{2x-5}{(x+2)(x^2-x+5)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2-x+5}$

$\Rightarrow 2x - 5 = A(x^2 - x + 5) + (Bx + C)\ (x + 2)$ (1)

Put $x = -2$ in Eq. (1),

$\Rightarrow -9 = A\ (11) + 0$

$A = \dfrac{-9}{11}$.

Again put $x = 0$ in Eq. (1), we have

$-5 = 5A + 2C$

$-5 = 5 \times \dfrac{-9}{11} + 2C$

$\Rightarrow C = \dfrac{-5}{11}$.

Comparing the coefficients of x^2 on both sides of Eq. (1), we have $A + B = 0$

$\Rightarrow B = -A = \dfrac{9}{11}$.

$$\therefore \frac{2x-5}{(x+2)(x^2-x+5)} = \frac{-9}{11(x+2)} + \frac{\frac{9}{11}x - \frac{5}{11}}{x^2-x+5} = \frac{1}{11}\left[\frac{9x-5}{x^2-x+5} - \frac{9}{x+2}\right].$$

Method 4

When $g(x)$ contains repeated and non-repeated irreducible quadratic factors of the form $(px^2 + qx + r)^n$: Corresponding to every repeated irreducible quadratic factor of $g(x)$ there exist partial fractions of the form $\dfrac{p_1x+q_1}{(px^2+qx+r)}, \dfrac{p_2x+q^2}{(px^2+qx+r)}, \ldots, \dfrac{p_nx+q_n}{(px^2+qx+r)^n}$, where p_1, $p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ are real numbers.

EXAMPLE 26.5

Resolve $\dfrac{2x+1}{(x+3)(x^2+1)^2}$ into partial fractions.

SOLUTION

Let, $\dfrac{2x+1}{(x+3)(x^2+1)^2} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$.

$\Rightarrow 2x + 1 = A(x^2 + 1)^2 + (Bx + C)(x + 3)(x^2 + 1) + (Dx + E)(x + 3)$ (1)

Put $x = -3$ in Eq. (1), we have

$$-5 = 100A \Rightarrow A = \frac{-1}{20}$$

Comparing the coefficients of x^4 on either sides of Eq. (1), we have

$$0 = A + B \Rightarrow B = \frac{1}{20}$$

Comparing the coefficients of x^3 on either sides of Eq. (1), we have

$$0 = 3B + C \Rightarrow C = \frac{-3}{20}$$

Put $x = 0$ in Eq. (1), we have

$$\Rightarrow 1 = A + 3C + 3E \Rightarrow 3E = 1 + \frac{1}{20} + \frac{9}{20}$$

$$\Rightarrow E = \frac{1}{2}$$

By putting $x = 1$ in Eq. (1), we have

$3 = 4A + 8B + 8C + 4D + 4E$

$$3 = \frac{-4}{20} + \frac{8}{20} + 8\left(\frac{-3}{20}\right) + 4D + 4\left(\frac{1}{2}\right)$$

$$3 + \frac{4}{20} - \frac{8}{20} + \frac{24}{20} - 2 = 4D$$

$$D = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \frac{2x+1}{(x+3)(x^2+1)^2} = \frac{\frac{-1}{20}}{x+3} + \frac{\frac{x}{20} - \frac{3}{20}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{-1}{20(x+3)} + \frac{(x-3)}{20(x^2+1)} + \frac{(x+1)}{2(x^2+1)^2}.$$

EXAMPLE 26.6

Resolve $\dfrac{3x^2 - 3x - 11}{(x+3)(3x+4)^2}$ into partial fractions.

(a) $\dfrac{1}{x+3} + \dfrac{2}{3x+4} - \dfrac{1}{(3x+4)^2}$

(b) $\dfrac{1}{x+3} - \dfrac{2}{3x+4} - \dfrac{1}{(3x+4)^2}$

(c) $\dfrac{1}{x+3} + \dfrac{2}{3x+4} - \dfrac{1}{2(3x+4)^2}$

(d) None of these

SOLUTION

Let, $\dfrac{3x^2 - 3x - 11}{(x+3)(3x+4)^2} = \dfrac{A}{x+3} + \dfrac{B}{3x+4} + \dfrac{C}{(3x+4)^2}$.

Consider

$3x^2 - 3x - 11 = A(3x + 4)^2 + B(x + 3)(3x + 4) + C(x + 3)$

$= A(9x^2 + 16 + 24x) + B(3x^2 + 13x + 12) + C(x + 3)$

Comparing coefficient of x^2, coefficient of x and constant terms, we get,

$$9A + 3B = 3 \tag{1}$$

$$24A + 13B + C = -3 \tag{2}$$

$$16A + 12B + 3C = -11 \tag{3}$$

Solving Eqs. (1), (2) and (3), we get

$A = 1,\ B = -2,\ C = -1$

$$\therefore \frac{3x^2 - 3x - 11}{(x+3)(3x+4)^2} = \frac{1}{x+3} - \frac{2}{3x+4} - \frac{1}{(3x+4)^2}.$$

EXAMPLE 26.7

If $\dfrac{x}{(x-1)(x^2+1)^2} = \dfrac{P}{x-1} + \dfrac{Qx+R}{x^2+1} + \dfrac{Sx+T}{(x^2+1)^2}$, then $P + Q - R - S + T =$ ______.

(a) $\dfrac{5}{4}$

(b) $\dfrac{3}{2}$

(c) $\dfrac{9}{7}$

(d) $\dfrac{8}{9}$

SOLUTION

$$\frac{x}{(x-1)(x^2+1)^2}$$

$$= \frac{P}{x-1} + \frac{Qx+R}{(x^2+1)} + \frac{Sx+T}{(x^2+1)^2}$$

Consider,

$x = P(x^2 + 1)^2 + (Qx + R)(x^2 + 1)(x - 1) + (sx + T)(x - 1)$

Put $x = 1$,

$$\Rightarrow 4P = 1 \Rightarrow P = \frac{1}{4}$$

Equating the coefficient of x^4, the coefficient of x^3, constant term and the coefficient of x^2 from both sides.

$$P + Q = 0, \text{ i.e., } P = -Q \ \therefore Q = -\frac{1}{4}$$

$$-Q + R = 0, \text{ i.e., } Q = R, \ \therefore R = \frac{-1}{4}$$

$P - R - T = 0$, i.e., $T = P - R$

$$\therefore T = \frac{1}{2}$$

$2P + Q - R + S = 0$

That is, $S = -2P = \dfrac{-1}{2}$

$$P + Q - R - S + T = \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} = \frac{5}{4}.$$

EXAMPLE 26.8

If $\dfrac{9x-13}{x^2-2x-15} = \dfrac{A}{x+3} + \dfrac{B}{x-5}$, then $A + B$ is ________.

(a) 9 **(b)** 13

(c) 12 **(d)** 8

SOLUTION

Given, $\dfrac{9x-13}{x^2-2x-15} = \dfrac{A}{x+3} + \dfrac{B}{x-5}$.

Consider, $9x - 13 = A(x - 5) + B(x + 3)$

$9x - 13 = Ax - 5A + Bx + 3B$

Comparing the coefficients of x, we get

$A + B = 9$.

EXAMPLE 26.9

Resolve $\dfrac{x}{(x-2)(x^2+3)^2}$ into partial fractions.

(a) $\dfrac{1}{49}\left[\dfrac{2}{x-2} - \dfrac{2x+4}{x^2+3} + \dfrac{12-2x}{(x^2+3)^2}\right]$

(b) $\dfrac{1}{49}\left[\dfrac{2}{x-2} - \dfrac{2x+4}{x^2+3} - \dfrac{12-2x}{(x^2+3)^2}\right]$

(c) $\dfrac{1}{49}\left[\dfrac{2}{x-2} + \dfrac{2x+4}{x^2+3} - \dfrac{12-2x}{(x^2+3)^2}\right]$

(d) $\dfrac{1}{49}\left[\dfrac{2}{x-2} - \dfrac{2x+4}{x^2+3} + \dfrac{21-14x}{(x^2+3)^2}\right]$

SOLUTION

Let $\dfrac{x}{(x-2)(x^2+3)^2} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+3} + \dfrac{Dx+E}{(x^2+3)^2}$

Consider,

$x = A\,(x^2 + 3)^2 + (Bx + C)\,(x - 2)\,(x^2 + 3) + (Dx + E)(x - 2)$

Put $x = 2$, we get

$A = \dfrac{2}{49}$

Comparing the coefficients of x^4, x^3, x^2 and constant terms, we get

$A + B = 0 \Rightarrow \therefore B = -\dfrac{2}{49}$.

$-2B + C = 0 \Rightarrow C = -\dfrac{4}{49}$.

$6A + 3B - 2C + D = 0$

$\Rightarrow \therefore D = \dfrac{-14}{49}$

and $-6C - 2E + 9A = 0$

$E = \dfrac{21}{49}$

$$\frac{x}{(x-2)(x^2+3)^2}$$
$$= \frac{\frac{2}{49}}{x-2} + \frac{\frac{-2}{49}x - \frac{4}{49}}{x^2+3} + \frac{-\frac{14}{49}x + \frac{21}{49}}{(x^2+3)^2}$$
$$= \frac{2}{49(x-2)} - \frac{2x+4}{49(x^2+3)} + \frac{21-14x}{49(x^2+3)^2}.$$

EXAMPLE 26.10

If $\dfrac{1}{(1-x)(1-2x)(1-3x)} = \dfrac{A}{(1-x)} + \dfrac{B}{(1-2x)} + \dfrac{C}{(1-3x)}$, then $A + B + C =$ ________.

(a) 0 **(b)** 1

(c) −1 **(d)** 2

SOLUTION

$$\frac{1}{(1-x)(1-2x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$$
$$= \frac{A(1-2x)(1-3x) + B(1-3x)(1-x) + C(1-x)(1-2x)}{(1-x)(1-2x)(1-3x)}$$

Consider,

$A(1 - 2x)(1 - 3x) + B(1 - 3x)(1 - x) + C(1 - x)(1 - 2x) = 1$

Comparing constant terms, we get $A + B + C = 1$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. Improper fraction is a fraction in which the degree of the numerator is _______ than or equal to the degree of denominator.

2. Improper fraction can always be written as the _______ of a polynomial and a proper fraction.

3. The partial fractions of $\frac{Ax+B}{x^4}$ has _______ terms.

4. A proper fraction is one in which the degree of numerator is _______ than the degree of denominator.

5. A proper fraction may be written as the sum of _______.

6. The number of terms of the partial fraction of $\frac{x^3}{(x-1)(2x-3)(x-4)}$ is _______.

7. The partial fraction of $\frac{2(x^2+1)}{x^3-1}$ is $\frac{1}{x-1}-\frac{f(x)}{g(x)}$. Then the degree of $f(x)$ is _______.

8. There are three terms of partial fractions of the proper fraction $\frac{f(x)}{x^3}$. Then the degree of the polynomial $f(x)$ would be _______.

9. If $\frac{x^2-2x-9}{(x^2+x+6)(x+1)}=\frac{2x-3}{x^2+x+6}+\frac{A}{x+1}$, then A is _______.

10. If the partial fractions of $\frac{3x^2+4}{(x-1)^3}$ is $\frac{3}{x-1}+\frac{A}{(x-1)^2}+\frac{7}{(x-1)^3}$, then $A =$ _______.

11. If $\frac{f(x)}{x^2-x+1}-\frac{1}{f(x)}$ is the partial fraction of $\frac{3x}{x^3+1}$, then $f(x)$ is _______.

12. If $\frac{ax^2+bx+c}{(x+1)(x^2-3x-4)}=\frac{f(x)}{x+1}+\frac{g(x)}{x+4}+\frac{h(x)}{x-1}$, then the degree of $f(x)$, $g(x)$ and $h(x)$ is _______.

13. If $\frac{2x^2+3x+4}{(x+2)^4}=\frac{A}{(x+2)}+\frac{2}{(x+2)^2}-\frac{5}{(x+2)^3}+\frac{6}{(x+2)^4}$, then $A =$ _______.

14. If $\frac{x^2}{(x-a)(x-b)}=1+\frac{a^2}{(a-b)(x-a)}+\frac{B}{(x-b)}$, then B is _______.

15. If $\frac{x^4}{(x+1)(x+2)}=f(x)+\frac{A}{x+1}+\frac{B}{x+2}$, then the degree of the polynomial $f(x)$ is _______.

Short Answer Type Questions

16. Resolve into partial fractions: $\frac{1}{x^2-a^2}$.

17. Resolve into partial fractions: $\frac{2x+5}{x^2+3x+2}$.

18. Resolve into partial fractions: $\frac{3x+4}{x^2+x-12}$.

19. Resolve into partial fractions: $\frac{2x^2+5x+8}{(x-2)^3}$.

20. Resolve into partial fractions: $\frac{x+1}{(x+2)(x^2+4)}$.

21. Find the value of A, B and C if
$$\frac{2x^2-x-10}{(x-2)^3}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}.$$

22. If $\frac{x+5}{(x-1)^2}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}$, then find the values of A and B.

23. Find the partial fractions of $\frac{ax+b}{(x-1)^2}$, where a and b are constants.

24. Resolve into partial fractions: $\frac{2x^2-3x+3}{x^3-2x^2+x}$.

25. Resolve into partial fractions: $\dfrac{x^2+5x+6}{x^3-7x-6}$.

26. Resolve into partial fractions: $\dfrac{x^3}{(x+1)(x+2)}$.

27. Resolve into partial fractions: $\dfrac{x+1}{x^3-1}$.

28. Resolve into partial fractions: $\dfrac{2x^2+4}{x^4+5x^2+4}$.

29. Find the partial fractions of $\dfrac{2x+3}{x^4+x^2+1}$.

30. Resolve into partial fractions: $\dfrac{1}{x^4+x}$.

Essay Type Questions

31. Resolve into partial fractions: $\dfrac{3x^2+4x+5}{x^3+9x^2+26x+24}$.

32. Resolve into partial fractions: $\dfrac{x^2-x-1}{x^4+x^2+1}$.

33. Resolve into partial fractions: $\dfrac{3x+4}{x^3-2x-4}$.

34. Resolve into partial fractions: $\dfrac{x^2-3}{x^3-2x^2-x+2}$.

35. Resolve into partial fractions: $\dfrac{x^4+1}{(x-2)(x+2)}$.

CONCEPT APPLICATION

Level 1

1. Resolve $\dfrac{1}{x^2-9}$ into partial fractions.

(a) $\dfrac{1}{3(x-3)}-\dfrac{1}{3(x+3)}$

(b) $\dfrac{1}{2(x-3)}-\dfrac{3}{2(x+3)}$

(c) $\dfrac{1}{6(x-3)}-\dfrac{1}{6(x+3)}$

(d) $\dfrac{1}{6(x-3)}+\dfrac{1}{6(x+3)}$

2. Resolve $\dfrac{2x+3}{x^2-6x+5}$ into partial fractions.

(a) $\dfrac{1}{4}\left(\dfrac{13}{x-5}-\dfrac{5}{x-1}\right)$

(b) $\dfrac{5}{x-5}-\dfrac{13}{4(x-1)}$

(c) $\dfrac{13}{5(x-1)}-\dfrac{4}{5(x-5)}$

(d) $\dfrac{5}{(x-5)}-\dfrac{4}{(x-1)}$

3. Resolve $\dfrac{x^2+x+1}{(x-1)^3}$ into partial fractions.

(a) $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}+\dfrac{1}{(x-1)^3}$

(b) $\dfrac{1}{(x-1)}+\dfrac{2}{(x-1)^2}+\dfrac{1}{(x-1)^3}$

(c) $\dfrac{1}{x-1}+\dfrac{3}{(x-1)^2}+\dfrac{3}{(x-1)^3}$

(d) $\dfrac{2}{x-1}+\dfrac{3}{(x-1)^2}+\dfrac{3}{(x-1)^3}$

4. Resolve $\dfrac{3x+2}{x^2+5x+6}$ into partial fractions.

(a) $\dfrac{7}{(x+2)}+\dfrac{4}{(x+3)}$

(b) $\dfrac{4}{(x+3)}+\dfrac{8}{(x+2)}$

(c) $\dfrac{4}{(x+3)}-\dfrac{7}{(x+2)}$

(d) $\dfrac{7}{(x+3)}-\dfrac{4}{(x+2)}$

5. Resolve $\dfrac{1}{x^2+x+42}$ into partial fractions.

(a) $\dfrac{1}{13(x-6)}+\dfrac{1}{13(x+7)}$

(b) $\dfrac{1}{(x-7)}+\dfrac{1}{(x-6)}$

(c) $\dfrac{1}{(x-7)}-\dfrac{1}{(x-6)}$

(d) $\dfrac{1}{13(x-6)}-\dfrac{1}{13(x+7)}$

6. Resolve $\dfrac{x^2+4x+6}{(x^2-1)(x+3)}$ into partial fractions

(a) $\dfrac{11}{8(x-1)}-\dfrac{3}{21(x+1)}+\dfrac{3}{8(x+3)}$

(b) $\dfrac{11}{8(x-1)}-\dfrac{3}{4(x+1)}+\dfrac{3}{8(x+3)}$

(c) $\dfrac{11}{4(x+1)}-\dfrac{3}{8(x+1)}+\dfrac{3}{8(x+3)}$

(d) $\dfrac{11}{8(x-1)}-\dfrac{3}{4(x+1)}+\dfrac{3}{8(x+3)}$

7. Resolve $\dfrac{x+1}{x(x-1)(x+3)}$ into partial fractions.

(a) $\dfrac{-1}{3x}+\dfrac{1}{2(x-1)}-\dfrac{1}{6(x+3)}$

(b) $\dfrac{1}{3x}+\dfrac{1}{2(x-1)}-\dfrac{1}{6(x+3)}$

(c) $\dfrac{1}{3x}-\dfrac{1}{2(x-1)}+\dfrac{1}{6(x+3)}$

(d) $\dfrac{1}{2x}-\dfrac{1}{3(x-1)}+\dfrac{1}{6(x+3)}$

8. Resolve $\dfrac{1}{x^2-(a+b)x+ab}$ into partial fractions.

(a) $\dfrac{1}{a+b}\left[\dfrac{1}{x-a}+\dfrac{1}{x-b}\right]$

(b) $\dfrac{1}{a+b}\left[\dfrac{1}{x-a}-\dfrac{1}{x-b}\right]$

(c) $\dfrac{1}{a-b}\left[\dfrac{1}{x-a}+\dfrac{1}{x-b}\right]$

(d) $\dfrac{1}{a-b}\left[\dfrac{1}{x-a}-\dfrac{1}{x-b}\right]$

9. Resolve $\dfrac{1}{x^2-5x+6}$ into partial fractions.

(a) $\dfrac{1}{x-3}-\dfrac{1}{x-2}$

(b) $\dfrac{1}{x-2}+\dfrac{1}{x-3}$

(c) $\dfrac{1}{x-3}+\dfrac{1}{x-2}$

(d) $\dfrac{2}{x-3}+\dfrac{2}{x-2}$

10. Resolve $\dfrac{x+2}{x^2-2x-15}$ into partial fractions.

(a) $\dfrac{7}{8(x-5)}+\dfrac{1}{8(x+3)}$

(b) $\dfrac{8}{7(x-5)}+\dfrac{1}{7(x+3)}$

(c) $\dfrac{5}{8(x-5)}+\dfrac{1}{8(x+3)}$

(d) None of these

11. Resolve $\dfrac{3x^2+2x+4}{(x-1)(x^2-4)}$ into partial fractions.

(a) $\dfrac{5}{x-2}+\dfrac{1}{x-1}+\dfrac{1}{x+2}$

(b) $\dfrac{5}{x-2}+\dfrac{2}{x-1}+\dfrac{1}{x+2}$

(c) $\dfrac{5}{x-2}+\dfrac{2}{x-1}-\dfrac{1}{x+2}$

(d) $\dfrac{5}{x-2}-\dfrac{3}{(x-1)}+\dfrac{1}{(x+2)}$

12. Resolve $\dfrac{2x+1}{x^2-2x-8}$ into partial fractions.

(a) $\dfrac{3}{2(x-4)}-\dfrac{1}{2(x+2)}$

(b) $\dfrac{3}{2(x-4)}+\dfrac{1}{2(x+2)}$

(c) $\dfrac{1}{2(x-4)}-\dfrac{3}{2(x+2)}$

(d) None of these

13. Resolve $\dfrac{2x^2+3x+18}{(x-2)(x+2)^2}$ into partial fractions.

(a) $\dfrac{2}{(x+2)}-\dfrac{5}{(x-1)^2}$

(b) $\dfrac{2}{x-2}+\dfrac{1}{(x+2)}+\dfrac{3}{(x+2)^2}$

(c) $\dfrac{2}{x-2}-\dfrac{5}{(x+2)^2}$

(d) $\dfrac{2}{x-2}-\dfrac{1}{(x+2)}+\dfrac{3}{(x+2)^2}$

14. Resolve $\dfrac{x^2-x+1}{x^3-1}$ into partial fractions.

(a) $\dfrac{1}{3(x-1)}+\dfrac{2x-2}{3(x^2+x+1)}$

(b) $\dfrac{1}{3(x-1)}-\dfrac{2x+2}{3(x^2+x+1)}$

(c) $\dfrac{1}{2(x-1)}+\dfrac{x+2}{2(x^2+x+1)}$

(d) $\dfrac{1}{3(x-1)}-\dfrac{x-2}{3(x^2+x+1)}$

15. Resolve $\dfrac{x-1}{x^3-3x^2+2x}$ into partial fractions.

(a) $\dfrac{1}{(x-2)}-\dfrac{1}{2x}$

(b) $\dfrac{1}{2(x-2)}-\dfrac{1}{2x}$

(c) $\dfrac{1}{(x-2)}+\dfrac{1}{x-1}+\dfrac{1}{2x}$

(d) $\dfrac{1}{x-2}-\dfrac{1}{x-1}+\dfrac{1}{2x}$

Level 2

16. Resolve $\dfrac{x-b}{x^2+(a+b)x+ab}$ into partial fractions.

(a) $\dfrac{2a}{(a+b)(x+a)}-\dfrac{2b}{(a+b)(x+b)}$

(b) $\dfrac{a+b}{(a-b)(x+a)}+\dfrac{2b}{(a-b)(x+b)}$

(c) $\dfrac{a+b}{(a-b)(x+a)}-\dfrac{2b}{(a-b)(x+b)}$

(d) None of these

17. Resolve $\dfrac{x+1}{x^2-4}$ into partial fractions.

(a) $\dfrac{3}{2(x-2)}+\dfrac{1}{2(x+2)}$

(b) $\dfrac{3}{2(x-2)}-\dfrac{1}{2(x+2)}$

(c) $\dfrac{3}{4(x-2)}+\dfrac{1}{4(x+2)}$

(d) $\dfrac{3}{4(x-2)}-\dfrac{1}{4(x+2)}$

18. Resolve $\dfrac{x}{(x+1)(x-1)^2}$ into partial fractions.

(a) $\dfrac{1}{2(x-1)^2}+\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}$

(b) $\dfrac{-1}{4(x+1)}+\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}$

(c) $\dfrac{-1}{4(x-1)}+\dfrac{1}{4(x+1)}+\dfrac{1}{2(x-1)^2}$

(d) None of these

19. Resolve $\dfrac{3x-5}{x^2+3x+2}$ into partial fractions.

(a) $\dfrac{7}{x+2}-\dfrac{5}{x+1}$

(b) $\dfrac{-8}{x+2}-\dfrac{11}{(x+1)}$

(c) $\dfrac{11}{x+2}-\dfrac{8}{x+1}$

(d) $\dfrac{7}{x+2}+\dfrac{5}{x+1}$

PRACTICE QUESTIONS

20. Resolve $\frac{3x+5}{x^2+8x-20}$ into partial fractions.

(a) $\frac{11}{24(x-2)}+\frac{25}{24(x+10)}$

(b) $\frac{11}{6(x-2)}+\frac{25}{6(x+10)}$

(c) $\frac{11}{12(x+2)}+\frac{25}{12(x-10)}$

(d) $\frac{11}{12(x-2)}+\frac{25}{12(x+10)}$

21. Resolve $\frac{ax+b^2}{x^2-(a+b)^2}$ into partial fractions.

(a) $\frac{a^2+ab+b^2}{2(a+b)(x-(a+b))}+\frac{a^2+ab-b^2}{2(a+b)(x+(a+b))}$

(b) $\frac{a^2+b^2}{2(a+b)(x-(a+b))}+\frac{a^2-b^2}{2(a+b)(x+(a+b))}$

(c) $\frac{a^2-b^2}{2(a+b)(x-(a-b))}+\frac{a^2+b^2}{2(a+b)(x+(a+b))}$

(d) $\frac{a^2+ab+b^2}{2(a+b)(x-(a+b))}-\frac{a^2+ab-b^2}{2(a+b)(x+(a+b))}$

22. Find the constants a, b, c and d, respectively, if $\frac{1}{x^4-x}=\frac{a}{x}+\frac{b}{x-1}+\frac{cx+d}{x^2+x+1}$.

(a) $-1, \frac{1}{3}, \frac{-1}{2}, \frac{-5}{6}$

(b) $-1, \frac{1}{3}, \frac{-1}{2}, \frac{5}{6}$

(c) $1, \frac{-1}{3}, \frac{1}{2}, \frac{5}{6}$

(d) None of these

23. If $\frac{x^3}{(x-1)(x-2)}=Ax+B+\frac{C}{x-1}+\frac{D}{x-2}$, then A, B, C and D, respectively are _______.

(a) 1, 3, 1, 8

(b) 1, −1, 3, 8

(c) 1, 3, −1, 8

(d) −1, −3, 1, 8

24. Resolve $\frac{2x^2+3}{x^4+8x^2+15}$ into partial fractions.

(a) $\frac{3}{2(x^2+3)}+\frac{5}{2(x^2+5)}$

(b) $\frac{3}{2(x^2-3)}+\frac{5}{2(x^2+5)}$

(c) $\frac{-3}{2(x^2+3)}+\frac{7}{2(x^2+5)}$

(d) $\frac{7}{2(x^2+3)}-\frac{3}{2(x^2+5)}$

25. Resolve $\frac{3x+1}{(x-1)^2}$ into partial fractions.

(a) $\frac{4}{(x-1)}-\frac{3}{(x-1)^2}$

(b) $\frac{3}{(x-1)}-\frac{4}{(x-1)^2}$

(c) $\frac{3}{(x-1)^2}-\frac{3}{(x-1)^2}$

(d) $\frac{3}{(x-1)}+\frac{4}{(x-1)^2}$

26. If $\frac{3x^2+14x+10}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$, then A, B and C, respectively are _______.

(a) $\frac{9}{2}, -5, \frac{3}{2}$

(b) $\frac{9}{2}, 5, \frac{-3}{2}$

(c) $\frac{9}{2}, \frac{3}{2}, -5$

(d) $\frac{-9}{2}, \frac{-3}{2}, 5$

27. If $\frac{2x-1}{(x+1)(x^2-1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$, then A, B, C, respectively are _______.

(a) $\frac{1}{4}, \frac{-1}{4}, \frac{3}{2}$

(b) $\frac{1}{4}, \frac{1}{4}, \frac{3}{2}$

(c) $\frac{1}{4}, \frac{-5}{2}, \frac{5}{4}$

(d) $\frac{1}{4}, \frac{5}{2}, \frac{5}{4}$

28. Resolve $\dfrac{ax-b}{(x+1)^2}$ into partial fractions.

(a) $\dfrac{a}{x+1}+\dfrac{a+b}{(x+1)^2}$

(b) $\dfrac{a}{x+1}+\dfrac{a-b}{(x+1)^2}$

(c) $\dfrac{a}{x+1}-\dfrac{a-b}{(x+1)^2}$

(d) $\dfrac{a}{x+1}-\dfrac{a+b}{(x+1)^2}$

29. Resolve $\dfrac{4x^2+3x+2}{(x-4)^3}$ into partial fractions.

(a) $\dfrac{25}{4(x-4)}+\dfrac{59}{2(x-4)}+\dfrac{78}{(x-4)^3}$

(b) $\dfrac{25}{4(x-4)}-\dfrac{69}{2(x-4)^2}+\dfrac{78}{(x-4)^3}$

(c) $\dfrac{35}{4(x-4)}+\dfrac{69}{2(x-4)^2}+\dfrac{78}{(x-4)^3}$

(d) $\dfrac{4}{x-4}+\dfrac{35}{(x-4)^2}+\dfrac{78}{(x-4)^3}$

30. Resolve $\dfrac{1}{x^2-7x-18}$ into partial fractions.

(a) $\dfrac{1}{11}\left(\dfrac{1}{x-9}-\dfrac{1}{x+2}\right)$

(b) $\dfrac{1}{11}\left(\dfrac{1}{x-9}+\dfrac{1}{x+2}\right)$

(c) $\dfrac{1}{11}\left(\dfrac{1}{x+9}-\dfrac{1}{x-2}\right)$

(d) $\dfrac{1}{11}\left(\dfrac{1}{x+9}+\dfrac{1}{x+2}\right)$

31. Resolve $\dfrac{10}{x^2-25}$ into partial fractions.

(a) $\dfrac{1}{x-5}-\dfrac{1}{x+5}$

(b) $\dfrac{1}{x-5}+\dfrac{2}{x+5}$

(c) $\dfrac{1}{x+5}+\dfrac{1}{x-5}$

(d) $\dfrac{1}{x-5}-\dfrac{2}{x+5}$

32. Resolve $\dfrac{2x}{3(x-1)(x-3)}$ into partial fractions.

(a) $\dfrac{1}{x+3}-\dfrac{1}{3(x+1)}$

(b) $\dfrac{1}{x-3}-\dfrac{1}{x+1}$

(c) $\dfrac{1}{x-3}-\dfrac{1}{3(x-1)}$

(d) $\dfrac{1}{x-3}-\dfrac{2}{3(x-1)}$

33. If $\dfrac{x}{(x+2)(x+3)}=\dfrac{A}{x+3}+\dfrac{B}{x+2}$, then $A-B$ is = ______.

(a) 4

(b) 5

(c) −6

(d) 2

34. Resolve $\dfrac{5}{(x+2)(x+3)}$ into partial fractions.

(a) $5\left(\dfrac{1}{x+2}+\dfrac{1}{x+3}\right)$

(b) $5\left(\dfrac{1}{x+2}-\dfrac{1}{x+3}\right)$

(c) $10\left(\dfrac{1}{x+2}-\dfrac{1}{x+3}\right)$

(d) $10\left(\dfrac{1}{x+2}+\dfrac{1}{x+3}\right)$

35. Resolve $\dfrac{1}{x^2-7x+12}$ into partial fractions.

(a) $\dfrac{1}{x-4}-\dfrac{1}{x-3}$

(b) $\dfrac{1}{x-3}+\dfrac{1}{x-4}$

(c) $\dfrac{2}{x-4}-\dfrac{1}{x-3}$

(d) $\dfrac{1}{x+3}-\dfrac{1}{x+4}$

Level 3

36. Resolve $\dfrac{1}{x^4 - 3x^2 - 4}$ into partial fractions.

(a) $\dfrac{1}{5(x^2-4)} + \dfrac{1}{5(x^2+1)}$

(b) $\dfrac{1}{5(x^2-4)} - \dfrac{1}{5(x^2+1)}$

(c) $\dfrac{1}{4(x^2+4)} - \dfrac{1}{4(x^2+1)}$

(d) $\dfrac{1}{4(x^2+4)} + \dfrac{1}{4(x^2+1)}$

37. Resolve $\dfrac{4x+3}{x^3 - 7x - 6}$ into partial fractions.

(a) $\dfrac{1}{4(x+1)} + \dfrac{3}{4(x-3)} - \dfrac{1}{(x+2)}$

(b) $\dfrac{-1}{4(x+1)} + \dfrac{3}{4(x-3)} + \dfrac{1}{(x+2)}$

(c) $\dfrac{1}{4(x+1)} - \dfrac{3}{4(x-3)} + \dfrac{1}{(x+2)}$

(d) $\dfrac{1}{4(x+1)} - \dfrac{3}{4(x-3)} - \dfrac{1}{x+2}$

38. If $\dfrac{1}{(a^2-bx)(b^2-ax)} = \dfrac{A}{a^2-bx} + \dfrac{B}{b^2-ax}$, then the value of A and B respectively would be _______.

(a) $\dfrac{b}{b^3-a^3}, \dfrac{a}{b^3-a^3}$

(b) $\dfrac{b}{b^3-a^3}, \dfrac{a}{a^3-b^3}$

(c) $\dfrac{b}{a^3-b^3}, \dfrac{a}{a^3-b^3}$

(d) $\dfrac{-b}{b^3-a^3}, \dfrac{-a}{a^3-b^3}$

39. Resolve $\dfrac{3x-5}{(x-1)^4}$ into partial fractions.

(a) $\dfrac{1}{(x-1)} + \dfrac{2}{(x-1)^2} - \dfrac{3}{(x-1)^3} + \dfrac{4}{(x-1)^4}$

(b) $\dfrac{3}{(x-1)^2} + \dfrac{2}{(x-1)^3} - \dfrac{1}{(x-1)^4}$

(c) $\dfrac{3}{(x-1)^3} + \dfrac{2}{(x-1)^4}$

(d) $\dfrac{3}{(x-1)^3} - \dfrac{2}{(x-1)^4}$

40. Resolve $\dfrac{x^2+x+1}{x^3+1}$ into partial fractions.

(a) $\dfrac{1}{3(x+1)} + \dfrac{x+1}{3(x^2-x+1)}$

(b) $\dfrac{-1}{3(x+1)} + \dfrac{2x+2}{3(x^2-x+1)}$

(c) $\dfrac{1}{3(x+1)} + \dfrac{2x+2}{3(x^2-x+1)}$

(d) None of these

41. Resolve $\dfrac{2x^2+8x+13}{(x+1)^4}$ into partial fractions.

(a) $\dfrac{2}{(x+1)^2} + \dfrac{5}{(x+1)^3} + \dfrac{7}{(x+1)^4}$

(b) $\dfrac{1}{(x+1)^2} + \dfrac{5}{(x+1)^3} + \dfrac{7}{(x+1)^4}$

(c) $\dfrac{2}{(x+2)^2} + \dfrac{4}{(x+1)^3} + \dfrac{7}{(x+1)^4}$

(d) $\dfrac{4}{(x+1)^2} + \dfrac{5}{(x+1)^3} + \dfrac{7}{(x+1)^4}$

42. Resolve $\dfrac{6x^2-14x+6}{x(x-1)(x-2)}$ into partial fractions.

(a) $\dfrac{2}{x} + \dfrac{3}{x-1} + \dfrac{1}{x-2}$

(b) $\dfrac{3}{x} + \dfrac{2}{x-1} + \dfrac{1}{x-2}$

(c) $\dfrac{1}{x} + \dfrac{2}{x-1} + \dfrac{3}{x-2}$

(d) $\dfrac{1}{x} + \dfrac{3}{x-1} + \dfrac{2}{x-2}$

43. Resolve $\dfrac{1}{(x-4)(x^2+3)}$ into partial fractions.

(a) $\dfrac{1}{19}\left[\dfrac{1}{x-4} + \dfrac{x+4}{x^2+3}\right]$

PRACTICE QUESTIONS

(b) $\frac{1}{19}\left[\frac{1}{x-4}+\frac{x+3}{x^2+3}\right]$

(c) $\frac{1}{19}\left[\frac{1}{x-4}+\frac{x+3}{x^2+3}\right]$

(d) $\frac{1}{19}\left[\frac{1}{x-4}-\frac{x+4}{x^2+3}\right]$

44. Resolve $\frac{3x^2+7}{x^4-3x^2+2}$ into partial fractions.

(a) $\frac{10}{x^2-2}+\frac{5}{x-1}-\frac{5}{x+1}$

(b) $\frac{13}{x^2-2}+\frac{5}{x+1}-\frac{5}{x-1}$

(c) $\frac{5}{x^2-2}-\frac{10}{x-1}+\frac{10}{x+1}$

(d) $\frac{5}{x-1}-\frac{5}{x+1}-\frac{13}{x^2-2}$

45. If $\frac{4x^2+5x+6}{x^2(x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}$, then $2A + 3B + 4C =$ _______.

(a) 10 (b) 20

(c) 30 (d) 40

PRACTICE QUESTIONS

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. greater
2. sum
3. two terms
4. smaller
5. partial fractions
6. four
7. zero or 1.
8. 2
9. -1
10. 6
11. $x+1$
12. zero
13. zero
14. $\dfrac{b^2}{b-a}$
15. two

Short Answer Type Questions

16. $\dfrac{1}{2a(x-a)} - \dfrac{1}{2a(x+a)}$
17. $\dfrac{3}{x+1} - \dfrac{1}{x+2}$
18. $\dfrac{8}{7(x+4)} + \dfrac{13}{7(x-3)}$
19. $\dfrac{2}{x-2} + \dfrac{13}{(x-2)^2} + \dfrac{26}{(x-2)^3}$
20. $\dfrac{-1}{8(x+2)} + \dfrac{\frac{1}{8}x + \frac{3}{4}}{x^2+4}$
21. $A = 2$, $B = 7$ and $C = -4$
22. $A = 1$ and $B = 6$
23. $\dfrac{a}{(x-1)} + \dfrac{a+b}{(x-1)^2}$
24. $\dfrac{3}{x} - \dfrac{1}{(x-1)} + \dfrac{2}{(x-1)^2}$
25. $\dfrac{3}{2(x-3)} - \dfrac{1}{2(x+1)}$
26. $\dfrac{9}{2(x+2)} - \dfrac{20}{x+3} + \dfrac{37}{2(x+4)}$
27. $\dfrac{2}{3(x-1)} - \dfrac{(2x+1)}{3(x^2+x+1)}$
28. $\dfrac{2}{3(x^2+1)} + \dfrac{4}{3(x^2+4)}$
29. $\dfrac{\frac{-3x}{2} + \frac{5}{2}}{x^2-x+1} + \dfrac{\frac{3}{2}x + \frac{1}{2}}{x^2+x+1}$
30. $\dfrac{1}{x} - \dfrac{1}{3(x+1)} + \dfrac{\frac{-2}{3}x + \frac{1}{3}}{x^2-x+1}$

Essay Type Questions

31. $x - 3 - \dfrac{1}{x+1} + \dfrac{8}{x+2}$
32. $\dfrac{x-1}{(x^2-x+1)} - \dfrac{x}{(x^2+x+1)}$
33. $\dfrac{1}{(x-2)} - \dfrac{x+1}{(x^2+2x+2)}$
34. $\dfrac{1}{3(x-2)} + \dfrac{1}{(x-1)} - \dfrac{1}{3(x+1)}$
35. $x^2 + 4 + \dfrac{17}{4(x-2)} - \dfrac{17}{4(x+2)}$

CONCEPT APPLICATION

Level 1

1. (c)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (a)
11. (d)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (b)					

Level 2

16. (c)	**17.** (c)	**18.** (b)	**19.** (c)	**20.** (d)	**21.** (a)	**22.** (d)	**23.** (c)	**24.** (c)	**25.** (d)
26. (b)	**27.** (a)	**28.** (d)	**29.** (d)	**30.** (a)	**31.** (a)	**32.** (c)	**33.** (b)	**34.** (b)	**35.** (a)

Level 3

36. (b)	**37.** (a)	**38.** (b)	**39.** (d)	**40.** (c)	**41.** (c)	**42.** (b)	**43.** (d)	**44.** (b)	**45.** (b)
46. (b)	**47.** (a)	**48.** (a)	**49.** (d)	**50.** (b)					

CONCEPT APPLICATION

Level 1

1. $\dfrac{1}{x^2-9} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$.

2. $\dfrac{2x+3}{x^2-6x+5} = \dfrac{A}{(x-5)} + \dfrac{B}{(x-1)}$.

3. $\dfrac{x^2+x+1}{(x-1)^3} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$.

4. $\dfrac{3x+2}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$.

5. $\dfrac{1}{x^2+x-42} = \dfrac{A}{x+7} + \dfrac{B}{x-6}$.

6. $\dfrac{x^2+4x+6}{(x^2-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$.

7. $\dfrac{x+1}{x(x-1)(x+3)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+3}$.

8. $\dfrac{1}{x^2-(a+b)x+ab} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$.

9. $\dfrac{1}{x^2-5x+6} = \dfrac{A}{x-3} + \dfrac{B}{x-2}$.

10. $\dfrac{x+2}{x^2-2x-15} = \dfrac{A}{x-5} + \dfrac{B}{x+3}$.

11. $\dfrac{3x^2+2x+4}{(x-1)(x^2-4)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$.

12. $\dfrac{2x+1}{x^2-2x-8} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$.

13. $\dfrac{2x^2+3x+18}{(x-2)(x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$.

14. Split the denominator into factors and use the relevant method to find its partial fractions.

15. (i) $x^3 - 3x^2 + 2x = x(x-1)(x-2)$.
    (ii) $\dfrac{x-1}{(x^3-3x^2+2x)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$.

Level 2

16. $\dfrac{x-b}{x^2+(a+b)x+ab} = \dfrac{A}{x+a} + \dfrac{B}{x+b}$.

17. $\dfrac{x+1}{x^2-4} = \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

18. $\dfrac{x}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

19. $\dfrac{3x-5}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$.

20. $\dfrac{3x+5}{x^2+8x-20} = \dfrac{A}{x+10} + \dfrac{B}{x-2}$.

21. $\dfrac{ax+b^2}{x^2-(a+b)^2} = \dfrac{A}{x-(a+b)} + \dfrac{B}{x+(a+b)}$.

22. (i) Simplify LHS and RHS.
    (ii) Put $x = 0$ and 1 and obtain the values of a and b.
    (iii) Compare the like terms and obtain the values of c and d.

23. (i) Simplify LHS and RHS by taking LCM.
    (ii) Compare the like terms and obtain the required values.

24. $\dfrac{2x^2+3}{x^4+8x^2+15} = \dfrac{Ax+B}{x^2+5} + \dfrac{Cx+D}{x^2+3}$.

25. $\dfrac{3x+1}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$.

26. (i) Simplify LHS and RHS by taking LCM.
    (ii) Put $x = 0$ and -2 and obtain the values of A and B.
    (iii) Compare the like terms and obtain the value of C.

27. (i) Simplify LHS and RHS by taking LCM.
    (ii) Compare the like terms and obtain the required values.

28. $\dfrac{ax-b}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$.

29. $\dfrac{4x^2+3x+2}{(x-4)^3} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{C}{(x-4)^3}$.

30. $\dfrac{1}{x^2-7x-18}=\dfrac{A}{x-9}+\dfrac{B}{x+2}.$

31. Let $\dfrac{10}{x^2-25}=\dfrac{A}{x-5}+\dfrac{B}{x+5}$

$$\frac{10}{x^2-25}=\frac{A(x+5)+B(x-5)}{(x-5)(x+5)}.$$

Consider, $A(x+5)+B(x-5)=10$

Put $x=5$, $10A=10 \Rightarrow A=1$

Put $x=-5$, $-10B=10 \Rightarrow B=-1$

$$\therefore \quad \frac{10}{x^2-5}=\frac{1}{x-5}-\frac{1}{x+5}.$$

32. Let $\dfrac{\frac{2x}{3}}{(x-1)(x-3)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-3)}$

$$\frac{\frac{2x}{3}}{(x-1)(x-3)}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}.$$

Consider, $A(x-3)+B(x-1)=\dfrac{2x}{3}$

Put $x=3$, $2B=2 \Rightarrow B=1$.

Put $x=1,-2A=\dfrac{2}{3} \Rightarrow A=-\dfrac{1}{3}$.

$$\therefore \frac{2x}{3(x-1)(x-3)}=\frac{1}{x-3}-\frac{1}{3(x-1)}.$$

33. Let $\dfrac{x}{(x+2)(x+3)}=\dfrac{A}{x+3}+\dfrac{B}{x+2}$

$$\frac{x}{(x+2)(x+3)}=\frac{A(x+2)+B(x+3)}{(x+2)(x+3)}.$$

Consider $A(x+2)+B(x+3)=x$

Put $x=-3$, $-A=-3 \Rightarrow A=3$

Put $x=-2$, $B=-2$.

$\therefore$ $A-B=3-(-2)=3+2=5$.

34. Let $\dfrac{5}{(x+2)(x+3)}=\dfrac{A}{(x+2)}+\dfrac{B}{(x+3)}$

$$\frac{5}{(x+2)(x+3)}=\frac{A(x+3)+B(x+2)}{(x+2)(x+3)}.$$

Consider $A(x+3)+B(x+2)=5$

If $x=-3$, $-B=5 \Rightarrow B=-5$

If $x=-2$, $A=5$.

$$\therefore \frac{5}{(x+2)(x+3)}=\frac{5}{x+2}-\frac{5}{x+3}.$$

35. $\dfrac{1}{x^2-7x+12}=\dfrac{1}{(x-3)(x-4)}$

Let $\dfrac{1}{(x-3)(x-4)}=\dfrac{A}{x-3}+\dfrac{B}{x-4}$ (1)

$$\frac{1}{(x-3)(x-4)}=\frac{A(x-4)+B(x-3)}{(x-3)(x-4)}.$$

Consider, $A(x-4)+B(x-3)=1$

Put $x=4$, we get $B=1$

Put $x=3$, we get $-A=1$

$\Rightarrow A=-1$.

Substitute these values in Eq. (1), we get

$$\frac{1}{(x-3)(x-4)}=\frac{1}{x-4}-\frac{1}{x-3}.$$

Level 3

36. $\dfrac{1}{x^4-3x^2-4}=\dfrac{Ax+B}{x^2-4}+\dfrac{Cx+D}{x^2+1}.$

37. (i) Use $x^3-7x-6=(x+1)(x+2)(x-3)$.

(ii) $\dfrac{4x+3}{x^3-7x-6}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x-3}.$

38. (i) Simplify LHS and RHS by taking LCM.
(ii) Compare the like terms and obtain the required values.

39. $\dfrac{3x-5}{(x-1)^4}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}+\dfrac{D}{(x-1)^4}.$

40. $\dfrac{x^2+x+1}{x^3+1}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}.$

41. Given, $\dfrac{2x^2+8x+13}{(x+1)^4}.$

Let $y=x+1 \Rightarrow x=y-1$

HINTS AND EXPLANATION

$$= \frac{2(y-1)^2 + 8(y-1) + 13}{y^4}$$

$$= \frac{2y^2 + 2 - 4y + 8y - 8 + 13}{y^4}$$

$$= \frac{2y^2 + 4y + 7}{y^4} = \frac{2}{y^2} + \frac{4}{y^3} + \frac{7}{y^4}$$

$$= \frac{2}{(x+1)^2} + \frac{4}{(x+1)^3} + \frac{7}{(x+1)^4}.$$

42. Let $\frac{6x^2 - 14x + 6}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$

$$\frac{6x^2 - 14x + 6}{x(x-1)(x+2)}$$

$$= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x(x-1)(x-2)}$$

$6x^2 - 14x + 6$

$= A\,(x-1)(x-2) + Bx(x-2) + C(x-1)x$

Put $x = 1$,

$6 - 14 + 6 = B1(1-2) \Rightarrow -2 = -B$

$\Rightarrow B = 2.$

Put $x = 2$, $6(2)^2 - 14(2) + 6 = C(2-1)(2)$

$2 = 2C$

$1 = C$

Put $x = 0$, $6 = A\,(-1)\,(-2) \Rightarrow A = 3$

$$\therefore \frac{6x^2 - 14x + 6}{x(x-1)(x-2)} = \frac{3}{x} + \frac{2}{x-1} + \frac{1}{x+2}.$$

43. $\frac{1}{(x-4)(x^2+3)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+3}$

$$\frac{1}{(x-4)(x^2+3)}$$

$$= \frac{A(x^2+3) + (x-4)(Bx+C)}{(x-4)(x^2+3)}$$

Consider, $Ax^2 + 3A + Bx^2 + Cx - 4Bx - 4C = 1$

$(A + B)x^2 + (C - 4B)x + 3A - 4C = 1$

Comparing the like terms, we get

$A + B = 0$, $C - 4B = 0$ and $(3A - 4C) = 1$

Solving the above equations, we get

$$A = \frac{1}{19}, B = \frac{-1}{19} C = \frac{-4}{19}.$$

$$\therefore \frac{1}{(x-4)(x^2+3)} = \frac{1}{19}\left[\frac{1}{x-4} - \frac{x+4}{x^2+3}\right].$$

44. Let $x^2 = p$, then $\frac{3x^2 + 7}{x^4 - 3x^2 + 2} = \frac{3p + 7}{p^2 - 3p + 2}$

Let $\frac{3p+7}{p^2 - 3p + 2} = \frac{A}{p-1} + \frac{B}{p-2}$

$$\Rightarrow \frac{3p+7}{(p-1)(p-2)} = \frac{A(p-2) + B(p-1)}{(p-1)(p-2)}$$

Consider,

$3p + 7 = A(p - 2) + B(p - 1)$

Put $p = 1$, $-A = 10 \Rightarrow A = -10$

$p = 2$, $B = 13$

$$\frac{3p+7}{p^2 - 3p + 2} = \frac{13}{p-2} - \frac{10}{p-1}.$$

But $p = x^2$,

$$\therefore \frac{3x^2 + 7}{x^4 - 3p^2 + 2} = \frac{13}{x^2 - 2} - \frac{10}{x^2 - 1}.$$

$$= \frac{13}{x^2 - 2} - 5\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{13}{x^2 - 2} + \frac{5}{x+1} - \frac{5}{x-1}.$$

45. $\frac{4x^2 + 5x + 6}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

$$\frac{4x^2 + 5x + 6}{x^2(x+3)} = \frac{Ax(x+3) + B(x+3) + Cx^2}{x^2(x+3)}$$

Consider,

$4x^2 + 5x + 6 = A\,x(x + 3) + B(x + 3) + cx^2$

Put $x = 0$, $6 = 3B \Rightarrow B = 2$

Put $x = -3$, $27 = 9C \Rightarrow C = 3$

Comparing the coefficient of x^2, we have

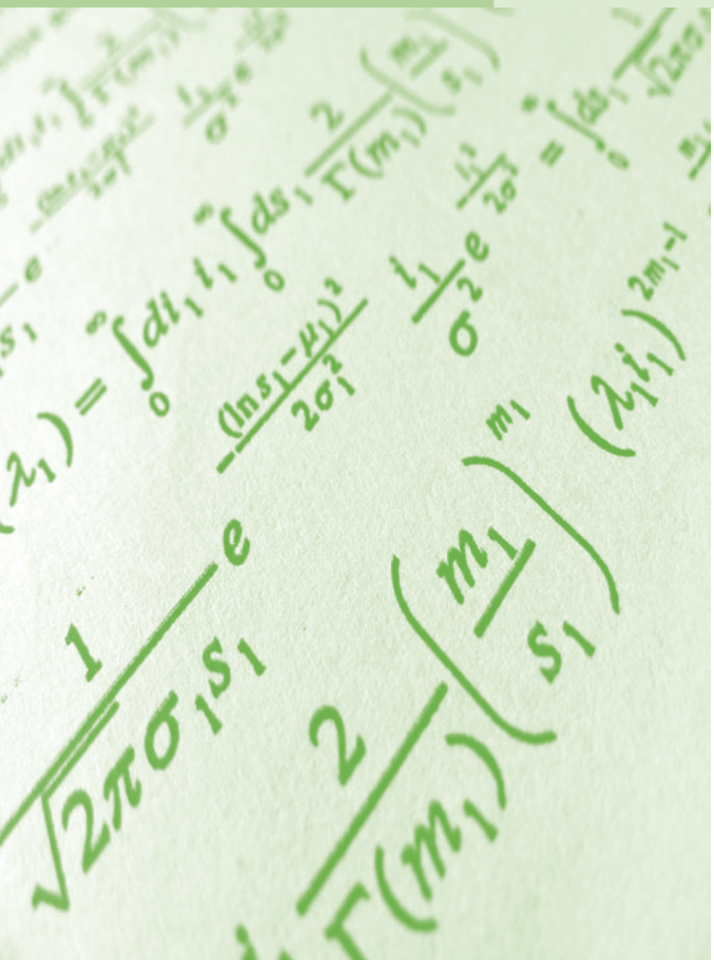
$A + C = 4 \Rightarrow A = 1.$

$2A + 3B + 4C = 2 + 6 + 12 = 20.$

Chapter

27

Logarithms



REMEMBER

Before beginning this chapter, you should be able to:

- Define logarithms
- Recall properties of logarithms

KEY IDEAS

After completing this chapter, you would be able to:

- Study system of logarithms
- Study use of signs for log values
- Find the log of a number to base 10
- Use log tables
- Find the antilog

INTRODUCTION

In earlier classes, we have learnt about indices. One of the results we have learnt is that, if $2^x = 2^3$, then $x = 3$ and if $4^x = 4^y$, then $x = y$, i.e., if two powers of the same base are equal and the base is not equal to -1, 0 or 1, then the indices are equal. But when $3^x = 5^2$, just by using the knowledge of indices, we cannot find the numerical value of x. The necessity of the concept of logarithms arises here. Logarithms are useful in long calculations involving multiplication and division. Logarithms also useful in the study of Probability, Statistics, Psychology, etc.

Definition

The logarithm of any positive number to a given base (a positive number not equal to 1) is the index of the power of the base which is equal to that number. If N and a ($\neq 1$) are any two positive real numbers and for some real x, $a^x = N$, then x is said to be the logarithm of N to the base a. It is written as $\log_a N = x$, i.e., if $a^x = N$, then $x = \log_a N$.

Examples:

1. $3^4 = 81 \Rightarrow 4 = \log_3 81$
2. $7^3 = 343 \Rightarrow 3 = \log_7 343$
3. $2^5 = 32 \Rightarrow 5 = \log_2 32$
4. $5^4 = 625 \Rightarrow 4 = \log_5 625$
5. $10^2 = 100 \Rightarrow 2 = \log_{10} 100$

If in a particular relation, all the log expressions are to the same base, we normally do not specify the base.

From the definition of logs, we get the following results:
When $a > 0$, $b > 0$ and $a \neq 1$,

1. $\log_a a^n = n$, e.g., $\log_6 6^3 = 3$
2. $a^{\log_a b} = b$, e.g., $9^{\log_9 5} = 5$

SYSTEM OF LOGARITHMS

Though we can talk of the logarithm of a number to any positive base as not equal to 1, there are two systems of logarithms, viz., natural logarithms and common logarithms, which are used most often.

1. **Natural logarithms:** These were discovered by Napier. They are calculated to the base 'e', which is approximately equal to 2.7828. These are used in higher mathematics.
2. **Common logarithms:** Logarithms to the base 10 are known as common logarithms. This system was introduced by Briggs, a contemporary of Napier. In the rest of this chapter, we shall use the short form 'log' instead of 'logarithm'.

Properties

1. Logs are defined only for positive real numbers.
2. Logs are defined only for positive bases (other then 1).
3. In $\boldsymbol{\log_b a}$ neither a is negative nor b is negative but the value of $\boldsymbol{\log_b a}$ can be negative.

Example: As $10^{-2} = 0.01$, $\log_{10} 0.01 = -2$.

4. Logs of different numbers to the same base are different, i.e., if $a \neq b$, then $\log_m a \neq \log_m b$. In other words, if $\log_m a = \log_m b$, then $a = b$.

   Example:

   $\log_{10} 2 \neq \log_{10} 3$

   $\log_{10} 2 = \log_{10} y \Rightarrow y = 2$.

5. Logs of the same number to different bases have different values, i.e., if $m \neq n$, then $\log_m a \neq \log_n a$. In other words, if $\log_m a = \log_n a$, then $m = n$.

   Example:

   $\log_2 16 \neq \log_4 16$

   $\log_2 16 = \log_n 16 \Rightarrow n = 2$.

6. Log of 1 to any base is 0.

   Example: $\log_2 1 = 0$ ($\because 2^\circ = 1$)

7. Log of a number to the same base is 1.

   Example: $\log_4 4 = 1$.

8. Log of 0 is not defined.

Laws

1. $\log_m (ab) = \log_m a + \log_m b$

   Example: $\log 56 = \log (7 \times 8) = \log 7 + \log 8$

2. $\log_m \left(\dfrac{a}{b}\right) = \log_m a - \log_m b$

   Example: $\log\left(\dfrac{81}{23}\right) = \log 81 - \log 23$

3. $\log a^m = m \log a$

   Example: $\log 216 = \log 6^3 = 3\log 6$

4. $\log_b a \log_c b = \log_c a$ (Chain Rule)

   Example: $\log_2 3 \times \log_8 2 \times \log_5 8 = \log_8 3 \times \log_5 8 = \log_5 3$

5. $\log_b a = \dfrac{\log_c a}{\log_4 b}$ (Change of Base Rule)

   Example: $\log_9 25 = \dfrac{\log_4 25}{\log_4 9}$

Note In this relation, if we replace c by a, then we get the following result:

$$\log_b a = \frac{1}{\log_a b}$$

Variation of $\log_a x$ with x

For $1 < a$ and $0 < p < q$, $\log_a p < \log_a q$.
For $0 < a < 1$ and $0 < p < q$, $\log_a p > \log_a q$.

Example: $\log_{10} 2 < \log_{10} 3$ and $\log_{0.1} 2 > \log_{0.1} 3$

Bases which are greater than 1 are called **strong bases** and bases which are less than 1 are called **weak bases**. Therefore, for strong bases, the log increases with the number and for weak bases, the log decreases with the number.

Sign of $\log_a x$ for Different Values of x and a

Strong Bases ($a > 1$)

1. If $x > 1$, $\log_a x$ is positive.

   For example, $\log_2 8$, $\log_4 81$ are positive.

2. If $0 < x < 1$, then $\log_a x$ is negative.

   For example, $\log_4 0.02 = \dfrac{\log 0.02}{\log 4} = \dfrac{\log 2 - \log 100}{\log 4}$

   $\log 2 < \log 100$ and $0 < \log 4$ for strong bases

   $\therefore \dfrac{\log 2 - \log 100}{\log 4} < 0$

   $\Rightarrow \log_4 0.02$ is negative.

Weak Bases ($0 < a < 1$)

1. If $x > 1$, then $\log_a x$ is negative.

   For example $\log_{0.3} 15$ and $\log_{0.4} 16$ are negative.

   Consider, $\log_{0.3} 15 = \dfrac{\log 15}{\log 0.3} = \dfrac{\log 15}{\log 3 - \log 10}$

   $\log 3 < \log 10$ (for any strong base)

   $\Rightarrow \dfrac{\log 15}{\log 3 - \log 10} < 0.$

2. If $0 < x < 1$, then $\log_a x$ is positive.

   For example, $\log_{0.1} 0.2$, $\log_{0.4} 0.3$ are positive.

Note

Logs of numbers (> 1) to strong bases and numbers (< 1) to weak bases are positive.

EXAMPLE 27.1

If $p = \log_{2a} a$, $q = \log_{3a} 2a$ and $r = \log_{4a} 3a$, then find the value of $qr(2-p)$.

(a) 1 **(b)** 0 **(c)** 2 **(d)** 3

SOLUTION

$qr = \log_{3a} 2a \log_{4a} 3a = \log_{4a} 2a$

$pqr = \log_{2a} a \log_{4a} 2a = \log_{4a} a$

Now $qr(2-p) = 2qr - pqr$

$= 2\log_{4a} 2a - \log_{4a} a$

$= \log_{4a} 4a^2 - \log_{4a} a$

$= \log_{4a} \dfrac{4a^2}{a} = \log_{4a} 4a = 1.$

EXAMPLE 27.2

If $3^x = (0.3)^y = 10000$, then find the value of $\dfrac{1}{x} - \dfrac{1}{y}$.

(a) 1 **(b)** $\dfrac{1}{2}$ **(c)** $\dfrac{1}{4}$ **(d)** $\dfrac{1}{3}$

SOLUTION

Given,, $3^x = (0.3)^y = 10000$

$3^x = 10^4$ $(0.3)^y = 10^4$

$x = 4 \log_3 10 \Rightarrow y = 4 \log_{0.3} 10$

$$\frac{1}{x} = \frac{1}{4\log_3 10} \qquad \frac{1}{y} = \frac{1}{4\log_{0.3} 10}$$

$$= \left(\frac{1}{4}\right)\log_{10} 3 \quad = \left(\frac{1}{4}\right)\log_{10} 0.3$$

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{4}[\log_{10} 3 - \log_{10} 0.3]$$

$$= \frac{1}{4}\left[\log_{10} \frac{3}{0.3}\right]$$

$$= \frac{1}{4}[\log_{10} 10]$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = \frac{1}{4}.$$

EXAMPLE 27.3

If $(2x)^{\log_{\sqrt{x}} x} = 16,$ then find the value of x.

(a) 2 **(b)** –2 **(c)** Both (a) and (b) **(d)** None of these

SOLUTION

$$(2x)^{\log_{\sqrt{x}} x} = 16$$

$$(2x)^{2\log_x x} = 16$$

$$(2x)^2 = 16$$

$$4x^2 = 16$$

$$x^2 = 4$$

$$x = \pm 2.$$

When $x = -2, \log_{\sqrt{x}} x$ is not defined.

$\therefore x = 2.$

EXAMPLE 27.4

Find the value of $3^{\frac{4}{\log_2 9}} + 27^{\frac{1}{\log_{49} 9}} + 81^{\frac{1}{\log_4 3}}$.

(a) 603 **(b)** 585 **(c)** 676 **(d)** 524

SOLUTION

$$3^{\frac{4}{\log_2 9}} + 27^{\frac{1}{\log_{49} 9}} + 81^{\frac{1}{\log_4 3}}$$

$$= 3^{4\log_9 2} + 27^{\log_9 49} + 81^{\log_3 4}$$

$$= 3^{2\log_3 2} + 3^{3\log_3 7} + 3^{4\log_3 4}$$

$$= 3^{\log_3 4} + 3^{\log_3 343} + 3^{4\log_3 256}$$

$$= 4 + 343 + 256 = 603.$$

EXAMPLE 27.5

If $p \in R$ and $q = \log_x \left(p + \sqrt{p^2 + 1}\right)$, then find the value of p in terms of x and q.

(a) $\dfrac{x^q + x^{-q}}{2}$ **(b)** $\dfrac{x^q - x^{-q}}{2}$ **(c)** $x^q + x^{-q}$ **(d)** $x^q - x^{-q}$

SOLUTION

Given, $q = \log_x \left(p + \sqrt{p^2+1}\right) x^q$

$$= p + \sqrt{p^2+1}$$

$$\Rightarrow x^{-q} = \frac{1}{x^q} = \frac{1}{p + \sqrt{p^2+1}}$$

$$= \frac{p - \sqrt{p^2+1}}{\left(p + \sqrt{p^2+1}\right)\left(p - \sqrt{p^2+1}\right)}$$

$$= \frac{p - \sqrt{p^2+1}}{p^2 - (p^2+1)} = -p + \sqrt{p^2+1}$$

$$\therefore x^q - x^{-q} = p + \sqrt{p^2+1} - \left(\sqrt{p^2+1} - p\right)$$

$$x^q - x^{-q} = 2p$$

$$\frac{x^q - x^{-q}}{2} = p.$$

To Find the log of a Number to the Base 10

Consider the following numbers:

2, 20, 200, 0.2 and 0.02.

We see that 20 = 10(2) and 200 = 100(2)

∴ log 20 = 1 + log 2 and log 200 = 2 + log 2.

Similarly, log 0.2 = –1 + log 2 and log 0.02 = –2 + log 2.

From the tables, we see that log 2 = 0.3010. ('Using the tables' is explained in greater detail in later examples).

∴ log 20 = 1.3010, log 200 = 2.3010, log 0.2 = –1 + 0.3010 and log 0.02 = –2 + 0.3010.

Notes

1. Multiplying or dividing by a power of 10 changes only the integral part of the log, not the fractional part.
2. For numbers less than 1, (for example log 0.2) it is more convenient to leave the log value as –1 + 0.3010 instead of changing it to –0.6090. We refer to the first form (in which the fraction is positive) as the standard form and the second form as the normal form. Both the forms represent the same number.

For numbers less than 1, it is convenient to express the log in the standard form. As the negative sign refers only to the integral part, it is written above the integral part, rather than in front, i.e., $\log 0.2 = \bar{1}.3010$ and not -1.3010.

The convenience of the standard form will be clear when we learn how to take the antilog, which will later be explained in detail.

antilog (–0.6090) = antilog (–1 + 0.3010) = antilog $\bar{1}.3010 = 0.2$.

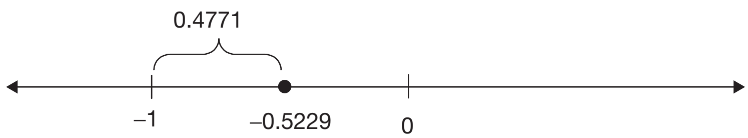
When the logs of numbers are expressed in the standard form (for numbers greater than 1, the standard form of the log is the same as the normal form), the integral part is called the characteristic and the fractional part (which is always positive) is called the mantissa.

EXAMPLE 27.6

Express –0.5229 in the standard form and locate it on the number line.

SOLUTION

$-0.5229 = -1 + 1 - 0.5229 = \bar{1}.4771.$



The Rule to Obtain the Characteristic of log *x*

1. If $x > 1$ and there are n digits in x, then the characteristic is $n - 1$.
2. If $x < 1$ and there are m zeroes between the decimal point and the first non-zero digit of x, then the characteristic is $(-m + 1)$ more commonly written as $(\overline{m+1})$.

Note

$-4 = \bar{4}$, but $-4.01 \neq \bar{4}.01$

To Find the log of a Number from the log Tables

EXAMPLE 27.7

Find the values of log 36, log 3600 and log 0.0036.

SOLUTION

In log tables, we find the number 36 in the first column. In this row in the next column (under zero), we find .5563 (the decimal point is dropped in other columns). This gives 5563 as the mantissa for the log of all numbers whose significant digits are 3 and 6.

$\therefore$ Prefixing characteristic, we have $\log 36 = 1.5563$

Similarly $\log 3600 = 3.5563$ and $\log 0.0036 = \bar{3}.5563$.

EXAMPLE 27.8

Find the values of $\log 3.74$, $\log 374000$ and $\log 0.3740$.

SOLUTION

In the log table, we locate 37 in the first column. In this row, in the column under 4, we find 5729. As in the earlier example, the same line as before gives the mantissa of logarithms of all numbers which begin with 37. From this line, we select the mantissa which is located in the column number 4. This gives 5729 as the mantissa for all numbers whose significant digits are 3, 7 and 4.

$\therefore \log 3.74 = 0.5729$

$\log 374000 = 5.5729$ and

$\log 0.3740 = \bar{1}.5729$.

EXAMPLE 27.9

Find the values of log 5.342 and log 0.05342.

SOLUTION

As found in the above example, we can find the mantissa for the sequence of digits 534 as 7275. Since there are four significant digits in 5342, in the same row where we found 7275 under the column 2 in the mean difference column, we can find the number 2.

$\therefore$ The mantissa of the logarithm of 5342 is 7275 + 2 = 7277.

Thus, log 5.342 = 0.7277

Similarly, log 0.05342 = $\bar{2}.7277$.

ANTILOG

As $\log_2 8 = 3$, 8 is the **antilogarithm** of 3 to the base 2, i.e., **antilog** of b to base m is $\boldsymbol{m^b}$.

In the above example, we have seen that log5.342 = 0.7277.

$\therefore$ Antilog 0.7277 = 5.342.

To Find the Antilog

EXAMPLE 27.10

Find the antilog of 2.421.

SOLUTION

Step 1: In the antilog table we find the number. 42 in the first column. In that row in the column under 1, we find 2636.

Step 2: As the characteristic is 2, we place the decimal after three digits from the left, i.e., antilog 2.421 = 263.6.

Note If the characteristic is n (a non-negative integer), then we would place the decimal after $(n + 1)$ digits from the left.

EXAMPLE 27.11

Find the antilog of 1.4215.

SOLUTION

We have to locate .42 in the first column and scan along the horizontal line and pick out the number in the column headed by 1. We see that the number is 2636. The mean difference for 5 in the same line is 3.

$\therefore$ The sum of these numbers is 2636 + 3 = 2639.

As the characteristic is 1, the required antilog is 26.39.

EXAMPLE 27.12

Find the value of $\frac{7.211 \times 0.084}{16.52 \times 0.016}$.

SOLUTION

log of a fraction = (log of numerator) – (log of denominator)

log of numerator = $\log 7.211 + \log 0.084 = 0.8580 + \bar{2}.9243 = \bar{1}.7823$

log of denominator = $\log 16.52 + \log 0.016 = 1.2180 + \bar{2}.2041 = \bar{1}.4221$

log of the given fraction = $\bar{1}.7823 - \bar{1}.4221 = 0.3602$

Value of the fraction = antilog (0.3602) = 2.292. (As the characteristic is 0, the decimal is kept after one digit from the left)

EXAMPLE 27.13

If $\log_{10} 4 = 0.6021$ and $\log_{10} 5 = 0.6990$, then find the value of $\log_{10} 1600$.

SOLUTION

$$\begin{aligned}
\log_{10} 1600 &= \log_{10}(64 \times 25) = \log_{10}(4^3 \times 5^2) \\
&= \log_{10} 4^3 + \log_{10} 5^2 \\
&= 3\log_{10} 4 + 2\log_{10} 5 \\
&= 3(0.6021) + 2(0.6990) \\
&= 1.8063 + 1.3980
\end{aligned}$$

$\log_{10} 1600 = 3.2043.$

EXAMPLE 27.14

Find the value of $\sqrt[3]{16.51}$ approximately.

SOLUTION

Let $P = \sqrt[3]{16.51}$

$$\begin{aligned}
\log P &= \log (16.51)^{1/3} \\
&= \frac{1}{3}\log 16.51 \\
&= \frac{1}{3}(1.2178) = 0.4059
\end{aligned}$$

$\log P = 0.4059$

$P = \text{antilog}\ (0.4059)$

$\therefore\ P = 2.546.$

TEST YOUR CONCEPT

Very Short Answer Type Questions

1. $\frac{1}{5}\log_2 32 + 3\log_{64} 4 =$ ________.
2. The characteristic of the logarithm of 3.6275 is ________.
3. If $4\log_x 8 = 3$, then $x =$ ________.
4. If $\log x - \frac{2}{3}\log x = 1$, then $x =$ ________.
5. If $a = \log\frac{3}{2}, b = \frac{4}{25}$ and $c = \log\frac{5}{9}$, then $a + b + c$ = ________.
6. The number of digits in the integral part of the number whose logarithm is 4.8345 is ________.
7. If $\log x = 32.756$, then $\log 10x =$ ________.
8. The characteristic of the logarithm of 0.0062 is ________.
9. If $\log_a x$ (where $a > 1$) is positive, then the range of x is ________.
10. If $\log 27.91 = 1.4458$, then $\log 2.791 =$ ________.
11. $\frac{\log 15 - \log 6}{\log 20 - \log 8} =$ ________.
12. If $\log 2 = 0.3010$, then $\log 5 =$ ________.
13. The value of $\log_{16} \sqrt[5]{64} =$ ________.
14. $\frac{\log 216}{\log 6} =$ ________.
15. If $\log_4 3 = x$, then $\log_{\sqrt[4]{3}} \sqrt[4]{64} =$ ________.
16. If $\log_x\left(\frac{1}{243}\right) = -5$, then find the value of x.
17. $7^{\log_{343} 27} =$ ________.
18. If $3^{\log_9 x} = 2$, then $x =$ ________.
19. If $\log_{xyz} x + \log_{xyz} y + \log_{xyz} z = \log_{10} p$, then $p =$ ________.
20. If $\log_{10} 4 + \log_{10} m = 2$, then $m =$ ________.
21. Simplify: $3\log_3 5 + \log_3 10 - \log_3 625$.
22. If $\log(a + 1) + \log(a - 1) = \log 15$, then $a =$ ________.
23. The value of $\log 10 + \log 100 + \log 1000 + \cdots + \log 10000000000 =$ ________.
24. If the number of zeroes between the decimal point and the first non-zero digit of a number is 2, then the characteristic of logarithm of that number is ________.
25. The value of $\log(\tan 10°) + \log(\tan 20°) + \log(\tan 45°) + \log(\tan 70°) + \log(\tan 80°) =$ ________.

Short Answer Type Questions

26. Simplify: $\log\left(\frac{3}{18}\right) + \log\left(\frac{45}{8}\right) - \log\left(\frac{15}{16}\right)$.
27. Show that $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$.
28. Solve for real value of x: $\log(x - 1) + \log(x^2 + x + 1) = \log 999$.
29. If $\frac{1}{1 + \log_a 10} = \frac{3}{2}$, then find the value of a.
30. If $x^2 + y^2 = 23xy$, then show that $2\log(x + y) = 2\log 5 + \log x + \log y$.
31. If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, then find the value $\log_{10} 135$.
32. If $\log_{10} 2 = x$ and $\log_{10} 3 = y$, then find $\log_{10} 21.6$.
33. If $\log_{10} 2 = 0.3010$, then find the number of digits in $(64)^{10}$.
34. Simplify $\frac{1}{\log_2 \log_2 \log_2 256}$.
35. Prove that $\log_3 810 = 4 + \log_3 10$.

PRACTICE QUESTIONS

Essay Type Questions

36. Solve: $x^{\log_4 3} + 3^{\log_4 x} = 18$.

37. If $p^2 + q^2 = 14pq$, then prove that $\log\left(\frac{p+q}{4}\right) = \frac{1}{2}[\log p + \log q]$.

38. Without using tables, find the value of $4\log_{10} 5 + 5\log_{10} 2 - \frac{1}{2}\log_{10} 4$.

39. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, then prove that $a^a b^b c^c = 1$.

40. Arrange the following numbers in the increasing order of their magnitude. $\log_7 9$, $\log_{18} 16$, $\log_6 41$, $\log_2 10$.

CONCEPT APPLICATION

Level 1

1. If $\log_{16x} = 2.5$, then $x =$ ________.
(a) 40 (b) 256
(c) 1024 (d) 1025

2. If $\log 5 = 0.699$ and $(1000)^x = 5$, then find the value of x.
(a) 0.0699 (b) 0.0233
(c) 0.233 (d) 10

3. The value of $\log\left(\frac{18}{14}\right) + \log\left(\frac{35}{48}\right) - \log\left(\frac{15}{16}\right) =$
(a) 0 (b) 1
(c) 2 (d) $\log_{16} 15$

4. If $\log_3 a + \log_9 a + \log_{81} a = \frac{35}{4}$, then $a =$ ________.
(a) 27 (b) 243
(c) 81 (d) 240

5. If $\log_9[(\log_8 x)] < 0$, then x belongs to ________.
(a) (1, 8) (b) $(-\infty, 8)$
(c) $(8, \infty)$ (d) (8, 1)

6. If $\log_3 \frac{x^3}{3} - 2\log_3 3x^3 = a - b\log_3 x$, then find the value of $a + b$.
(a) 6 (b) −6
(c) 0 (d) −3

7. The value of $\log_{40} 5$ lies between ________.
(a) $\frac{1}{3}$ and $\frac{1}{2}$ (b) $\frac{1}{4}$ and $\frac{1}{3}$
(c) $\frac{1}{2}$ and 1 (d) 2 and 3

8. If $x = \log_{\frac{1}{2}} \frac{4}{3} \cdot \log_2 \frac{1}{3} \cdot \log_{\frac{2}{3}} 0.8$, then ________.
(a) $x > 0$ (b) $x < 0$
(c) $x = 0$ (d) $x \geq 0$

9. If $\log_{144} 729 = x$, then the value of $\log_{36} 256$ is ________.
(a) $\frac{4(3-x)}{(3+x)}$ (b) $\frac{4(3+x)}{(3-x)}$
(c) $\frac{(3+x)}{4(3-x)}$ (d) $\frac{(3-x)}{4(3+x)}$

10. The solution set of the equation $\log(2x - 5) - \log 3 = \log 4 - \log(x + 9)$ is ________.
(a) $\left\{\frac{-19}{2}, 3\right\}$ (b) $\left\{-3, \frac{19}{2}\right\}$
(c) $\left\{3, \frac{19}{2}\right\}$ (d) $\{3\}$

11. If $\log_{10} \tan 19° + \log_{10} \tan 21° + \log_{10} \tan 37° + \log_{10} \tan 45° + \text{long}_{10} \tan 69° + \log_{10} \tan 71° + \log_{10} \tan 53° = \log_{10} \frac{x}{2}$, then $x =$ ________.
(a) 0 (b) 1
(c) 2 (d) 4

12. The solution set of the equation $\log(x + 6) - \log 8 = \log 9 - \log(x + 7)$ is ________.
(a) $\{-15, 2\}$ (b) $\{2\}$
(c) $\{-15, 0, 2\}$ (d) $\{0, 2\}$

PRACTICE QUESTIONS

13. If $\log 40\ 4 = x$ and $\log_{40} 5 = y$, then express $\log_{40} 32$ in terms of x and y.
(a) $5(1 + x + y)$ (b) $5(1 - x + y)$
(c) $5(1 - x - y)$ (d) $5(1 + x - y)$

14. If $\log_{10} 11 = p$, then $\log_{10}\left(\frac{1}{110}\right) =$ ________.
(a) $(1 + p)^{-1}$ (b) $-(1 + p)$
(c) $1 - p$ (d) $\frac{1}{10p}$

15. If $\log_4 \frac{x^4}{4} + 3\log_4 4x^4 = p + q\log_4 x$, then the value of $\log_p (q)$ is ________.
(a) 4 (b) −4
(c) 3 (d) 2

16. If $\log_4 x + \log_8 x^2 + \log_{16} x^3 = \frac{23}{2}$, then $\log_x 8 =$
(a) 2 (b) $\frac{1}{2}$
(c) 3 (d) $\frac{3}{4}$

17. If $\log_{(x+y)}(x - y) = 7$, then the value of $\log_{(x^2-y^2)}(x^2 + 2xy + y^2)$ is ________.
(a) 14 (b) $\frac{2}{7}$
(c) $\frac{7}{2}$ (d) $\frac{1}{4}$

18. The value of $\log_{35} 3$ lies between ________.
(a) $\frac{1}{4}$ and $\frac{1}{3}$ (b) $\frac{1}{3}$ and $\frac{1}{2}$
(c) $\frac{1}{2}$ and 1 (d) 3 and 4

19. If $\log\left(\frac{a+b}{6}\right) = \frac{1}{2}(\log a + \log b)$, then $\frac{a}{b} + \frac{b}{a} =$ ________.
(a) 30 (b) 31
(c) 32 (d) 34

20. If $\log_p q = x$, then $\log_{1/p}\left(\frac{1}{q}\right) =$ ________.
(a) $\frac{1}{x}$ (b) $-x$
(c) x (d) x^2

21. If $\log_{(x-y)} (x + y) = 5$, then what is the value of $\log_{x^2-y^2}(x^2 - 2xy + y^2)$?
(a) 1 (b) $\frac{\sqrt{5}}{3}$
(c) $\frac{1}{3}$ (d) 0

22. The value of $\log_a 1 + \log_2 2^2 + \log_3 3^3 \log_a 1 + \log_2 2^2 + \log_3 3^3$ (where a is a positive number and $a \neq 1$) is ________.
(a) 210 (b) 209
(c) 145 (d) 89

23. $\log_{1/2} \frac{2}{3}$ ______ $\log_{2/3} \frac{1}{2}$. The appropriate symbol in the blank is
(a) > (b) <
(c) = (d) Cannot be determined

24. The value of $\log_3[\log_2\{\log_4(\log_5 625^4)\}]$ is ________.
(a) 0 (b) 1
(c) 2 (d) 3 log 4

25. If $\log(x - 3) + \log(x + 2) = \log(x^2 + x - 6)$, then the real value of x, which satisfies the above equation is
(a) is any value of x.
(b) is any value of x except $x = 0$.
(c) is any values of x except $x = 3$.
(d) Does not exist

Level 2

26. If $\log_{\left(\sqrt{b\sqrt{b\sqrt{b\sqrt{b}}}}\right)}\left(\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}}\right) = x\log_b a$, then $x =$
(a) $\frac{32}{16}$ (b) $\frac{31}{15}$
(c) $\frac{31}{30}$ (d) $\frac{1}{2}$

27. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10} \sqrt{x}$, then $\frac{a}{b} + \frac{b}{a} =$
(a) 47 (b) 51
(c) 14 (d) 49

PRACTICE QUESTIONS

28. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10}\sqrt{x} =$ _______.

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) 0

29. The value of $\log_b a + \log_{b^2} a^2 + \log_{b^3} a^3 + \cdots + \log_{b^n} a^n$ is _______.

(a) n (b) $\log_b a$

(c) $\frac{n(n+1)}{2}\log_b a$ (d) $\log_b a^n$

30. If $(\log_2 x) + \log_2(\log_4 x) = 2$, then find $\log_x 4$.

(a) 2 (b) $\frac{1}{2}$

(c) 1 (d) 0

31. If $pqr = 1$ then find the value of $\log_{rq} p + \log_{rp} q + \log_{pq} r$.

(a) 0 (b) -1

(c) -3 (d) 1

32. If $\log_3[\log_2\{\log_x(\log_6 216^3)\}] = 0$, then $\log_3(3x) =$ _______.

(a) $\log_3 12$ (b) 1

(c) 2 (d) $\log_3 6$

33. If a^x, b^x and c^x are in GP, then which of the following is/are true?

(A) a, b, c are in GP

(B) $\log a, \log b, \log c$ are in GP

(C) $\log a, \log b, \log c$ are in AP

(D) a, b, c are in AP

(a) A and B (b) A and C

(c) B and D (d) Only A

34. The value of $\frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \frac{1}{\log_5 n} + \cdots + \frac{1}{\log_8 n}$ is __________.

(a) $\log_n 8!$ (b) $\log_{n!} 8$

(c) $\log_n\left(\frac{8!}{2}\right)$ (d) $\log_{n!} 8!$

35. If $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y}$, then $abc =$ ________.

(a) $a^x b^y c^z$ (b) $a^{y+z}\, b^{z+x}\, c^{x+y}$

(c) 1 (d) All of these

36. If $\frac{1}{\log_x 10} = \frac{3}{\log_p 10} - 3$, then $x =$ ________.

(a) $100p^2$ (b) $\frac{p^2}{100}$

(c) $1000p^3$ (d) $\frac{p^3}{1000}$

37. If $\log_8 m = 3.5$ and $\log_2 n = 7$, then the value of m in terms of n is ________.

(a) $n\sqrt{n}$ (b) $2n$

(c) n^2 (d) $\sqrt[3]{n}$

38. If $\log_{12}(\log_7 x) < 0$, then x belongs to ______.

(a) $(1, \infty)$ (b) $(1, 7)$

(c) $(1, \infty)$ (d) $(1, 7)$

39. The value of $\log_{381} 7$ lies between ________.

(a) $\frac{1}{3}$ and $\frac{1}{2}$ (b) $\frac{1}{4}$ and $\frac{1}{3}$

(c) $\frac{1}{5}$ and $\frac{1}{4}$ (d) $\frac{1}{6}$ and $\frac{1}{5}$

40. If $\log_{10}\tan 31° \cdot \log_{10}\tan 32° \cdots \log_{10}\tan 60° = \log 10a$, then $a =$ _________.

(a) 10 (b) 1

(c) 4 (d) 2

Level 3

41. The solution set for $|1-x|^{\log_{10}(x^2-5x+5)} = 1$, is _____.

(a) $\{0, 1, 4\}$ (b) $\{1, 4\}$

(c) $\{0, 4\}$ (d) $\{0, 2, 4\}$

42. The value of $\log\sqrt{2\sqrt{2\sqrt{2\ldots\infty \text{ times}}}} + \log\sqrt{3\sqrt{3\sqrt{3\ldots\infty \text{ times}}}}$ is

(a) 1 (b) 2

(c) $\log 5$ (d) $\log 6$

43. The least positive integral value of the expression $\frac{1}{2}\log_{10} m - \log_{m^{-2}} 10$ is _________.

(a) 0 (b) 1

(c) 2 (d) -1

PRACTICE QUESTIONS

44. The domain of log $(3 - 5x)$ is ________.

(a) $\left(\frac{3}{5}, \infty\right)$ (b) $\left(0, \frac{3}{5}\right)$

(c) $\left(-\infty, \frac{3}{5}\right)$ (d) $\left(-\frac{3}{5}, 0\right)$

45. If $\log_7 x + \log_7 y \geq 2$, then the smallest possible integral value of $x + y$ (given $x \neq y$) is ________.

(a) 7 (b) 14

(c) 15 (d) 20

46. If p, q, r are in GP and $a^p = b^q = c^r$, then which of the following is true?

(a) $\log_c b = \log_a c$ (b) $\log_c b = \log_b a$

(c) $\log_c a = \log_b c$ (d) None of these

47. The value of $\log_5 \sqrt{5\sqrt{5\sqrt{5\ldots^{\circ}}}}$ $+\log\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\infty\right)$ is ________.

(a) 1 (b) 25

(c) 10 (d) 20

48. The solution set of $|x + 2|^{\log_{10}(x^2+6x+9)} = 1$ is ________.

(a) $\{-3, -4\}$ (b) $\{0, -3\}$

(c) $\{-4, -1\}$ (d) $\{-3, -1\}$

49. If $\log_p pq = x$, then $\log_q pq =$ ______.

(a) $\frac{x}{x-1}$ (b) $\frac{x-1}{x}$

(c) $\frac{x}{x+1}$ (d) $\frac{x+1}{x}$

50. If $\log_{1/8}(\log_4(x^2 - 5)) > 0$, then ________.

(a) $x \in (-\infty, -3) \cup (3, \infty)$

(b) $x \in \left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$

(c) $x \in \left(-3, \sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$

(d) $x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$

51. If $p = \log_a bc$, $q = \log_b ca$ and $r = \log_c ab$, then which of the following is true?

(a) $p + q + r + 2 = pqr$ (b) $pqr = 2$

(c) $p + q + r = pqr$ (d) $pqr = 1$

52. If $\log_2 p + \log_8 p + \log_{32} p = \frac{46}{5}$, then $p =$ ________.

(a) 128 (b) 64

(c) 32 (d) 256

53. If p and q are positive numbers other than 1, then the least value of $|\log_q p + \log_p q|$ is ______.

(a) 3 (b) 1

(c) 2 (d) 4

54. If $\log_{48} 81 = x$, then $\log_{12} 3 =$ ________.

(a) $\frac{x+4}{2x}$ (b) $\frac{x+4}{x}$

(c) $\frac{x}{x+4}$ (d) $\frac{2x}{x+4}$

55. If $\log_l p$, $\log_m p$ and $\log_n p$ are in AP, then $(ln)^{\log_l m} =$ ________.

(a) n^2 (b) m^2

(c) l^2 (d) p^2

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. 2
2. 0
3. 16
4. 10^3
5. $\log \frac{2}{15}$
6. 5
7. 33.756
8. $\bar{3}$
9. $(1, \infty)$
10. 0.4458
11. 1
12. 0.6990
13. $\frac{3}{10}$
14. 3
15. $\frac{3}{x}$
16. 3
17. 3
18. 4
19. 10
20. 25
21. $\log_3 2$
22. 4
23. 55
24. $\bar{3}$
25. 0

Short Answer Type Questions

26. 0.
28. $x = 10$
29. 10^{-3}
31. 2.1303
32. $3(x + y) - 1$.
33. 19
34. $\log_3 2$

Essay Type Questions

36. 16
38. 4
40. $\log_{18} 16, \log_7 9, \log_6 41, \log_2 10$

CONCEPT APPLICATION

Level 1

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (a) **6.** (c) **7.** (a) **8.** (a) **9.** (a) **10.** (d)
11. (c) **12.** (b) **13.** (c) **14.** (b) **15.** (a) **16.** (b) **17.** (d) **18.** (a) **19.** (d) **20.** (c)
21. (c) **22.** (b) **23.** (b) **24.** (a) **25.** (d)

Level 2

26. (c) **27.** (b) **28.** (a) **29.** (d) **30.** (b) **31.** (c) **32.** (c) **33.** (b) **34.** (c) **35.** (d)
36. (d) **37.** (a) **38.** (b) **39.** (b) **40.** (b)

Level 3

41. (c) **42.** (d) **43.** (b) **44.** (c) **45.** (c) **46.** (b) **47.** (a) **48.** (c) **49.** (a) **50.** (d)
51. (a) **52.** (b) **53.** (c) **54.** (d) **55.** (a)

CONCEPT APPLICATION

Level 1

1. $\log_b a = n \Rightarrow a = b^n$.
2. Taking logarithms for $10^{3x} = 5$ and substituting in the given equation we get the value of x.
3. $\log a + \log b - \log c = \log\left(\frac{ab}{c}\right)$ and $\log 1 = 0$.
4. $\log_{b^n} a = \frac{1}{n}\log_b a$.
5. Given, $\log_9(\log_8 x) < 0$
   $\log_8 x < 9^0$
   $\log_7 x < 1$
   $x < 7^1$
   There fore $x \in (1, 7)$.
6. Use $\log a - \log b = \log\frac{a}{b}$ and $\log a + \log b = \log ab$.
7. Consider $5^2 < 40 < 5^3$ take logarithm with base 5.
8. $\log_b a > 0$; when $a > 1$ and $b < 1$
   $\log_b a < 0$ when $a < 1$ and $b > 1$
   $\log_b a > 0$ when $a < 1$ and $b < 1$.
9. Find the values of $\log_{12} 4$, $\log_{12} 36$ and using $\log_{12} 27 = x$.
10. $\log a - \log b = \log\left(\frac{a}{b}\right)$.
11. $\tan\theta \cdot \tan(90 - \theta) = 1$.
12. $\log a - \log b = \log\frac{a}{b}$.
13. Express $\log_{40} 32$ in terms of x and y.
14. $\log\frac{1}{a} = \log a^{-1}$; $\log mn = \log m + \log n$ and $\log_a a = 1$.
15. $\log a + \log b = \log ab$
    $\log a - \log b = \log\frac{a}{b}$.
16. $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
17. Adding '1' on both sides and the '1' on the left side is expressed as $\log_{x+y}(x + y)$.
18. Consider the inequality and $3^3 < 35 < 3^4$ taking logarithm with base 3.
19. (i) Use, $\log a + \log b = \log ab$. remove the logarithms on both sides and evaluate $(a + b)^2$.
    (ii) Use $m(\log a + \log b) = \log(ab)^m$.
    (iii) Eliminate logarithms on both sides and obtain equations in terms of a and b.
    (iv) Divide both sides of the equation with ab and obtain the required answer.
20. $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
21. Adding '1' on both sides, the '1' on the left side is expressed as $\log_{x-y}(x - y)$.
22. (i) $\log_b a^m = m\log_b a$ and $\sum n = \frac{n(n+1)}{2}$.
    (ii) $\log_a 1 = 0$, $\log_2 2^2 = 2$, $\log_3 3^3 = 3$, and so on.
    (iii) The required answer is the sum of first 20 natural numbers except 1.
23. (i) Use $a > b \Rightarrow \log_b a > 1$ and $a < b \Rightarrow \log_b a < 1$.
24. (i) $\log a^m = m\log a$.
    (ii) Express 625 in terms of base 5 and simplify from the extreme right logarithm.
25. $\log a + \log b = \log ab$.

Level 2

26. (i) Use, $\sqrt{a\sqrt{a\sqrt{a\cdots n \text{ terms}}}} = a^{(2^n - 1/2^n)}$ and then $\log_{b^n} a^m = \frac{m}{n}\log_b a$ and simplify LHS.
    (ii) Compare LHS and RHS and find the value of x.
27. (i) Use $\log a + \log b = \log ab$ and remove logarithms on both the sides and evaluate $(a - b)^2$.
    (ii) Express RHS into single logarithm with coefficient 1.
    (iii) Apply antilog and cancel the logarithms on both sides.
    (iv) Divide LHS and RHS by ab and obtain the required value.
28. (i) $x^{\log y} = y^{\log x}$.
    (ii) $7^{\log x} = x^{\log 7}$.
    (iii) Convert LHS into $7^{\log x}$ (or) $x^{\log 7}$ and solve for x.
29. (i) $\log_{b^n} a^m = \frac{m}{n}\log_b a$.
    $\log a + \log b = \log ab$.

(ii) Each term of the given expression is equal to $\log_b a$.
(iii) There are n terms in the expression.
(iv) Use the above information and find the required sum.

30. (i) Assume $\log_2 x = a$ then $\log_4 x = \frac{a}{2}$.
(ii) Take $\log_2 (\log_4 x)$ as $2 \log_4 (\log_4 x)$ and convert LHS into single logarithm.
(iii) Express the result in terms of $\log_2 x$ and solve for $\log_2 x$.
(iv) Find $\log_x 2$ and then $\log_x 4$.

31. (i) Use $\log_a a = 1$.
(ii) Replace $rq = p^{-1}$, $rp = q^{-1}$ and $pq = r^{-1}$ in the given expression.
(iii) Simplify and eliminate logarithms.

32. (i) Remove logarithm one by one by using
$$\log_a x = b \Rightarrow x = a^b.$$
(ii) Now substitute the value of x in $\log_3 3x$ and simplify.

33. (i) Verify from options.
(ii) Use if a, b and c are in GP, then $b^2 = ac$.
(iii) Substitute a^x, b^x and c^x in the above equation and simplify.
(iv) Apply logarithm for the above result and proceed.

34. (i) $\text{Log} \log_b a = \frac{1}{\log_a b}$.
(ii) Take all the logarithms to the numerators by using the formula $\frac{1}{\log_b a} = \log_a b$.
(iii) Use, $\log a + \log b + \cdots + \log n = \log (abc\ldots n)$ and simplify.

35. (i) Equate the given ratios to k and get the values of $\log a$, $\log b$ and $\log c$.
(ii) Add $\log a$, $\log b$ and $\log c$ and solve for abc.

36. Given, $\frac{1}{\log_x 10} = \frac{3}{\log_p 10} - 3$

$$\log_{10} x = 3\log_{10} p - 3\left(\therefore \log_b a = \frac{1}{\log_a b}\right)$$

$$\log_{10} x = 3(\log_{10} p - 1)$$
$$= 3(\log_{10} p - \log_{10} 10)$$
$$\log_{10} x = \log_{10}\left(\frac{p}{10}\right)^3$$
$$x = \left(\frac{p}{10}\right)^3 = \frac{p^3}{1000}.$$

37. Given, $\log_8 m = 3.5 \log_2 n = 7$
$\Rightarrow m = 8^{3.5}$ (1)
and $n = 2^7$ (2)
$$m = 8^{35/10}$$
$$m = (2^3)^{7/2}$$
$$= (2^7)^{3/2}$$
$$= n^{3/2} \quad (\because \text{ from Eq. (1)})$$
$$\Rightarrow m = n\sqrt{n}.$$

38. Given, $\log_{12} (\log_7 x) < 0$ is defined when $\log_7 x > 0$
$x > 7^0 \Rightarrow x > 1$ (1)
$\log_{12}(\log_7 x) < 0 \Rightarrow \log_7 x < 12°$
$\Rightarrow \log_7 x < 1 = x < 7^1 \Rightarrow x < 7$ (2)
From Eqs. (1) and (2), we get $x \in (1, 7)$.

39. We know that $7^3 < 381 < 7^4$
$\Rightarrow \log_7 7^3 < \log_7 381 < \log_7 7^4$
$\Rightarrow 3 < \log_7 381 < 4$
$$\Rightarrow \frac{1}{4} < \frac{1}{\log_7 381} < \frac{1}{3}$$
$$\frac{1}{4} < \log_{381} 7 < \frac{1}{3}\left(\because \log_b a = \frac{1}{\log_a b}\right).$$

40. Given, $\log_{10} a = \log_{10} \tan 31° \log_{10} \tan 32° \ldots \log_{10} \tan 45° \ldots \log_{10} \tan 60°$
$= \log_{10} \tan 31° \ldots \log_{10} 1 \ldots \log_{10} \tan 60°$
$= 0 \; (\because \log_{10} 1 = 0)$
$\log_{10} a = 0 \Rightarrow a = 1.$

Level 3

41. (i) Use $a^0 = 1$ and $1^m = 1$.
(ii) Consider RHS, i.e., 1 as $|1 - x|^0$ and equate the exponents.
(iii) Convert the logarithm in the exponential form by using $\log_b a = N \Rightarrow a = b^N$.
(iv) Solve the quadratic equation for x.

42. (i) $\log a + \log b = \log ab$ and $\log\sqrt{x\sqrt{x\ldots\infty}} = x$.

(ii) Use, $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\ldots\infty}}}} = a$ and then use $\log a + \log b = \log ab$.

43. (i) The least positive integral value of $x + \dfrac{1}{x} = 2$.

(ii) Let $\log_m 10 = x$ then $\log_m 10 = \dfrac{1}{x}$.

(iii) The given expression becomes $\dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$.

(iv) Now the least positive value is obtained if $x + \dfrac{1}{x}$ is minimum.

44. (i) $\log f(x)$ is defined only when $f(x) > 0$.

(ii) logarithms take only positive values. That is, $(3 - 5x) > 0$.

(iii) Solve the above inequation for x.

45. $\log_7 x + \log_7 y \geq 2$

$\log_7 xy \geq 2$

$xy \geq 7^2$

$xy \geq 49$

$\therefore$ The possible values of (x, y) are (1, 49), (2, 25), (3, 17), (4, 14), (5, 10), (6, 9), (7, 8), …

$\therefore$ Smallest possible value of $5 + 10 = 6 + 9 = 7 + 8 = 15$

$\therefore$ Hence, option (c) is the correct answer.

46. Let $ap = bq = cr = k$

Then $p = \log_a k$, $q = \log_b k$ and $r = \log_c k$

Also given $q^2 = pr$

$(\log_b k)^2 = (\log_a k)(\log_c k)$

$(\log_b k)(\log_b k) = (\log_a k)(\log_c k)$

$$\frac{\log_b k}{\log_a k} = \frac{\log_c k}{\log_b k}$$

$\log_b a = \log_c b$.

47. Let $x = \sqrt{5\sqrt{5\sqrt{5\ldots\infty}}}$

$x = \sqrt{5x}$

$x^2 = 5x$

$\Rightarrow x = 5\ (\because x \neq 0)$

$\therefore \log_5\sqrt{5\sqrt{5\ldots\infty}} = \log_5 5 = 1$.

Consider $\dfrac{1}{2} + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^3 + \cdots + \infty$

Clearly it is a GP, $s_\infty = \dfrac{a}{1-r}$

Here $a = \dfrac{1}{2}$ and $r = \dfrac{1}{2}$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$$\therefore \log\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right) = \log 1 = 0.$$

$\therefore$ The required value $= 1 + 0 = 1$.

48. Given, $|x+2|^{\log_{10} x^2 + 6x + 9} = 1$

As the RHS = 1

The exponent of $x + 2$ should be 0 ($a^0 = 1$)

or $|x + 2| = 1$

$\Rightarrow \log_{10} x^2 + 6x + 9 = 0$ or $x + 2 = \pm 1$

$x^2 + 6x + 9 = 1$ or $x = -1$ or -3

$x = -2$ or -4

$x = -1$ or -3

when $x = -3$, $\log_{10}(x^2 + 6x + 9)$ is not defined.

When $x = -2$, $|x + 2| = 0$ which is not possible.

$x = -4$, or -1.

49. Given, $\log_p pq = x$

$\log_p p + \log_p q = x$

$1 + \log_p q = x$

$\log_p q = x - 1$

$\log_q pq = \log_q p + \log_q q$

$$= \frac{1}{x-1} + 1 = \frac{1 + x - 1}{x - 1} = \frac{x}{x-1}.$$

50. Given, $\log_{1/8}(\log_4(x^2 - 5)) > 0$

Here $\log_4 x^2 - 5 > 0$

$x^2 - 5 > 4^0 \Rightarrow x^2 - 5 > 1$

$\Rightarrow x^2 - 6 > 0 \Rightarrow x^2 - \left(\sqrt{6}\right)^2 > 0$

$\left(x - \sqrt{6}\right)\left(x + \sqrt{6}\right) > 0$

$\therefore x \in \left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$ (1)

And $\log_{1/8}(\log_4 x^2 - 5) > 0$

$$\Rightarrow \log_4(x^2 - 5) < \left(\frac{1}{8}\right)^0$$

$x^2 - 5 < 4^1$

$\Rightarrow x^2 - 9 < 0$

$(x-3)(x+3) < 0$

$\therefore x \in (-3, 3)$ (2)

From Eqs. (1) and (2), we have

$x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$.

51. $pqr = (\log_a b + \log_a c)(\log_b c + \log_b a)(\log_c a + \log_c b)$

$= (\log_a c + 1 + \log_a c \cdot \log_b c + \log_b c)(\log_c a + \log_c b)$

$= 1 + \log_a b + \log_c a + \log_c b + \log_b c + \log_a c + \log_b a + 1$

$= 2 + (\log_a b + \log_a c) + (\log_c a + \log_c b) + (\log_b c + \log_b a)$

$= 2 + \log_a bc + \log_b ac + \log_c ab$

$= 2 + p + q + r$.

52. Given, $\log_2 p + \log_8 p + \log_{32} p = \frac{46}{5}$

$$\log_2 p + \left(\frac{1}{3}\right)\log_2 p + \left(\frac{1}{5}\right)\log_2 p = \frac{46}{5}$$

$$\log_2 p\left(1 + \frac{1}{3} + \frac{1}{5}\right) = \frac{46}{5}$$

$$\left(\frac{15+5+3}{15}\right)\log_2 p = \frac{46}{5}$$

$$\left(\frac{23}{15}\right)\log_2 p = \frac{46}{5}$$

$$\log_2 p = \frac{46}{5} \times \frac{15}{23}$$

$\log_2 p = 6$

$p = 2^6 = 64$.

53. We know that AM $(a, b) \geq$ GM (a, b)

AM $(\log_q p, \log_p q) \geq$ GM $(\log_q p, \log_p q)$

$$\frac{|\log_q p + \log_p q|}{2} \geq \sqrt{\log_p q \cdot \log_q p}$$

$\Rightarrow |\log_q p + \log_p q| \geq 2$.

$\therefore$ The least value is 2.

54. Given, $\log_{48} 81 = x$

$\log_{48} 3^4 = x$

$4 \log_{48} 3 = x$

$$\log_3 48 = \frac{4}{x}$$

$$\log_3 3(16) = \frac{4}{x}$$

$$\log_3 3 + \log_3 16 = \frac{4}{x}$$

$$1 + \log_3 4^2 = \frac{4}{x}$$

$$2\log_3 4 = \left(\frac{4}{x}\right) - 1$$

$$\log_3 4 = \frac{4-x}{2x}.$$

Consider $\log_3 12 = \log_3 3(4)$

$= \log_3 3 + \log_3 4$

$$\Rightarrow 1 + \frac{4-x}{2x}$$

$$\Rightarrow \log_3 12 = \frac{x+4}{2x}$$

$$\log_{12} 3 = \frac{2x}{x+4}.$$

55. Given, $\log_l p$, $\log_m p$, $\log_n p$ are in AP.

$\Rightarrow \log_p l$, $\log_p m$ and $\log_p n$ are in HP.

$$\log_p m = \frac{2\log_p l \cdot \log_p n}{\log_p l + \log_p n}$$

$$\frac{\log_p m}{\log_p l} = \frac{2\log_p n}{\log_p ln}$$

$\Rightarrow \log_l m = \log_{ln} n^2$

$\Rightarrow (ln)^{\log_l m} = n^2$.

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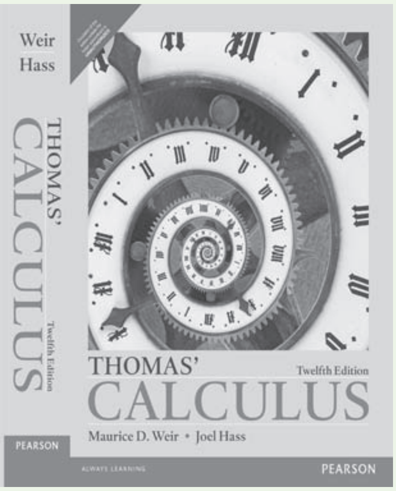

Thomas' Calculus, 12/e

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About the Book

Calculus hasn't changed, but students have. Today's students have been raised on immediacy and the desire for relevance, and they come to calculus with varied mathematical backgrounds. Thomas' Calculus, Twelfth Edition, helps students successfully generalize and apply the key ideas of calculus through clear and precise explanations, clean design, thoughtfully chosen examples, and superior exercise sets. This book offers the right mix of basic, conceptual, and challenging exercises, along with meaningful applications. This significant revision features more examples, more mid-level exercises, more figures, improved conceptual flow, and the best in technology for learning and teaching.

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- Strong exercise sets feature a great breadth of problems-progressing from skills problems to applied and theoretical problems-to encourage students to think about and practice the concepts until they achieve mastery.
- Figures are conceived and rendered to provide insights for students and support conceptual reasoning.
- The flexible table of contents divides complex topics into manageable sections, allowing instructors to tailor their course to meet the specific needs of their students. For example, the precise definition of the limit is contained in its own section and may be skipped.
- Complete and precise multivariable coverage enhances the connections of multivariable ideas with their single-variable analogues studied earlier in the book.

Contents

Appendices

About the Author

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